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ON SOME CONSTANTS OF QUASICONFORMAL DEFORMATION AND ZYGMUND CLASS

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Abstract

A real-valued function $f(x)$ on $\Re$ belongs to Zygmund class $\Lambda_*(\Re)$ if its Zygmund norm $\|f\|_z = \inf\limits_{x,t}\left|\frac{f(x+t)-2f(x)+f(x-t)}{t}\right|$ is finite. It is proved that when $f \in \Lambda_*(\Re)$, there exists an extension $F(z)$ of f to $H = \{\mathrm{Im} z > 0\}$ such that

$$\|\overline{\partial}F\|_\infty \le \frac{\sqrt{1+53^2}}{72}\|f\|_z.$$

It is also proved that if $f(0) = f(1) = 0$, then

$$\max_{x\in[0,1]} |f(x)| \le \frac{1}{3}\|f\|_z.$$

Keywords Quasiconformal deformation, Zygmund class, Beurling-Ahlfors extension.
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§1. Introduction

Let $f(z)$ be a continous real-valued function on $\Re$. If it satisfies

$$|f(x+t) - 2f(x) + f(x-t)| \le C|t| \tag{1.1}$$

for all $x, t \in \Re$ and some constant C, we say it belongs to Zygmund class $\Lambda_*(\Re)$. If $f(x) \in \Lambda_*(\Re)$, we denote the infimum of the values C in (1.1) by $\|f\|_z$. A continous complex-valued function $F(z)$ is called a quasiconformal deformation in the terminology of [1] if it has generalized derivative $\overline{\partial}F$ and $\|\overline{\partial}F\|_\infty < +\infty$. Let $Q_*(H)$ be the class of quasiconformal deformations on the upper half plane H. It was proved independently by Gardiner and Sullivan in [3] and Reich and the first author of this paper in [4] that the necessary and sufficient condition for a real-valued function $f(x)$ on $\Re$ to have an extension $F(z) \in Q_*(H)$ is $f(x) \in \Lambda_*(\Re)$. In [3], Gardiner and Sullivan proved that when $f(x) \in \Lambda_*(\Re)$, the Beuling-Ahlfors extension $F_{BA}(z) = u(x,y) + iv(x,y)$, where

$$\begin{cases} u(x,y) = \dfrac{1}{2y}\displaystyle\int_{x-y}^{x+y} f(t)dt, \\ v(x,y) = \dfrac{1}{y}\Big(\displaystyle\int_{x}^{x+y} f(t)dt - \int_{x-y}^{x} f(t)dt\Big) \end{cases} \tag{1.2}$$

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is a quasiconformal deformation on H. From their proof, it is not difficult to know that

$$\|\overline{\partial}F_{BA}\|_\infty \leq \frac{\sqrt{5}}{2}\|f\|_z. \tag{1.3}$$

Define $m_0(f) = \inf\{ \| \overline{\partial}F\|_\infty : F|_\Re = f \text{ and } F \in Q_*(H) \}$. Reich discussed in [5] the following two constants:

$$\mu = \sup_f\Big\{\frac{m_0(f)}{\|f\|_z}\Big\}, \tag{1.4}$$

and

$$\mu_{BA} = \sup_f\Big\{\frac{\|\overline{\partial}F_{BA}\|_\infty}{\|f\|_z}\Big\}, \tag{1.5}$$

where F_{BA} is the Beuling-Ahlfors extension of f. He pointed out that

$$0.28 \leq \mu \leq \mu_{BA} \leq \frac{\sqrt{5}}{2}. \tag{1.6}$$

We know well from [6] that quasiconformal mappings of H onto itself are closely related to quasiconformal deformations on H. Let $F(z)$ be a quasiconformal deformation on H. Then the solution $f(z,t)$ of the differential equation $\frac{dw}{dt} = F(w)$ with initial conditiobn $w(0) = z$ are quasiconformal mappings of H onto itself, and their dilatations $K(z,t)$ are bounded by $\exp\{2\|\overline{\partial}F\|_\infty t\}$. So for a given $f \in \Lambda_*(\Re)$, it is of interest and importance to find how small the L^∞-norm of the $\overline{\partial}$-derivative of its extension of quasiconformal deformation can be. In §2 we will improve the upper bound in (1.6), and obtain

Theorem 1.1. *Suppose $f(x) \in \Lambda_*(\Re)$. Then*

$$\mu \leq \mu_{BA} \leq \frac{\sqrt{1+53^2}}{72} = 0.736. \tag{1.7}$$

We are also interesed in the problem of the estimation of max $\{|f(x)| : 0 \leq x \leq 1\}$ when $f \in \Lambda_*(\Re)$ is normalized by $f(0) = f(1) = 0$.

Gardiner and Sullivan proved in [3] that

$$M = \max\{|f(x)| : 0 \leq x \leq 1\} \leq \frac{1}{2}\|f\|_z. \tag{1.8}$$

In §3, we will improve the estimation and prove

Theorem 1.2. *Suppose $f(x) \in \Lambda_*(\Re)$, and $f(0) = f(1) = 0$. Then*

$$M = \max\{|f(x)| : 0 \leq x \leq 1\} \leq \frac{1}{3}\|f\|_z. \tag{1.9}$$

§2. $\bar{\partial}$-Derivative of Beurling-Ahlfors Extension

Let $f \in \Lambda_*(\Re)$, and $F_{BA}(x,y) = u(x,y) + iv(x,y)$ is the Beurling-Ahlfors Extension of f. By (1.2) we have

$$\begin{cases} u_x = \dfrac{1}{2y}[f(x+y) - f(x-y)], \\ u_y = -\dfrac{1}{2y^2}\displaystyle\int_{x-y}^{x+y} f(t)dt + \dfrac{1}{2y}[f(x+y) - f(x-y)], \\ v_x = \dfrac{1}{y}[f(x+y) - 2f(x) + f(x-y)], \\ v_y = -\dfrac{1}{y^2}[\displaystyle\int_x^{x+y} f(t)dt - \int_{x-y}^{x} f(t)dt] + \dfrac{1}{y}[f(x+y) - f(x-y)]. \end{cases} \tag{2.1}$$

First we notice the fact that if $f(x) \in \Lambda_*(\Re)$, then $f_*(x) = \frac{1}{a} f(ax+b) + cx + d \in \Lambda_*(\Re)$ and $\|f_*\|_z = \|f\|_z$. So without loss of generality, we assume $f(0) = f(1) = 0$. Furthermore,

$$\begin{aligned} \mu_{BA} &= \sup_f \Big\{ \frac{\|\bar\partial F_{BA}\|_\infty}{\|f\|_z} \Big\} = \sup_{f,x,y} \Big\{ \frac{|\bar\partial F_{BA}(x,y)|}{\|f\|_z} \Big\} \\ &= \sup_f \Big\{ \frac{|\bar\partial F_{BA}(\frac{1}{2}, \frac{1}{2})|}{\|f\|_z} \Big\}. \end{aligned} \tag{2.2}$$

It follows that

$$|\bar\partial F_{BA}\Big(\frac{1}{2}, \frac{1}{2}\Big)|^2 = H(X,Y,Z) = 4(X-Y)^2 + (X+Y+2Z)^2, \tag{2.3}$$

where

$$\begin{cases} X = \displaystyle\int_0^{\frac{1}{2}} f(t)dt, \\ Y = \displaystyle\int_{\frac{1}{2}}^1 f(t)dt, \\ Z = f(\dfrac{1}{2}). \end{cases} \tag{2.4}$$

Now we have the following lemmas.

Lemma 2.1. *Let $f \in \Lambda_*(\Re)$, and $f(0) = f(1) = 0$. Then*

$$-\frac{1}{4}\|f\|_z \le Z \le \frac{1}{4}\|f\|_z, \tag{2.5}$$

$$-\frac{1}{4}\|f\|_z \le Y - 3X \le \frac{1}{4}\|f\|_z, \tag{2.6}$$

$$-\frac{1}{4}\|f\|_z \le X - 3Y \le \frac{1}{4}\|f\|_z, \tag{2.7}$$

where X, Y, and Z are defined by (2.4).

Proof. Inequality (2.5) is obvious.

Let $x \in (0, \frac{1}{2})$. Then

$$-x\|f\|_z \le f(2x) - 2f(x) + f(0) \le x\|f\|_z.$$

Inequality (2.6) follows from integrating the above inequality with respect to x from 0 to $\frac{1}{2}$.

Let $x \in (\frac{1}{2}, 1)$. Then

$$-(1-x)\|f\|_z \le f(1) - 2f(x) + f(2x-1) \le (1-x)\|f\|_z.$$

Inequality (2.7) follows from integrating the above inequality with respect to x from $\frac{1}{2}$ to 1.

Lemma 2.2. *Let $f \in \Lambda_*(\Re)$, and $f(0) = f(1) = 0$. Then*

$$-\frac{17}{72}\|f\|_z \le X + Y \le \frac{17}{72}\|f\|_z. \tag{2.8}$$

Proof. Let $x \in (0, \frac{1}{6})$. Then

$$-x\|f\|_z \le f(\frac{1}{2} + x) - 2f(\frac{1}{2}) + f(\frac{1}{2} - x) \le x\|f\|_z.$$

Integrating the above inequality with respect to x from 0 to $\frac{1}{6}$ leads to

$$-\frac{1}{72}\|f\|_z \le \int_{\frac{1}{3}}^{\frac{2}{3}} f(t)dt - \frac{1}{3} f(\frac{1}{2}) \le \frac{1}{72}\|f\|_z.$$

By (2.5), we get

$$-\frac{7}{72}\|f\|_z \le \int_{\frac{1}{3}}^{\frac{2}{3}} f(t)dt \le \frac{7}{72}\|f\|_z. \tag{2.9}$$

Now let $x \in (0, \frac{1}{3})$. Then

$$-x\|f\|_z \le f(2x) - 2f(x) + f(0) \le x\|f\|_z.$$

Integrating the above inequlity with respect to x from 0 to $\frac{1}{3}$ leads to

$$-\frac{1}{9}\|f\|_z \le \int_{\frac{1}{3}}^{\frac{2}{3}} f(t)dt - 3\int_0^{\frac{1}{3}} f(t)dt \le \frac{1}{9}\|f\|_z,$$

so we obtain

$$-\frac{5}{72}\|f\|_z \le \int_0^{\frac{1}{3}} f(t)dt \le \frac{5}{72}\|f\|_z. \tag{2.10}$$

For the same reason, we can obtain

$$-\frac{5}{72}\|f\|_z \le \int_{\frac{2}{3}}^{1} f(t)dt \le \frac{5}{72}\|f\|_z. \tag{2.11}$$

The lemma is proved by combining (2.9), (2.10) and (2.11).

Proof of Theorem 1.1. We are going to find the maximum value of expression (2.3) with fixed $\|f\|_z$. By Lemmas 2.1 and 2.2, we know that the point (X, Y, Z), where X, Y and Z are defined by (2.4), lies in the closed domain D bounded by planes

$$X - 3Y = \pm\frac{1}{4}\|f\|_z, \quad Y - 3X = \pm\frac{1}{4}\|f\|_z,$$

$$X + Y = \pm\frac{17}{72}\|f\|_z, \quad Z = \pm\frac{1}{4}\|f\|_z.$$

It is easy to know that the quadratic form

$$H_{X^2}\Delta X^2 + H_{Y^2}\Delta Y^2 + H_{Z^2}\Delta Z^2 + 2H_{XY}\Delta X\Delta Y + 2H_{YZ}\Delta Y\Delta Z + 2H_{ZX}\Delta Z\Delta X$$

is positive definite. So $H(X, Y, Z)$ is convex and reaches its maximum at one of the twelve vertexes of domain D.

With some computation, we obtain

$$H(X, Y, Z) \le H\Big(\frac{35}{288}\|f\|_z, \frac{33}{288}\|f\|_z, \frac{1}{4}\|f\|_z\Big) = \frac{1 + 53^2}{72^2}\|f\|_z^2.$$

Hence

$$\mu_{BA} = \sup_f\Big\{\frac{|\bar\partial F_{BA}(\frac{1}{2}, \frac{1}{2})|}{\|f\|_z}\Big\} \le \frac{\sqrt{1 + 53^2}}{72} = 0.736.$$

The proof of Theorem 1.1 is complete.

In order to obtain an estimation of μ_{BA} from below, we construct a piecewise linear function $f_*(x)$ which equals zero when $x < 0$ and $x > 1$. The dividing points in [0,1] and the values of f_* at the dividing points are listed below:

$$f_*(x) = \begin{cases} 0, & x = 0, 1, \\ \frac{1}{4}, & x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \\ \frac{5}{16}, & x = \frac{3}{8}, \frac{5}{8}. \end{cases} \tag{2.12}$$

It is not difficult to check that $\|f_*\|_z = 1$. Let $F_*(z)$ be the Beurling-Ahlfors Extension of f_*. After some computation, we obtain

$$|\bar{\partial} F_*(\frac{1}{2}, \frac{1}{2})| = \frac{45}{64} \doteq 0.703,$$

which implies

$$0.703 \le \mu_{BA} \le 0.736. \tag{2.13}$$

We do not know whether the estimation in Theorem 1.1 is sharp. So what is the exact value of μ_{BA} is still open.

§3. Maximum Value in $[0, 1]$

We still assume $f \in \Lambda_*(\Re)$ and $f(0) = f(1) = 0$. For the proof of Theorem 1.2, we need the following lemma.

Lemma 3.1. *Let* $f \in \Lambda_*(\Re)$,*and* $\max\ \{|f(a)|, |f(b)|\} \le A$. *Then we have*

$$\begin{cases} \left|f\Big(\dfrac{a+b}{2}\Big)\right| \le A + \dfrac{b-a}{4}\|f\|_z, \\ \left|f\Big(\dfrac{3a+b}{4}\Big)\right| \le A + \dfrac{b-a}{4}\|f\|_z, \end{cases} \tag{3.1}$$

and

$$\max_{x\in[a,b]} |f(x)| \le A + \frac{b-a}{2}\|f\|_z. \tag{3.2}$$

Proof. Inequalities (3.1) can be obtained from

$$\left|f(b) - 2f\Big(\frac{a+b}{2}\Big) + f(a)\right| \le \frac{b-a}{2}\|f\|_z,$$

and

$$\left|f\Big(\frac{a+b}{2}\Big) - 2f\Big(\frac{3a+b}{4}\Big) + f(a)\right| \le \frac{b-a}{4}\|f\|_z.$$

Because of symmetry, we may as well assume $|f(a+t_0)| = \max\limits_{x\in[a,b]} |f(x)|$, where $t_0 \in [0, \frac{b-a}{2}]$. From

$$|f(a) - 2f(a+t_0) + f(a+2t_0)| \le t_0\|f\|_z,$$

we obtain

$$\max_{x\in[a,b]} |f(x)| = |f(a+t_0)| \le 2|f(a+t_0)| - |f(a+2t_0)| \le A + \frac{b-a}{2}\|f\|_z.$$

Proof of Theorem 1.2. Now we set $a_0 = 0$, $b_0 = 1$, and $A_0 = 0$. Denote by Λ_0 the Zygmund class on the interval $[a_0, b_0]$ with $f(a_0) = f(b_0) = A_0$ and $\|f\|_z \le B$. By Lemma 3.1, we have

$$\left|f\Big(\frac{a_0+b_0}{2}\Big)\right| \le \frac{B}{4} = A_1,$$

$$\left|f\Big(\frac{3a_0+b_0}{4}\Big)\right| \le \frac{B}{4} = A_1$$

and

$$\sup_{x\in[a_0,b_0], f\in\Lambda_0} |f(x)| \le \frac{B}{2} = M_0.$$

Denote by Λ_1 the Zygmund class on the interval

$$[a_1, b_1] = \Big[\frac{3a_0 + b_0}{4}, \frac{a_0 + b_0}{2}\Big]$$

with $\max\{|f(a_1)|, |f(b_1)|\} \le A_1$ and $\|f\|_z \le B$. Then

$$\sup_{x\in[a_0,b_0], f\in\Lambda_0} |f(x)| \le \sup_{x\in[a_1,b_1], f\in\Lambda_1} |f(x)|. \tag{3.3}$$

By Lemma 3.1, we have for $f \in \Lambda_1$,

$$\Big|f\Big(\frac{a_1 + b_1}{2}\Big)\Big| \le \frac{B}{4} + \frac{B}{16} = A_2, \quad \Big|f\Big(\frac{3a_1 + b_1}{4}\Big)\Big| \le \frac{B}{4} + \frac{B}{16} = A_2$$

and

$$\sup_{x\in[a_1,b_1], f\in\Lambda_1} |f(x)| \le \frac{B}{4} + \frac{B}{8} = M_1.$$

Again we denote by Λ_2 the Zygmund class on the interval $[a_2, b_2] = [\frac{3a_1+b_1}{4}, \frac{a_1+b_1}{2}]$ with $\max\ \{|f(a_2)|, |f(b_2)|\} \le A_2$ and $\|f\|_z \le B$. With the same discussion as above, we have

$$\sup_{x\in[a_0,b_0], f\in\Lambda_0} |f(x)| \le \sup_{x\in[a_1,b_1], f\in\Lambda_1} |f(x)| \le \sup_{x\in[a_2,b_2], f\in\Lambda_2} |f(x)|. \tag{3.4}$$

By Lemma 3.1, we have again

$$\sup_{x\in[a_2,b_2], f\in\Lambda_2} |f(x)| \le \frac{B}{4} + \frac{B}{16} + \frac{B}{32} = M_2.$$

This procedure can be continued for any times. So we have

$$\sup_{x\in[0,1], f\in\Lambda_0} |f(x)| \le M_n = \Big(\sum_{k=0}^{n} \frac{1}{2^{2k+2}}\Big)B + \frac{B}{2^{2n+2}}. \tag{3.5}$$

Since M_n is decreasing and (3.5) holds for any n and any $B \ge \|f\|_z$, we obtain for $f \in \Lambda_*(\Re)$ and $f(0) = f(1) = 0$,

$$\max_{x\in[0,1]} |f(x)| \le \Big(\sum_{k=0}^{\infty} \frac{1}{2^{2k+2}}\Big)\|f\|_z = \frac{1}{3}\|f\|_z,$$

which completes the proof of Theorem 1.2.

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