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UNIFORM CONVERGENCY FOR WEIGHTED PERIODOGRAM OF STATIONARY LINEAR RANDOM FIELDS**

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Abstract

Let $\{X_n; n \in \mathcal{N}^2\}$ be a two dimensionally indexed linear stationary random field generated by a 1/4 martingale difference white noise. The logarithm uniform convergency resulte for the weighted periodogram of $\{X_k; 1 \le k \le n\}$ is proved.

Keywords Periodogram, Random field, Martingale difference, Log-convergency.
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§1. Introduction

Let $\{\xi_n; n \ge 1\}$ be a one dimensional stationary sequence, the periodogram of $\{\xi_1, \xi_2, \cdots, \xi_N\}$ is defined by $I_N(\xi, \lambda) = \frac{1}{2N\pi}\left|\sum\limits_{j=1}^{N} \xi_j e^{ij\lambda}\right|^2$. With some conditions, it is true that

$$\sup_{\lambda} I_N(\xi, \lambda) = O(\log N), \quad \text{a.s. as } N \to \infty \tag{1.1}$$

(cf. [1–3]). Result (1.1) is important for spectral analysis in one dimensional time series.

For two dimensionally indexed random field $\left\{X_k; k \in \mathcal{N}_+^2\right\}$, where $\mathcal{N}_+$ is the set of all positive integers, the periodogram of $\{X_k; 1 \le k \le n\}$ has the same form:

$$I_n(X, \lambda) = \frac{1}{4|n|\pi^2}\Big|\sum_{1 \le k \le n} X_k e^{ik\lambda}\Big|^2, \tag{1.2}$$

where $\lambda = (\lambda_1, \lambda_2) \in [-\pi, \pi]^2$, $\quad k\lambda = k_1\lambda_1 + k_2\lambda_2$. Just like one dimensional case, periodogram for random fields possesses important position in spectral analysis (cf. [4]). The uniform convergency result for $I_n(X, \lambda)$ is the basic tool for discrete spectral analysis of random field just like one dimensional case (cf. [5–7]). In Section 2 we gave the uniform convergency result for $\left\{X_k; k \in \mathcal{N}^2\right\}$ being a martingale difference white noise random field and in Section 3 we gave the same result for $\left\{X_k; k \in \mathcal{N}^2\right\}$ being a linear random field generated by a martingale difference white noise.

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§2. Periodogram of White Noise Random Fields

Denote by $\mathcal{N}^2$ ($\mathcal{N}_+^2$) the set of all 2-dimensional (positive) integer lattice. For $n=(n_1,n_2)$, $m=(m_1,m_2)\in\mathcal{N}^2$, assume the usual partial order, i.e., $n\le m$ means $n_i\le m_i,\ i=1,2$ and $n<m$ means $n\le m$ but $n\ne m$. For $i=1$ or 2, $n\le^{(i)}m$ means that $n_i<m_i$ or $n_i=m_i$ and $n_{3-i}\le m_{3-i}$. We use $|n|$ for $\max(1,|n_1n_2|)$. For $N=(N_1,N_2)\in\mathcal{N}_+^2$, by $N\to\infty$ we mean $N_1,N_2\to\infty$. $\{\mathcal{F}_t;t\in\mathcal{N}^2\}$ is said to be an increasing array of σ-fields under $\le$, if $s\le t$ implies $\mathcal{F}_s\subset\mathcal{F}_t$. For any increasing array of σ-fields $\{\mathcal{F}_t;t\in\mathcal{N}^2\}$, define

$$\mathcal{F}_{t-}=\vee_{s<t}\mathcal{F}_s,\ \mathcal{F}_i(t)=\vee_{s\le^{(i)}t}\mathcal{F}_s,\ i=1,2,\ \mathcal{F}_1(t-)=\mathcal{F}_1(t_1,t_2-1),$$

$$\mathcal{F}_2(t-)=\mathcal{F}_2(t_1-1,t_2),\quad \mathcal{F}_{t_1}^1=\vee_{s_1\le t_1,s_2<\infty}\mathcal{F}_s,\ \mathcal{F}_{t_2}^2=\vee_{s_2\le t_2,s_1<\infty}\mathcal{F}_s.$$

According to [8], if for any integrable random variable ξ it is true that $E(E(\xi|\mathcal{F}_n)|\mathcal{F}_m)=E(\xi|\mathcal{F}_{n\wedge m}),\ \forall n,m\in\mathcal{N}^2$, where $n\wedge m=(n_1\wedge m_1,n_2\wedge m_2)$, then we say that $\{\mathcal{F}_n;n\in\mathcal{N}^2\}$ satisfies F_4-condition. A condition equivalent to F_4-condition is, for any integrable ξ,

$$E\left(E\left(\xi|\mathcal{F}_{k_1}^1\right)|\mathcal{F}_{k_2}^2\right)=E\left(E\left(\xi|\mathcal{F}_{k_2}^2\right)|\mathcal{F}_{k_1}^1\right)=E(\xi|\mathcal{F}_k),\quad \text{a.s.},\quad k=(k_1,k_2)$$

(see [8] or [9]).

A random field $\{X_n;\ n\in\mathcal{N}^2\}$ is said adapted to $\{\mathcal{F}_n;\ n\in\mathcal{N}^2\}$ (or simply $\{X_n,\mathcal{F}_n;\ n\in\mathcal{N}^2\}$ is an adapted random field) if X_n is $\mathcal{F}_n$-measurable for each $n\in\mathcal{N}^2$. An adapted random field $\{X_n,\mathcal{F}_n;n\in\mathcal{N}^2\}$ is called a 1/4 MD (martingale difference) if $E(X_n|\mathcal{F}_{n-})=0$, a.s., $\forall\ n\in\mathcal{N}^2$, is called a 1/2 LMD (leftward MD) if $E(X_n|\mathcal{F}_1(n-))=0$, a.s. and a 1/2 RMD (rightward MD) if $E(X_n|\mathcal{F}_2(n-))=0$, a.s., $\forall\ n\in\mathcal{N}^2$.

It is seen that 1/2 LMD or 1/2 RMD requires more than 1/4 MD. If $\{X_n,\mathcal{F}_n;n\in\mathcal{N}^2\}$ is a 1/2 LMD (or RMD) with finite identical variance, it will be a white noise: $EX_n=0$, $EX_nX_m=\delta_{n,m}\sigma^2$, $\forall n,\ m\in\mathcal{N}^2$. In this case we call it a 1/2 LMD (or RMD) white noise. But for a 1/4 MD with finite identical variance to be a white noise, F_4-condition is required. For this reason, by calling $\{X_n,\mathcal{F}_n;n\in\mathcal{N}^2\}$ a 1/4 MD white noise we mean that $\{X_n,\mathcal{F}_n;n\in\mathcal{N}^2\}$ is a 1/4 MD with the σ-field array $\{\mathcal{F}_n;n\in\mathcal{N}^2\}$ satisfying F_4-condition and $EX_n^2=\sigma^2$, $\forall n\in\mathcal{N}^2$. If $\{X_n;n\in\mathcal{N}^2\}$ is an independent array of random variables, write $\mathcal{F}_n=\sigma\{X_k;k\le n\}$, the information obtained by observing $\{X_n;n\in\mathcal{N}^2\}$ up to time n; then the F_4-condition is ensured for $\{\mathcal{F}_n;n\in\mathcal{N}^2\}$ (cf. [8]). Therefore an independent white noise is a 1/4 MD white noise.

For two dimensional random field $\mathbf{W}=\{W_k;k\in\mathcal{N}^2\}$, the periodogram of $\{W_k;1\le k\le n\}$ is defined by

$$I_n(W,\lambda)=\frac{1}{4|n|\pi^2}\Big|\sum_{1\le k\le n}W_ke^{ik\lambda}\Big|^2,\quad \lambda=(\lambda_1,\lambda_2)\in[-\pi,\pi]^2. \tag{2.1}$$

Simple calculation will show that $I_N(W,\lambda)$ is a polynomial of $\lambda=(\lambda_1,\lambda_2)$ with order less than $N=(N_1,N_2)$. From Vol. 2, p.11 of [10] it is seen that for any trigonometric polynomial $T(x)$ of order $N\in\mathcal{N}_+=\{1,2,\cdots\}$, $\sup_x|\frac{d}{dx}T(x)|\le N\sup_x|T(x)|$, which leads to the following lemma.

Lemma 2.1. *Let $h(\lambda)$ be a real function defined on $[-\pi,\pi]^2$. If $h(\lambda)$ is continuously differentiable and strictly positive, write $M_N=\sup_\lambda\{I_N(W,\lambda)/h(\lambda)\}$. Then for any $\lambda=$*

(λ_1, λ_2), $\mu = (\mu_1, \mu_2) \in [-\pi, \pi]^2$, *it is true that*

$$\Big|\frac{I_N(W,\lambda)}{h(\lambda)} - \frac{I_N(W,\mu)}{h(\mu)}\Big| \le CN_1|\lambda_1 - \mu_1|M_N + CN_2|\lambda_2 - \mu_2|M_N, \tag{2.2}$$

where C is determined by $h(\lambda)$.

Denote by $[x]$ the integer part of real x. Let $\log x$ have its usual meaning except near zero: we take $\log x$ to be 1 for $x \in [0, e)$. For fixed $N \in \mathcal{N}_+^2$ and positive integers $\phi(N_1), \phi(N_2)$, divide $[-\pi, \pi]^2$ into $\phi(N_1) \times \phi(N_2)$ disjoint rectangles denoted by $\mathcal{I}_{i,j}$, $i = 1, 2, \cdots, \phi(N_1)$, $j = 1, 2, \cdots, \phi(N_2)$, each with length $2\pi/\phi(N_1)$ and width $2\pi/\phi(N_2)$. Let $\lambda_{i,j}$ be the left-upper element of $\mathcal{I}_{i,j}$. Then Lemma 2.1 leads to the following lemma.

Lemma 2.2. *With the same conditions as that of Lemma* 2.1, *for*

$$\phi(N_i) = 2\big[2\pi CN_i \log|N| + 2\big], \quad i = 1, 2, \quad \theta_N = 1 - 1/\log|N|, \tag{2.3}$$

it is true that

$$M_N \le \frac{1}{\theta_N} \max_{i,j} \frac{I_N(W, \lambda_{i,j})}{h(\lambda_{i,j})},$$

where C is given by Lemma 2.1.

Proof. Suppose $\lambda \in \mathcal{I}_{i,j}$ such that $M_N = I_N(W,\lambda)/h(\lambda)$. For $\mu \in I_{i,j}$, if $\theta_N M_N > I_N(W,\mu)/h(\mu)$, we have $0 < (1-\theta_N) M_N < I_N(W,\lambda)/h(\lambda) - I_N(W,\mu)/h(\mu)$. Lemma 2.1 shows that $N_1|\lambda_1 - \mu_1| + N_2|\lambda_2 - \mu_2| > (C\log|N|)^{-1}$. That is in contradiction with

$$N_1|\lambda_1 - \mu_1| + N_2|\lambda_2 - \mu_2| \le N_1 2\pi/\phi(N_1) + N_2 2\pi/\phi(N_2) \le (C\log|N|)^{-1}.$$

Hence Lemma 2.2 must be true.

For white noise random field $\mathbf{W} = \{W_k; k \in \mathcal{N}^2\}$ with $EW_k^2 = \sigma^2$, define

$$\mathcal{F}_n = \sigma\{W_k; k \le n\}, \quad \forall n \in \mathcal{N}^2,$$

$$\mathcal{F}_{(-\infty,0)} = \bigcap_{k_1<0} \mathcal{F}_{(k_1,0)}, \quad \mathcal{F}_{(0,-\infty)} = \bigcap_{k_2<0} \mathcal{F}_{(0,k_2)}, \quad \mathcal{F}_{-\infty} = \bigcap_{k<0} \mathcal{F}_k.$$

We introduce 5 conditions for further use.

(1) $\mathbf{W}$ is independent and there exists a nonnegative r.v. Y such that $EY^2 \log Y < \infty$, and for all $k \in \mathcal{N}_+^2$, $x \ge 0$, $P(|W_k| \ge x) \le CP(Y \ge x)$.

(2) $\mathbf{W}$ is a strictly stationary 1/4 MD white noise with

$$E(W_0 \log|W_0|)^2 < \infty, \quad E\left(W_0^2 | \mathcal{F}_{(-\infty,0)}\right) = E\left(W_0^2 | \mathcal{F}_{(0,-\infty)}\right) = \sigma^2.$$

(3) $\mathbf{W}$ is a strictly stationary ergodic 1/4 MD white noise with

$$EW_0^2 \log|W_0| < \infty. \tag{2.4}$$

(4) $\mathbf{W}$ is a strictly stationary ergodic 1/2 LMD white noise satisfying (2.4).

(5) $\mathbf{W}$ is a strictly stationary ergodic 1/2 RMD white noise satisfying (2.4).

Lemma 2.3. *Let $\mathbf{W}$ be a white noise random field satisfying condition* (1) *or* (2). *Then for any nonnegative integer u, v, $\phi(N_1), \phi(N_2), \lambda_{i,j}$ being defined by* (2.3) *and $\theta \in [0, \pi]$,*

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^{1+u} N_2^{1+v} \log|N|}} \max_{i,j} \Big| \sum_{1 \le k \le N} (k_1^u k_2^v)^{1/2} W_k \cos(k\lambda_{i,j} - \theta)\Big|$$

$$\le 2\sqrt{\frac{\sigma^2}{(1+u)(1+v)}}, \quad \text{a.s.} \tag{2.5}$$

Proof. Define $U_k = (k_1^u k_2^v)^{1/2} W_k$. Using methods similar to those of proving Theorem 2.19 of [11] and ergodic theory for random field (cf. [12]), we can prove that

$$\lim_{N\to\infty} \frac{1}{N_1^u N_2^v |N|} \sum_{1\le k\le N} U_k^2 = \lim_{N\to\infty} \frac{1}{N_1^u N_2^v |N|} \sum_{1\le k\le N} E\left(U_k^2 | \mathcal{F}_{k-}\right) = \frac{\sigma^2}{(1+u)(1+v)}, \quad \text{a.s.} \tag{2.6}$$

Without loss of generality we will suppose $\sigma^2 = 1$. Write

$$Y_k = \left(\frac{\log |N|}{|N|}\right)^{1/2} \frac{a}{\sqrt{N_1^u N_2^v}} U_k, \quad 1 \le k \le N,$$

where $a = 2\sqrt{(1+u)(1+v)}$. Using inequalities

$$\exp\left(x - x^2/6\right) \le 1 + x + x^2/3 \text{ and } e^{-x} \le 1/(1+x) \text{ for } x \ge 0 \tag{2.7}$$

we obtain for any $b > 0$

$$\begin{aligned} p_N = P\Big(&\max_{i,j} \sum_{1\le k\le N} \Big\{\frac{U_k \cos(k\lambda_{i,j} - \theta)}{\sqrt{N_1^u N_2^v |N| \log |N|}} \\ &- \frac{a}{6N_1^u N_2^v |N|}\left(U_k^2 + 2E\left(U_k^2 | \mathcal{F}_{k-}\right)\right) \cos^2(k\lambda_{i,j} - \theta)\Big\} \ge b\Big) \\ \le& \sum_{i,j} P\Big(\sum_{1\le k\le N} \big\{Y_k \cos(k\lambda_{i,j} - \theta) - \frac{1}{6}\big(Y_k^2 \\ &+ 2E\left(Y_k^2 | \mathcal{F}_{k-}\right)\big) \cos^2(k\lambda_{i,j} - \theta)\big\} \ge \log |N|^{ab}\Big) \\ \le& \sum_{i,j} P\left(T_N(i,j) \ge |N|^{ab}\right), \end{aligned}$$

where $T_N(i,j) = \prod_{1\le k\le N} U_k(i,j)/V_k(i,j)$ with

$$U_k(i,j) = 1 + Y_k \cos(k\lambda_{i,j} - \theta) + Y_k^2 \cos^2(k\lambda_{i,j} - \theta)/3$$

and

$$V_k(i,j) = 1 + E\left(Y_k^2 | \mathcal{F}_{k-}\right) \cos^2(k\lambda_{i,j} - \theta)/3.$$

Using $ET_N(i,j) = 1$ we get $p_N \le \phi(N_1)\phi(N_2)|N|^{-ab} \le C_0 |N|^{1-ab} (\log |N|)^2$.

Now Borel-Cantelli lemma implies that for $b > 2/a$,

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^u N_2^v |N| \log |N|}} \max_{i,j} \sum_{1\le k\le N} (k_1^u k_2^v)^{1/2} W_k \cos(k\lambda_{i,j} - \theta) \\ &\le b + \frac{a}{2(1+u)(1+v)}, \text{ a.s.} \end{aligned}$$

Noticing that this inequality is also true for $\left\{-W_k; k \in \mathcal{N}^2\right\}$ since it is also a white noise satisfying condition (1) or (2), we obtain that the left hand side of (2.5) is less than $b + a/\{2(1+u)(1+v)\}$, a.s. Let $b \downarrow 1/\sqrt{(1+u)(1+v)}$. Then (2.5) follows.

If $\left\{W_k; k \in \mathcal{N}^2\right\}$ is a stationary ergodic random field with $EW_0^2 \log |W_0| < \infty$, according to ergodic theory (cf. [12])

$$\limsup_{N\to\infty} \frac{1}{N_1^u N_2^v |N|} \sum_{1\le k\le N} U_k^2 \le \sigma^2,$$

$$\lim_{N\to\infty} \frac{1}{N_1^u N_2^v |N|} \sum_{1\le k\le N} E\left(U_k^2 | \mathcal{F}_{k-}\right) \le \sigma^2, \quad \text{a.s.} \tag{2.8}$$

So using the same procedure as used in the proof of Lemma 2.3 we can show the following lemma.

Lemma 2.4. *Let* $\mathbf{W}$ *satisfy condition* (3). *Then for any nonnegative integers* u, v, $\phi(N_1), \phi(N_2)$, $\lambda_{i,j}$ *being defined by* (2.3) *and* $\theta \in [0, \pi]$,

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^{1+u} N_2^{1+v} \log|N|}} \max_{i,j} \Big| \sum_{1\le k\le N} (k_1^u k_2^v)^{1/2} W_k \cos(k\lambda_{i,j} - \theta) \Big| \le 2\sqrt{\sigma^2}, \quad \text{a.s.} \tag{2.9}$$

For a strictly stationary ergodic random field without the requirement of F_4-condition for $\{\mathcal{F}_n; n \in \mathcal{N}^2\}$, similar result is also true but until now only for 1/2 LMD or 1/2 RMD.

Lemma 2.5. *Let* $\mathbf{W}$ *satisfy condition* (4) *or* (5). *Then the result of Lemma* 2.4 *is true.*

Define $\beta_m = 2\pi/m, \quad m \ge 4$. According to Lemma 2.2 and

$$|a + ib| \le \max_{1\le j\le m} \{a\cos(j\beta_m) - b\sin(j\beta_m)\} / \cos(\beta_m)$$

for $\theta_N = 1 - 1/\log|N|$, we have

$$\begin{aligned} &\sup_{\lambda} \Big| \sum_{1\le k\le N} (k_1^u k_2^v)^{1/2} W_k e^{ik\lambda} \Big| \\ &\le \frac{1}{\sqrt{\theta_N}} \max_{i,j} \Big| \sum_{1\le k\le N} (k_1^u k_2^v)^{1/2} W_k e^{ik\lambda_{i,j}} \Big| \\ &\le \frac{1}{\sqrt{\theta_N}\cos\beta_m} \max_{i,j} \max_{1\le l\le m} \Big| \sum_{1\le k\le N} (k_1^u k_2^v)^{1/2} W_k \cos(k\lambda_{i,j} - l\beta_m) \Big|. \end{aligned}$$

Now Lemmas 2.3, 2.4 and 2.5 lead to the following theorems.

Theorem 2.1. *Under conditions of* (1) *or* (2), *for any nonnegative integers* u, v

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^u N_2^v |N| \log|N|}} \sup_{\lambda} \Big| \sum_{1\le k\le N} (k_1^u k_2^v)^{1/2} W_k e^{ik\lambda} \Big| \\ &\le \frac{2\sigma}{\sqrt{(1+u)(1+v)}}, \quad \text{a.s.} \end{aligned} \tag{2.10}$$

Theorem 2.2. *Under conditions of* (3) *or* (4) *or* (5), *for any nonnegative integers* u, v,

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^u N_2^v |N| \log|N|}} \sup_{\lambda} \Big| \sum_{1\le k\le N} (k_1^u k_2^v)^{1/2} W_k e^{ik\lambda} \Big| \le 2\sigma, \quad \text{a.s.} \tag{2.11}$$

§3. Periodogram of Linear Random Fields

Let $\{W_k; k \in \mathcal{N}^2\}$ be a white noise random field: $EW_k = 0, \quad EW_nW_m = \sigma^2\delta_{n,m}$, and $\{X_n; n \in \mathcal{N}^2\}$ be a linear array generated from $\{W_k; k \in \mathcal{N}^2\}$:

$$X_n = \sum_{k\ge 0} d_k W_{n-k}, \quad n \in \mathcal{N}^2, \quad \text{with} \quad \sum_{k\ge 0} d_k^2 < \infty. \tag{3.1}$$

Then $\{X_n; n \in \mathcal{N}^2\}$ is a weakly stationary random field having spectral density

$$f(\lambda) = f(\lambda_1, \lambda_2) = \frac{1}{4\pi^2} \Big| \sum_{k\ge 0} d_k e^{ik\lambda} \Big|^2. \tag{3.2}$$

In this section we will study the uniform convergency of the periodogram produced by $\{X_n; n \in \mathcal{N}^2\}$.

Write $b_t = d_t e^{-it\lambda}$, $V_t = W_t e^{-it\lambda}$, where $\lambda = (\lambda_1, \lambda_2) \in [-\pi, \pi]^2$. Then it can be checked that for any nonnegative integers u, v, and $\lambda \in [-\pi, \pi]^2$

$$\begin{aligned}
r_N(\lambda) &\equiv \sum_{1\le k\le N} k_1^u k_2^v X_k e^{-ik\lambda} - \left(\sum_{1\le k\le N} k_1^u k_2^v W_k e^{-ik\lambda}\right)\left(\sum_{k\ge 0} a_k e^{-ik\lambda}\right) \\
&= \sum_{1\le k\le N}\sum_{0\le t\le k-1} b_t k_1^u k_2^v V_{k-t} + \sum_{1\le k\le N}\sum_{t\ge k} b_t k_1^u k_2^v V_{k-t} \\
&\quad + \sum_{1\le k\le N}\sum_{(t_1\ge k_1, 0\le t_2\le k_2-1)} b_t k_1^u k_2^v V_{k-t} + \sum_{1\le k\le N}\sum_{(t_2\ge k_2, 0\le t_1\le k_1-1)} b_t k_1^u k_2^v V_{k-t} \\
&\quad - \sum_{0\le t\le N-1}\sum_{1\le k\le N-t} b_t k_1^u k_2^v V_k - \sum_{0\le t\le N-1}\sum_{(N_1-t_1+1\le k_1\le N_1, 1\le k_2\le N_2)} b_t k_1^u k_2^v V_k \\
&\quad - \sum_{0\le t\le N-1}\sum_{(N_2-t_2+1\le k_2\le N_2, 1\le k_1\le N_1-t_1)} b_t k_1^u k_2^v V_k - \sum_{t\ge N}\sum_{1\le k\le N} b_t k_1^u k_2^v V_k \\
&\quad - \sum_{(t_1\ge N_1, 0\le t_2\le N_2-1)}\sum_{1\le k\le N} b_t k_1^u k_2^v V_k - \sum_{(t_2\ge N_2, 0\le t_1\le N_1-1)}\sum_{1\le k\le N} b_t k_1^u k_2^v V_k \\
&= \sum_{i=1}^{10} R_i(N, \lambda).
\end{aligned} \tag{3.3}$$

Suppose

$$\sum_{k\ge 0} (k_1|d_k| + k_2|d_k|) < \infty. \tag{3.4}$$

We will prove

$$\sup_\lambda |R_j(N,\lambda)| = o\left(N_1^u N_2^v \sqrt{|N|\log|N|}\right), \quad \text{a.s., for } j \ne 1, \quad 5, \tag{3.5}$$

$$\sup_\lambda |R_1(N,\lambda) + R_5(N,\lambda)| = o\left(N_1^u N_2^v \sqrt{|N|\log|N|}\right), \quad \text{a.s.} \tag{3.6}$$

Lemma 3.1. *For $N = (N_1, N_2) \in \mathcal{N}_+^2$, $k = 1$ or 2, let*

$$R(N, \lambda_{i,j}, l) = \sum_{n=1}^{l} \xi(N, \lambda_{i,j}, n), \quad 1 \le l \le N_k.$$

Here $\xi(N, \lambda_{i,j}, l) \in \mathcal{F}_l^k = \underset{s_k\le l, s_{3-k}\le<\infty}{\vee} \mathcal{F}_{(s_1,s_2)}$, $E\left(\xi(N,\lambda_{i,j},l)\,|\mathcal{F}_{l-1}^k\right) = 0$, and $\left\{\mathcal{F}_l^k; l\ge 0\right\}$ is a one parameter indexed increasing sequence of σ-fields. Suppose that $\phi(N_1), \phi(N_2)$ and $\lambda_{i,j}$ are given by (2.3). Then

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{\sqrt{N_k \log|N|}} \max_{i,j} \max_{1\le l\le N_k} |R(N, \lambda_{i,j}, l)| \\
&\le 2 + \limsup_{N\to\infty} \frac{1}{6N_k} \max_{i,j} \left\{\sum_{n=1}^{N_k} \xi^2(N, \lambda_{i,j}, n) + 2E\left(\xi(N,\lambda_{i,j},n)\,|\mathcal{F}_{n-1}^k\right)\right\}, \quad \text{a.s.}
\end{aligned} \tag{3.7}$$

Proof. We give the proof for the case of $k = 2$.

For $n \in \mathcal{N}_+$, write $U(N, \lambda_{i,j}, n) = \sqrt{\frac{\log|N|}{N_2}}\,\xi(N, \lambda_{i,j}, n)$, $\quad 1 \le n \le N_2$. Then

$$U(N, \lambda_{i,j}, n) \in \mathcal{F}_n^2, \quad E\left(U(N,\lambda_{i,j},n)\,|\mathcal{F}_{n-1}^2\right) = 0.$$

According to the proof of Lemma 2.3, using Corollary 2.1 of [11] and the fact that

$$Z_l = \prod_{n=1}^{l} \frac{1 + U\left(N, \lambda_{i,j}, n\right) + U^2\left(N, \lambda_{i,j}, n\right)/3}{1 + E\left(U^2\left(N, \lambda_{i,j}, n\right) |\mathcal{F}^2_{n-1}\right)/3} \in \mathcal{F}_l, \quad E(Z_l|\mathcal{F}_{l-1}) = Z_{l-1}, \quad 1 \le l \le N_2,$$

we obtain

$$\begin{aligned} & P\Big(\max_{i,j} \max_{1\le l\le N_2} \sum_{n=1}^{l} \Big\{ \frac{\xi\left(N, \lambda_{i,j}, n\right)}{\sqrt{N_2 \log |N|}} \\ & - \frac{1}{6N_2}\left(\xi^2\left(N, \lambda_{i,j}, n\right) + 2E\left(\xi^2\left(N, \lambda_{i,j}, n\right)|\mathcal{F}^2_{n-1}\right)\right)\Big\} \ge b\Big) \\ \le & \sum_{i,j} P\left(\max_{1\le l\le N_2} Z_l \ge |N|^b\right) \le C_0 |N|^{b-1}\left(\log |N|\right)^2. \end{aligned}$$

Hence, whenever $b > 2$, the left hand side of (3.7)

$$\le b + \limsup_{N\to\infty} \frac{1}{6N_2} \max_{i,j} \sum_{n=1}^{N_2} \left\{\xi^2\left(N, \lambda_{i,j}, n\right) + 2E\left(\xi\left(N, \lambda_{i,j}, n\right)|\mathcal{F}^2_{n-1}\right)\right\}, \quad a.s.$$

Now $b \downarrow 2$ ensures (3.7).

Proof of (3.5) for j=2. Firstly, $\frac{|R_2(N,\lambda)|}{N_1^u N_2^v \sqrt{|N|}} \le \sum_{k\ge 1} \frac{1}{\sqrt{|k|}} \sum_{t\ge k} |d_t||W_{k-t}| \equiv R_2$. Since

$$ER_2 \le \sigma \sum_{k\ge 1} \frac{1}{\sqrt{|k|}} \sum_{t\ge k} |d_t| \le 2\sigma \sum_{t\ge 1} \sqrt{|t|}|d_t|,$$

it is seen that (3.4) leads to $R_2 < \infty$, a.s. and, for $j = 2$, (3.5) follows.

Proof of (3.5) for j=3,4. $R_3\left(N, \lambda\right) = \sum_{1\le k\le N} \widetilde{W}_k e^{-ik\lambda}$, with

$$\widetilde{W}_k = \sum_{t_1 \ge k_1, 0\le t_2 \le k_2 - 1} d_t k_1^u k_2^v W_{k-t}.$$

By Lemma 2.2, for $\theta_N = 1 - 1/\log |N|$

$$\sup_{\lambda} |R_3\left(N, \lambda\right)| \le \frac{1}{\sqrt{\theta_N}} \max_{i,j} |R_3\left(N, \lambda_{i,j}\right)|. \tag{3.8}$$

Rewrite $R_3\left(N, \lambda_{i,j}\right) = \sum_{k_2=1}^{N_2} N_1^u N_2^v \xi\left(N, \lambda_{i,j}, k_2\right)$, with

$$\xi\left(N, \lambda_{i,j}, k_2\right) = \frac{1}{N_1^u N_2^v} \sum_{k_1=0}^{\infty} \sum_{t_1=k_1+1}^{k_1+N_1} \sum_{t_2=0}^{N_2-k_2} b_t \left(t_1 - k_1\right)^u \left(k_2 + t_2\right)^v V_{(-k_1,k_2)},$$

$\xi\left(N, \lambda_{i,j}, k_2\right) \in \mathcal{F}^2_{k_2}, \quad E\left(\xi\left(N, \lambda_{i,j}, k_2\right)|\mathcal{F}^2_{k_2-1}\right) = 0$, and

$$|\xi\left(N, \lambda_{i,j}, k_2\right)| \le \xi_{k_2} \equiv \sum_{k_1=0}^{\infty} \sum_{t_1=k_1+1}^{\infty} \sum_{t_2=0}^{\infty} |d_t W_{(-k_1,k_2)}|, \quad E\xi^2_{k_2} < \infty.$$

If $\left\{W_n; n \in \mathcal{N}^2\right\}$ is strictly stationary, $\{\xi_{k_2}; k_2 \in \mathcal{N}\}$ is also strictly stationary. Hence Lemma 3.1 leads to

$$\max_{i,j} |R_3\left(N, \lambda_{i,j}\right)| = o\left(N_1^u N_2^v \sqrt{|N| \log |N|}\right), \quad \text{a.s. as } N \to \infty.$$

Therefore (3.8) ensures that (3.5) is true for $j = 3$.

Symmetrically it can be shown that

$$\max_{\lambda} |R_4\left(N, \lambda\right)| = o\left(N_1^u N_2^v \sqrt{|N| \log |N|}\right), \quad \text{a.s. as } N \to \infty.$$

Proof of (3.5) for j=6.

$$R_6(N,\lambda)=\sum_{0\le t\le N-1}\ \sum_{N_1-t_1+1\le k_1\le N_1,1\le k_2\le N_2} d_t k_1^u k_2^v W_k e^{-i(t+k)\lambda}=\sum_{1\le k\le 2N} U_k e^{-ik\lambda},$$

where U_k is a random variable which has nothing to do with λ. Using Lemma 2.2, we get

$$\sup_{\lambda}|R_6(N,\lambda)|\le \frac{1}{\sqrt{\theta_{2N}}}{\max_{i,j}}'|R_6(N,\lambda_{i,j})|, \tag{3.9}$$

where ${\max_{i,j}}' = \max\limits_{1\le i\le\phi(2N_1),1\le j\le\phi(2N_2)}$.

$$\begin{aligned} R_6(N) &= \sum_{k_2=1}^{N_2}\sum_{k_1=2}^{N_1-N_1^\delta}\left\{\sum_{t_2=0}^{N_2-1}\ \sum_{t_1=N_1-k_1+1}^{N_1-1} b_t k_1^u k_2^v V_k\right\} \\ &\quad + \sum_{k_2=1}^{N_2}\ \sum_{k_1=N_1-N_1^\delta+1}^{N_1}\left\{\sum_{t_2=0}^{N_2-1}\ \sum_{t_1=N_1-k_1+1}^{N_1-1} b_t k_1^u k_2^v V_k\right\} \\ &\equiv p_1(N)+p_2(N). \end{aligned}$$

Write

$$Q_1(N)=N_1^\delta p_1(N)=\sum_{k_2=1}^{N_2}\sum_{k_1=2}^{N_1-N_1^\delta} B(N,k)V_k,$$

where $B(N,k)=N_1^\delta\sum\limits_{t_2=0}^{N_2-1}\ \sum\limits_{t_1=N_1-k_1+1}^{N_1-1} b_t k_1^u k_2^v$, with

$$|B(N,k)|\le N_1^u N_2^v\sum_{t_2=0}^{\infty}\sum_{t_1=1}^{\infty} t_1|d_t|.$$

Using the same procedure as the proof of Lemma 2.4 or Lemma 2.5, we can obtain

$${\max_{i,j}}'|Q_1(N,\lambda_{i,j})|=O\left(N_1^u N_2^v\sqrt{|N|\log|N|}\right),\quad \text{a.s.}$$

Hence

$${\max_{i,j}}'|p_1(N,\lambda_{i,j})|=o\left(N_1^u N_2^v\sqrt{|N|\log|N|}\right),\quad \text{a.s.} \tag{3.10}$$

Again

$$p_2(N,\lambda)=\sum_{k_2=1}^{N_2}\ \sum_{k_1=N_1-N_1^\delta+1}^{N_1} N_1^u N_2^v\left(\xi_1(N,\lambda,k)+i\xi_2(N,\lambda,k)\right),$$

where $\xi_1(N,\lambda,k)=D(N,\lambda,k)W_k$, $\quad\xi_2(N,\lambda,k)=B'(N,\lambda,k)W_k$, are all real valued and there exists a constant C such that $|D(N,\lambda,k)|\le C$, $|B'(N,\lambda,k)|\le C$. By the usual procedure, we can prove that, under the same condition as that of Lemma 2.3 or Lemma 2.4, for any $b>2$

$$\begin{aligned} &\limsup_{N\to\infty}\frac{d}{\sqrt{|N|\log|N|}}{\max_{i,j}}'\left|\sum_{k_2=1}^{N_2}\ \sum_{k_1=N_1-N_1^\delta+1}^{N_1}\xi_1(N,\lambda_{i,j},k)\right| \\ &\le b+\limsup_{N\to\infty}\frac{d^2C^2}{6|N|}\sum_{k_2=1}^{N_2}\ \sum_{k_1=N_1-N_1^\delta+1}^{N_1}\left(W_k^2+2E\left(W_k^2|\mathcal{F}_{k-}\right)\right),\quad \text{a.s.} \end{aligned} \tag{3.11}$$

For the same reason under conditions of Lemma 2.5, for $l = 1$ or 2,

$$\limsup_{N\to\infty} \frac{d}{\sqrt{|N|\log|N|}} \max_{i,j}{}' \Big| \sum_{k_2=1}^{N_2} \sum_{k_1=N_1-N_1^\delta+1}^{N_1} \xi_1(N,\lambda_{i,j},k) \Big|$$
$$\le b + \limsup_{N\to\infty} \frac{d^2C^2}{6|N|} \sum_{k_2=1}^{N_2} \sum_{k_1=N_1-N_1^\delta+1}^{N_1} \left(W_k^2 + 2E\left(W_k^2|\mathcal{F}_l(k-)\right)\right), \quad \text{a.s.} \tag{3.12}$$

Because as $N\to\infty$, $\left(N_1 - N_1^\delta\right)/N_1 \to 1$, by using Theorems 1 and 2 of [11], it follows that under the conditions of Lemma 2.3 or Lemma 2.4, $\lim\limits_{N\to\infty} \frac{1}{3|N|}\sum\limits_k \left(W_k^2 + 2E\left(W_k^2|\mathcal{F}_{k-}\right)\right) = \sigma^2$, a.s. Therefore $\limsup\limits_{N\to\infty} \sum\limits_{k_2=1}^{N_2} \sum\limits_{k_1=N_1-N_1^\delta+1}^{N_1} \left(W_k^2 + 2E\left(W_k^2|\mathcal{F}_{k-}\right)\right) = 0$, a.s. while under the conditions of Lemma 2.5, by using ergodic theorem for $l = 1$ or 2,

$$\limsup_{N\to\infty} \sum_{k_2=1}^{N_2} \sum_{k_1=N_1-N_1^\delta+1}^{N_1} \left(W_k^2 + 2E\left(W_k^2|\mathcal{F}_l(k-)\right)\right) = 0, \quad \text{a.s.}$$

Hence, it is true that under the conditions of Lemma 2.3 or Lemma 2.4 or Lemma 2.5

$$\limsup_{N\to\infty} \frac{d}{\sqrt{|N|\log|N|}} \max_{i,j}{}' \Big| \sum_{k_2=1}^{N_2} \sum_{k_1=N_1-N_1^\delta+1}^{N_1} \xi_1(N,\lambda_{i,j},k) \Big| \le b, \quad \text{a.s.}$$

For the same reason and under the same condition

$$\limsup_{N\to\infty} \frac{d}{\sqrt{|N|\log|N|}} \max_{i,j}{}' \Big| \sum_{k_2=1}^{N_2} \sum_{k_1=N_1-N_1^\delta+1}^{N_1} \xi_2(N,\lambda_{i,j},k) \Big| \le b, \quad \text{a.s.}$$

Since in the two inequalities d is independent of b, it must be true that

$$\max_{i,j}{}' |p_2(N,\lambda_{i,j})| = o\left(N_1^u N_2^v \sqrt{|N|\log|N|}\right), \quad \text{a.s. as } N\to\infty.$$

At last, combining (3.10) and (3.9), we see that (3.5) is true for $j = 6$.

Symmetrically it can be shown that (3.5) is true for $j = 7$.

It is clear that under the conditions of Theorem 2.1 or Theorem 2.2, (3.5) is true for $j = 8, 9, 10$.

Proof for (3.6). $R_1(N,\lambda) + R_5(N,\lambda) = N_1^u N_2^v \sum\limits_{1\le k\le N} B(N,k) W_k e^{-i(k+t)\lambda}$, with

$$B(N,k) = \sum_{1\le t\le N-k} d_t \frac{(k_1+t_1)^u (k_2+t_2)^v - k_1^u k_2^v}{N_1^u N_2^v} = O\left(N_1^{-1} + N_2^{-1}\right).$$

Therefore, using the same methods as used in the proof of Lemma 2.3 we can show that (3.6) is valid.

Now we give the following theorems.

Theorem 3.1. *Suppose that* $\mathbf{W}$ *satisfies condition* (2) *and* $\left\{X_n; n\in\mathcal{N}^2\right\}$ *is a linear array generated from* $\left\{W_k; k\in\mathcal{N}^2\right\}$*:*

$$X_n = \sum_{k\ge 0} d_k W_{n-k}, \quad n\in\mathcal{N}^2 \text{ with } \sum_{k\ge 0}(k_1+k_2)|d_k| < \infty. \tag{3.13}$$

Denote by $f(\lambda)$ *the spectral density of* $\left\{X_n; n\in\mathcal{N}^2\right\}$*. Then for any nonnegative integers*

u, v,

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^{1+2u} N_2^{1+2v} \log |N|}} \sup_{\lambda} \Big| \sum_{1\le k\le N} (k_1^u k_2^v) X_k e^{ik\lambda} \Big| \le 4\pi \sup_{\lambda} \sqrt{\frac{f(\lambda)}{(1+2u)(1+2v)}}, \text{ a.s.} \tag{3.14}$$

If $f(\lambda)$ is strictly positive, (3.14) *can be strengthened to*

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^{1+2u} N_2^{1+2v} \log |N|}} \sup_{\lambda} \left\{ \Big| \sum_{1\le k\le N} (k_1^u k_2^v) X_k e^{ik\lambda} \Big| \Big/ \sqrt{f(\lambda)} \right\} \le \frac{4\pi}{\sqrt{(1+2u)(1+2v)}}, \quad \text{a.s.} \tag{3.15}$$

Theorem 3.2. *Let* **W** *satisfy condition* (3) *or* (4) *or* (5) *and* $\{X_n; n \in \mathcal{N}^2\}$ *be a linear array generated from* $\{W_k; k \in \mathcal{N}^2\}$ *with* (3.13) *being required. Then for any nonnegative integers u, v,*

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^{1+2u} N_2^{1+2v} \log |N|}} \sup_{\lambda} \Big| \sum_{1\le k\le N} (k_1^u k_2^v) X_k e^{ik\lambda} \Big| \le 4\pi \sup_{\lambda} \sqrt{f(\lambda)}, \quad \text{a.s.} \tag{3.16}$$

If $f(\lambda)$ is strictly positive, (3.16) *can be strengthened to*

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N_1^{1+2u} N_2^{1+2v} \log |N|}} \sup_{\lambda} \left\{ \Big| \sum_{1\le k\le N} (k_1^u k_2^v) X_k e^{ik\lambda} \Big| \Big/ \sqrt{f(\lambda)} \right\} \le 4\pi, \quad \text{a.s.} \tag{3.17}$$

Corollary. *With the same conditions as that of Theorem* 3.1 *or Theorem* 3.2,

$$\limsup_{N\to\infty} \frac{1}{\log |N|} \sup_{\lambda} I_N(X, \lambda) \le 4 \sup_{\lambda} f(\lambda), \quad \text{a.s.} \tag{3.18}$$

If $f(\lambda)$ is strictly positive, (3.18) *can be strengthened to*

$$\limsup_{N\to\infty} \frac{1}{\log |N|} \sup_{\lambda} \{I_N(X, \lambda)/f(\lambda)\} \le 4, \quad \text{a.s.} \tag{3.19}$$

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