Chin. Ann. of Math.
16B: 3(1995),341-350.

FATOU PROPERTY ON HARMONIC MAPS FROM COMPLETE MANIFOLDS WITH NONNEGATIVE CURVATURE AT INFINITY INTO CONVEX BALLS

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Abstract

The author considers harmonic maps on complete noncompact manifolds, solves the Dirichlet problem in manifolds with nonnegative sectional curvature out of a compact set, and proves the Fatou theorem for harmonic maps into convex balls.

Keywords Complete manifold, Harmonic map, Convex ball, Fatou property.
1991 MR Subject Classification 58E20.

§1. Introduction

In recent three decades, harmonic maps have been attracting the attention of many geometers and analysts. In particular, in recent years, the theory of harmonic maps of complete noncompact manifolds is becoming one of the active areas in differential geometry increasingly (see [15], [8], [2]). In the present paper, motivated by [2], we consider the existence and boundary behavior of harmonic maps from complete noncompact manifolds with nonnegative sectional curvature at infinity to convex balls. In fact, we generalize Fatou property on harmonic functions (see [13, Theorem 3]) to harmonic maps with certain conditions on the domain manifolds. The reason is that in the present case, not as in [2], the Green's functions do not decrease rapidly at infinity, and we can not find Harnack inequality at infinity as in [3]. This makes us add certain conditions on Green's functions, and we assume that the energy of the harmonic maps discussed is finite. It should be pointed out that the energy of the harmonic maps constructed in Section 3 is finite. Combining these facts, we characterize all such harmonic maps. We also consider the existence of harmonic maps with prescribed boundary data at infinity from certain Cartan-Hadamard manifolds to convex balls. It should be noticed that the boundary at infinity is a geometric boundary, not necessarily a Martin boundary. Finally we should point out that the results on existence can be considered as a special case of [2], but we must firstly consider the solvability of Dirichlet problem on harmonic functions, and then the existence of harmonic maps can be obtained by using an approximation process by harmonic functions, which is analogous to [2].

This paper is organized as follows: Section 2 provides some preliminaries. In particular Lemma 2.1 is used frequently. We shall refer the readers to [13], [14] for many properties of the complete noncompact manifolds with nonnegative sectional curvature at infinity. Section

Manuscript received July 15, 1993.
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3 gives an existence theorem of harmonic maps from complete noncompact manifolds with nonnegative sectional curvature at infinity to convex balls. Section 4 gives Fatou property of harmonic maps in Section 3. Section 5 gives an existence theorem of harmonic maps from certain Cartan-Hadamard manifolds to convex balls.

§2. Preliminaries

Throughout this paper, we assume that $B_\tau(p)$ is a convex ball in Riemannian manifold N^n, $\dim N = n$, i.e., the geodesic ball centered at p with radius τ, $\tau < \frac{\pi}{2\sqrt{\kappa}}$, and $B_p(\tau)$ lies inside the cut-locus of p, where κ is an upper bound of the sectional curvature of N, $\kappa \geq 0$. In addition, Ω always denotes a bounded domain with smooth boundary in a complete noncompact manifold M.

We firstly state a lemma which is a key in the following development.

Lemma 2.1 (see [2, Lemma 3.1]). *Given $\varphi \in C^0(\partial\Omega, B_\tau(p))$, let $u \in C^0(\bar{\Omega}, B_\tau(p)) \cap C^\infty(\Omega, B_\tau(p))$ be a harmonic map on Ω which equals φ on $\partial\Omega$. With respect to geodesic normal coordinates centered at p, φ may also be viewed as being R^n-valued. Let $h : \bar{\Omega} \to R^n$ be the harmonic extention of φ, i.e., $h = (h^1, \cdots, h^n)$, where h^i is a harmonic function for each i and $h|_{\partial\Omega} = \varphi$. Let $v : \bar{\Omega} \to R$ be the harmonic extention of $\frac{1}{2}|\varphi|^2 = \frac{1}{2}\sum_{i=1}^{n}(\varphi^i)^2$. Then, there exists a constant $C > 0$, depending only on the geometry of $B_\tau(p)$, such that*

$$[\rho(u(x), h(x))]^2 \leq C(v(x) - \frac{1}{2}|h(x)|^2), \quad x \in \Omega, \tag{2.1}$$

where ρ is the distance function on N.

By complete noncompact manifold M^m, dim $M = m$, with nonnegative sectional curvature at infinity we mean that there exists a compact subset D in M such that the sectional curvature on $M \setminus D$ is nonnegative. Without loss of generality we may assume $D = B_{x_0}(1)$, a geodesic ball in M centered at x_0 with radius 1. From [13] (also see [5]), we know that M is of finite topological type. More precisely, M has finite ends, i.e., there exists a compact subset $C_0 \subset M$ such that for any compact subset C containing C_0, $M \setminus C$ has the same number of components. In particular we call each component an end. We also know that any end is diffeomorphic to the product of a compact manifold and the half line.

Let E be an end and denote the volume of $E \cap B_{x_0}(t)$ by $V(t)$. If

$$\int_{t_0}^{\infty} \frac{t}{V(t)} dt < \infty,$$

we call E a large end. Otherwise, we call E a small end. In this paper we always assume that all ends are large and that the number of ends is greater than 1. Denote all ends by $E_1, \cdots, E_l, l \geq 2$. By means of [13] there exist l positive harmonic functions $f_1, \cdots, f_l$ such that

$$\begin{cases} 0 < f_A(x) < 1, & x \in M, \\ f_A(x) \to 1, & x \to \infty, \quad x \in E_A, \\ f_A(x) \to 0, & x \to \infty, \quad x \notin E_A, \end{cases} \tag{2.2}$$

and any bounded harmonic function on M is a linear combination of f_A's, and there does not exist any nonnegative unbounded harmonic function on M.

To compare with harmonic maps, we state the following proposition.

Proposition 2.1. *Let M be a complete noncompact manifold with nonnegative sectional curvature at infinity, E be an end of M (not necessarily a large end). Assume that f is a harmonic function bounded below on E. Then*

$$f(x) \to a, \quad x \to \infty, \quad x \in E,$$

where $-\infty < a \leq \infty$. When E is a large end, a is a finite number.

Proof. See [13, Theorem 3.3, Corollary 4.4].

Because we assume that M has at least a large end, from [14, Remark 1] we know that there exists a unique minimal positive Green function $G(x, y)$ on M, which can be constructed as follows: Let $B_{x_0}(R_\mu)$ be the geodesic ball on M centered at x_0 with radius $R_\mu, R_\mu \to \infty$, denote the Green's function corresponding to Dirichlet boundary condition on $B_{x_0}(R_\mu)$ by $G_\mu(x, y)$. By passing to a subsequence of $G_\mu(x, y)$'s, still denoted by $G_\mu(x, y), G_\mu(x, y)$'s converge uniformly to $G(x, y)$ on arbitrary compact subset of $M \setminus \{x\}$ with respect to y. In addition, from [14], we know that the number of Martin boundary points is one of large ends. More precisely, an arbitrary large end corresponds to a unique Martin boundary point (for details, see [14, Corrollary 2]). From the point of view of Martin boundary, Proposition 2.1 is natural.

For the sake of convenience in the sequel, we state two lemmas.

Lemma 2.2 (See [10]). *Let $h = (h^1, \cdots, h^n)$ be normal coordinates on $B_p(2\tau)$ such that p has coordinates $(0, \cdots, 0)$. Denote by $g_{ik}(h), \Gamma^l_{ik}(h)$, and $\Gamma_{ikl}(h)$ the metric and Christoffel symbols, respectively, in this coordinates system. Then for all h satisfying*

$$|h| = (\sum_{i=1}^{n} h^i h^i)^{\frac{1}{2}} \leq 2\tau < \frac{\pi}{\sqrt{\kappa}}$$

and all $\xi \in R^n$ we have the following estimates

$$\Gamma^l_{ik}(h) h^l \xi^i \xi^k \leq \{\delta_{ik} - a_\kappa(|h|) g_{ik}(h)\} \xi^i \xi^k, \tag{2.3}$$

where

$$a_\kappa(t) = \begin{cases} t\sqrt{\kappa}\,\mathrm{ctg}(t\sqrt{\kappa}), & \kappa > 0, \quad 0 \leq t < \frac{\pi}{\sqrt{\kappa}}, \\ 1, & \kappa = 0, \quad 0 \leq t < \infty. \end{cases}$$

Lemma 2.3 (See [4]). *Let $u : M \to N$ be a harmonic map such that $u(M) \subset B_p(\tau)$. Then $e(u)$ is bounded by a constant depending only on κ, τ, and the lower bound of the Ricci curvature of M.*

§3. Harmonic Maps on Noncompact Complete Manifolds with Nonnegative Sectional Curvature at Infinity

Let M^m be a noncompact complete manifold with nonnegative sectional curvature at infinity, and its ends be large, which are denoted by $E_1, \cdots, E_l, l \geq 2$. Thus, from Section 2, there exist positive harmonic functions such that

$$\begin{cases} 0 < f_A(x) < 1, & x \in M, \\ f_A(x) \to 1, & x \to \infty, \quad x \in E_A, \\ f_A(x) \to 0, & x \to \infty, \quad x \notin E_A, 1 \leq A \leq l. \end{cases}$$

From now on, we fix normal coordinates $(h^1, \cdots, h^n)$ on $B_p(\tau)$ satisfying the condition in Lemma 2.3, the notations are also as in Lemma 2.3. Let $p^1, \cdots, p^l$ be in $B_p(\tau)$, their coordinates be $(h_A^1, \cdots, h_A^n)$ respectively. We construct n functions $\sum_{A=1}^{l} h_A^k f_A$, $1 \leq k \leq n$, which are harmonic. Set $h = \big(\sum_{A=1}^{l} h_A^1 f_A, \cdots, \sum_{A=1}^{l} h_A^n f_A\big)$, which defines a map from M to $B_p(\tau)$ under the above fixed coordinates of $B_p(\tau)$ denoted still by h, this is because $h(x) \to p_B$, as $x \to \infty, x \in E_B$, and by means of Maximum principle one has

$$\sum_{k=1}^{n} \left(\sum_{A=1}^{l} h_A^k f_A(x) \right)^2 < \tau^2, \quad \forall x \in M.$$

Let R_μ's be a real number sequence such that $R_\mu \to \infty$, as $\mu \to \infty$, $B_{x_0}(R_\mu)$ be the geodesic ball in M. Considering the boundary map φ_μ: $\partial B_{x_0}(R_\mu) \to B_p(\tau)$ with $\varphi_\mu = h|_{\partial B_{x_0}(R_\mu)}$, according to [10], we can find a unique harmonic map u_μ: $B_{x_0}(R_\mu) \to B_p(\tau)$ with $u_\mu|_{\partial B_{x_0}(R_\mu)} = h|_{\partial B_{x_0}(R_\mu)}$. By means of Theorem 4 in [9], we see that u_μ's converge uniformly to a harmonic map u: $M \to B_p(\tau)$ on arbitrary compact subset of M. On the other hand, by Lemma 2.1, we have

$$[\rho(u_\mu(x), h(x))]^2 \leq C\big(v_\mu(x) - \frac{1}{2}|h(x)|^2\big), \quad \forall x \in \overline{B_{x_0}(R_\mu)},$$

where v_μ is the harmonic extenstion of $\frac{1}{2}|h(x)|^2|_{\partial B_{x_0}(R_\mu)}, C$ depends only on the geometry of $B_p(\tau)$.

We now set $\tilde{v} = \frac{1}{2} \sum_{A=1}^{l} \sum_{i=1}^{n} (h_A^i)^2 f_A$, which is harmonic on M and has the same boundary value as $\frac{1}{2}|h(x)|^2$ at infinity. So, by Maximum principle, $\tilde{v} - \frac{1}{2}|h(x)|^2 > 0$ on M. Obviously $v_\mu(x) - \frac{1}{2}|h(x)|^2 < \tilde{v} - \frac{1}{2}|h(x)|^2$ on $B_{x_0}(R_\mu)$. Thus

$$[\rho(u_\mu(x), h(x))]^2 \leq C(\tilde{v} - \frac{1}{2}|h(x)|^2), \forall \mu.$$

Hence, $[\rho(u(x) - h(x))]^2 \leq C(\tilde{v} - \frac{1}{2}|h(x)|^2)$, i.e.,$u$ is a harmonic map with $u(x) \to p_A$, as $x \in E_A, x \to \infty$.

In the following, we shall show that the energy of $u(x)$ is finite. The method is similar to that of [12], as was pointed out to me by Professor P.Li, after the author had completed this paper. From [11], we know that u_μ with the above prescribed boundary value is unique, so it is an energy minimizing harmonic map. Thus

$$\int_{B_{x_0}(R_\mu)} |\nabla u_\mu|^2 \leq \int_{B_{x_0}(R_\mu)} |\nabla h|^2, \quad \forall \mu.$$

Hence we obtain

$$\int_{B_{x_0}(R)} |\nabla u_\mu|^2 \leq \int_M |\nabla h|^2$$

for any $R > 0$ and $R_\mu > R$. Therefore $\int_{B_{x_0}(R)} |\nabla u|^2 \leq \int_M |\nabla h|^2$. So, if we can prove $\int_M |\nabla h|^2 < \infty$, we have $\int_M |\nabla u|^2 < \infty$. For this, we only need to prove $\int_M |\nabla f_A|^2 < \infty, 1 \leq A \leq l$. For the purpose, we outline the construction of f_A as follows (for details see [13]). Construct a harmonic function u_R on $B_{x_0}(R)$ with $u_R|_{\partial B_{x_0}(R) \cap E_A} = 1, u_R|_{\partial B_{x_0}(R) \cap E_B} = 0, B \neq A$. Using the barriers on the ends, one can prove that there exists a sequence R_μ's such that

$R_\mu \to \infty$ as $\mu \to \infty$, and u_{R_μ}'s converge uniformly to f_A on any compact subset of M, which satisfies the properties stated before.

Considering the integral $\int\limits_{B_{x_0}(R_\mu)\cap E_A} |\nabla(1-u_{R_\mu})|^2$, we have

$$\begin{aligned}\int\limits_{B_{x_0}(R_\mu)\cap E_A} |\nabla u_{R_\mu}|^2 &= \int\limits_{B_{x_0}(R_\mu)\cap E_A} \nabla((1-u_{R_\mu})\nabla(1-u_{R_\mu}))\\ &= \int\limits_{\partial E_A} (1-u_{R_\mu})\frac{\partial u_{R_\mu}}{\partial\gamma}\\ &\le \int\limits_{\partial E_A} |1-u_{R_\mu}|\left|\frac{\partial u_{R_\mu}}{\partial\gamma}\right|,\end{aligned}$$

where γ is the outer unit normal vector of $M\setminus E_A$. Setting $\mu\to\infty$, we have

$$\int\limits_{E_A} |\nabla f_A|^2 \le \int\limits_{\partial E_A} (1-f_A)\left|\frac{\partial f_A}{\partial\gamma}\right| < \infty.$$

For $B\neq A$, we can consider the integral $\int\limits_{B_{x_0}(R_\mu)\cap E_B} |\nabla u_{R_\mu}|^2$.

$$\int\limits_{B_{x_0}(R_\mu)\cap E_B} |\nabla u_{R_\mu}|^2 = \int\limits_{B_{x_0}(R_\mu)\cap E_B} \nabla(u_{R_\mu}\nabla u_{R_\mu}) = -\int\limits_{\partial E_B} u_{R_\mu}\frac{\partial u_{R_\mu}}{\partial\gamma}.$$

Taking $\mu\to\infty$, we obtain

$$\int\limits_{E_B} |\nabla f_A|^2 \le -\int\limits_{\partial E_B} f_A\frac{\partial f_A}{\partial\gamma} < \infty.$$

Hence we have $\int\limits_M |\nabla f_A|^2 < \infty$.

We now are in a position to state our result.

Theorem 3.1. *Let M be a complete noncompact manifold with nonnegative sectional curvature at infinity, the ends of which are large, denoted by $E_1,\cdots,E_l, l\ge 2$. Let $p_1,\cdots,p_l$ be l points in $B_p(\tau)$. Then there exists a unique harmonic map $u: M\to B_p(\tau)$ with $u(x)\to p_A$, as $x\in E_A, x\to\infty$, the energy of which is finite.*

Proof. We only need to prove the uniqueness, which can be done as in [11] if we use Maximum principle on complete manifolds.

To conclude the section we give the following remark.

Remark 3.1. From [4], we know that the energy density of u is bounded, as we will use in the next section.

§4. Fatou's Property for Harmonic Maps

In this section, we consider the converse of Section 3, i.e., the boundary behavior of harmonic maps into convex balls, the domain manifold M satisfying the conditions in Theorem 3.1. According to [14], the Green's function constructed in Section 1 is a unique minimal positive Green's function (see [7]) and

$$\max_{y\in M\setminus B_x(R)} G(x,y) = \max_{y\in\partial B_x(R)} G(x,y)$$

for any $x \in M, R > 0$. Due to the lack of Harnack inequality at infinity, we add some conditions to $G(x,y)$ as follows: $\exists R_0 > 0, q > 1$, s.t.

$$\sup_{x\in M}\Big(\max_{\partial B_x(R_0)} G(x,y)\Big) < \infty, \tag{4.1}$$

$$\int_{B_x(R_0)} |G(x,y)|^q < C < \infty, \quad \forall x \in M, \tag{4.2}$$

for some constant $C > 0$. It is easy to check that the minimal positive Green's function on $R^n, n \geq 3$, satisfies (4.1), (4.2).

We can now state Fatou's property of harmonic maps in the present case.

Theorem 4.1 *Let M satisfy the conditions in Theorem* 3.1, *and its minimal positive Green's function satisfy* (4.1), (4.2). *Then for any harmonic map with finite energy u : $M \to B_p(\tau)$ Fatou's property holds, i.e., $u(x) \to p_A \in B_p(\tau)$ as $x \in E_A, \quad x \to \infty$.*

Proof. As in Section 3, we fix normal coordinates $(h^1, \cdots, h^n)$ on $B_p(\tau)$ with the coordinates of p being $(0, \cdots, 0)$. Set $|u(x)|^2 = \sum_{i=1}^{n} |u^i(x)|^2$. A direct computation shows

$$\frac{1}{2}\triangle|u|^2 = |\nabla u|^2 + u^l \triangle u^l = |\nabla u|^2 - u^l \Gamma^l_{ij}(u) u^i_\alpha u^j_\beta \gamma^{\alpha\beta}$$

where $(\gamma^{\alpha\beta})^{-1}$ is the Riemannian metric of M, Γ^l_{ij} is the Christoffel symbols on $B_p(\tau)$ with respect to the fixed normal coordinates. Using Lemma 2.2, we have

$$\frac{1}{2}\triangle|u|^2 \geq |\nabla u|^2 - \{\delta_{ij} - a_\kappa(|u|) g_{ij}(u)\} u^i_\alpha u^j_\beta \gamma^{\alpha\beta} = a_\kappa(|u|) e(u) \geq 0, \tag{4.3}$$

where (g_{ij}) is the Riemannian metric of $B_p(\tau)$, $a_\kappa(|u|) > 0$ since $|u| < \frac{\pi}{2\sqrt{\kappa}}$, and $e(u)$ is the energy density.

Let $G_\mu(x,y)$ be the Green's function with respect to Dirichlet boundary value on $B_{x_0}(R_\mu)$. Consider $\int_{B_{x_0}(R_\mu)} \triangle|u|^2(y) G_\mu(x,y)dy$, denoted by f_μ, which stisfies

$$\begin{cases} \triangle f_\mu = -\triangle|u|^2, \quad \text{on} \quad B_{x_0}(R_\mu), \\ f_\mu|_{\partial B_{x_0}(R_\mu)} = 0. \end{cases}$$

Maximum principle implies $f_\mu \leq \tau^2$. Hence $\int_M \triangle|u|^2(y)G(x,y)dy$, denoted by f, is not greater than τ^2. By means of (4.3), we obtain $\int_M e(u)(y)G(x,y)dy < \infty$.

On the other hand, the harmonicity of u implies

$$\triangle u^l + \gamma^{\alpha\beta}\Gamma^l_{ij} u^i_\alpha u^j_\beta = 0.$$

Thus, $|\triangle u^l| \leq Ce(u)$, C depending only on the geometry of $B_p(\tau)$. Therefore, we can set

$$h^l = \int_M \triangle u^l(y) G(x,y)dy + u^l, \qquad 1 \leq l \leq n. \tag{4.4}$$

Obviously, h^l's are harmonic functions. Proposition 2.1 implies that for any large end E_A there exists a real number a^l_A such that $h^l(x) \to a^l_A$ *as* $x \in E_A, x \to \infty$.

If we can prove that $\lim_{x\in E_A, x\to\infty} \int_M \triangle u^l(y)G(x,y)dy$ exists, the theorem above is obtained.

To this aim, we consider more general case. Let $f \in L^\infty(M) \cap L^1(M)$, s.t.

$$\int_M f(y)G(x,y)dy \in L^\infty(M).$$

We claim that under the condition of the theorem, when $x \in E_A, x \to \infty$.

$$\int\limits_M f(y)G(x,y)dy \to 0, \qquad 1 \le A \le l. \tag{4.5}$$

Since $f \in L^1(M)$, there exists an R sufficiently large such that $\int\limits_{E_A \setminus B_{x_0}(R)} |f| < \varepsilon$ for any sufficiently small $\varepsilon > 0$. Fix $R > 0$, and consider the integral

$$\int\limits_{E_A \setminus B_{x_0}(R)} f(y)G(x,y)dy.$$

We assume $\mathrm{dist}(x_0, x)$ sufficiently large such that $B_x(R_0) \subset E_A \setminus B_{x_0}(R)$. So we have

$$\begin{aligned}
&\Big| \int\limits_{E_A \setminus B_{x_0}(R)} f(y)G(x,y)dy \Big| \\
&\le \Big| \int\limits_{(E_A \setminus B_{x_0}(R)) \setminus B_x(R_0)} f(y)G(x,y)dy \Big| + \Big| \int\limits_{B_x(R_0)} f(y)G(x,y)dy \Big| \\
&\le \max_{\partial B_x(R_0)} G(x,y) \int\limits_{(E_A \setminus B_{x_0}(R)) \setminus B_x(R_0)} |f(y)|dy \\
&\quad + \Big(\int\limits_{B_x(R_0)} |f(y)|^r dy \Big)^{\frac{1}{r}} \Big(\int\limits_{B_x(R_0)} |G(x,y)|^q dy \Big)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{r} + \frac{1}{q} = 1$. Since $f \in L^\infty$, using (4.1), (4.2) we see that the right hand side of the inequality above is sufficiently small.

In the following, we consider the integral $\int\limits_{M \setminus (E_A \setminus B_{x_0}(R))} f(y)G(x,y)dy$. We firstly observe the behavior of $G(x,y)$ on $\partial B_{x_0}(R) \cap E_A$ and $\partial B_{x_0}(R+R_0) \cap E_A$. Fixing $x_R \in \partial B_{x_0}(R) \cap E_A$, $x_{R_0} \in \partial B_{x_0}(R+R_0) \cap E_A$, we claim that the following inequality

$$\frac{G(y,x)}{G(x_R,x)} \le C \frac{G(y,x_{R_0})}{G(x_R,x_{R_0})}, \quad \forall y \in M \setminus (E_A \setminus B_{x_0}(R)), \tag{4.6}$$

holds, where $x \in E_A$ with $\mathrm{dist}(x,x_0) > 2(R+R_0)$, and $C > 0$ depends only on the geometry of M. This is because $G(y,x)$ is harmonic on $M \setminus (E_A \setminus B_{x_0}(R))$ with respect to y and the upper and lower bounds of $G(y,x)|_{\partial B_{x_0}(R) \cap E_A}$ is controlled by the multiplicities of $G(x_R,x)$, as can be obtained by using Theorem 3.2 in [13], i.e, Harnack inequality. On the other hand, the upper and lower bounds of $\frac{G(y,x_{R_0})}{G(x_R,x_{R_0})} G(x_R,x)$ are also controlled by the multiplicities of $G(x_R,x)$ restricted to $\partial B_{x_0}(R) \cap E_A$ with respect to y, based on the same reason. So, by using Maximum principle, (4.6) is obtained. Thus, we have

$$\int_{M \setminus (E_A \setminus B_{x_0}(R))} f(y)G(x,y)dy \le C \int\limits_{M \setminus (E_A \setminus B_{x_0}(R))} f(y) \left(\frac{G(x_R,x)}{G(x_R,x_{R_0})} \right) G(y,x_{R_0})dy.$$

So, when $x \in E_A$ and $x \to \infty$, we have (using the estimates for $G(x,y)$ in [14])

$$\int\limits_{M \setminus (E_A \setminus B_{x_0}(R))} f(y)G(x,y)dy \to 0. \tag{4.7}$$

Combining (4.7) and (4.6), we obtain (4.5). Now come back to our present case. From

Lemma 2.3, we have $e(u) \in L^\infty(M)$, so $e(u) \in L^\infty(M) \cap L^1(M)$. Finally, from (4.4) we obtain for some a^l_A, $u^l(x) \to a^l_A$, as $x \in E_A, x \to \infty$. The proof of the theorem is completed.

We give the following remarks to conclude this section.

Remark 4.1. From Theorem 4.1 and Theorem 3.1, we see that under the conditions of Theorem 4.1 all harmonic maps into convex balls can be obtained from Theorem 3.1.

Remark 4.2. The conditions (4.1), (4.2) are completely technical, which should be able to be deleted.

§5. Harmonic Maps from Cartan-Hadamard Manifolds

In this section, we use the work of W. Y. Ding and Y. D. Wang[8] (also see [15]) to derive an existence theorem of harmonic maps from a class of Cartan-Hadamard manifolds into convex balls. Let M^m be a Cartan-Hadamard manifold with $\mathrm{Ricc}^M \geq -K^2$, $K > 0$ and $\lambda_1(M) > 0$. Fixing a point O in M, we can construct the geometric boundary of M, denoted by S^{m-1}, $(m-1)$-dimension sphere, which formly consists of all geodesic rays from O (for details see [3]). We still assume

$$\theta = O(\rho^{-k}(x)) \qquad \text{for some} \quad k > 0, \tag{5.1}$$

where $\rho(x) = \mathrm{dist}(x, O)$, $\theta = \angle xOy$, $x, y \in M$, $\mathrm{dist}(x,y) = 1, \rho(x) = \rho(y)$ sufficiently large, and

$$\int_{M \setminus B_1(O)} \rho^{-k(\frac{m}{2}+\varepsilon)} dV_M < \infty \quad \text{for some} \quad \varepsilon > 0, \tag{5.2}$$

where dV_M is the volume element of M, $B_1(O)$ is the geodesic ball centered at O with radius 1.

Theorem 5.1. *Let M be a Cartan-Hadamard manifold with $m \geq 4$,* $\mathrm{Ricc}^M \geq -K^2$ *for some constant $K > 0$, and $\lambda(M) > 0$. M satisfies* (5.1), (5.2). *Let $\phi \in C^0(S^{m-1}, B_p(\tau))$. Then there exists a unique harmonic map $u : M \to B_p(\tau), u|_{S^{m-1}} = \phi$.*

In order to prove the theorem, we firstly establish the following lemma.

Lemma 5.1. *Let M satisfy the conditions of Theorem* 5.1. *Let $\phi \in C^\infty$ (S^{m-1}). Then there exists a harmonic function h on M with $h|_{S_{m-1}} = \phi$.*

Remark 5.1. If M satisfies $-b^2 \leq \mathrm{Riem}^M \leq -a^2, a \geq b > 0$, Lemma 5.1 was proved by M. Anderson[1] and D. Sullivan[16] independently. Recently, Q. Ding[6] also obtained a similar theorem with the conditions $-a^2 \leq \mathrm{Ricc}^M \leq -b^2$, $a \geq b > 0$ and $\mathrm{Riem}^M \leq 0$ using his new Laplace comparison theorem. Here we use the work of W .Y. Ding and Y. D. Wang (which generalizes the result in [15]) to prove Lemma 5.1.

Proof of Lemma 5.1. Let $\{(r,\theta) | \theta \in S_O(1) \cong S^{m-1}\}$ be normal geodesic coordinates at O, where $S_O(1)$ is the unit sphere of $T_O M$. Thus ϕ can be written as $\phi = \phi(\theta), \theta \in S_O(1)$. Extend ϕ to $M \setminus \{O\}$ in radius directions, i.e., set $\phi(r,\theta) = \phi(\theta), \forall r > 0$, and the function extented is still denoted by ϕ, then ϕ is a bounded smooth function on $M \setminus \{O\}$.

Introduce the notation $\mathrm{osc}_{B_x(1)}\phi = \sup\limits_{y \in B_x(1)} |\phi(y) - \phi(x)|$, which expresses the osillation of ϕ in $B_x(1)$. By means of the definition of ϕ, we have

$$|\phi(y) - \phi(x)| = |\phi(\theta') - \phi(\theta)| \leq C|\theta' - \theta|, \quad \forall y \in B_x(1),$$

where θ, θ' are the geodesic sphere coordinates of x and y, respectively, C depends only on $\phi|_{S^{m-1}}$ (and so do all the constants below). So, when $\mathrm{dist}(x, O)$ is sufficiently large, we have $\mathrm{osc}_{B_x(1)}\phi \le C\rho^{-k}(x)$.

We now average ϕ, denoted by $\bar{\phi}$, and show $\triangle\bar{\phi} = O(\rho^{-k})$. Choose the cut-off function $\chi \in C_O^\infty(R), 0 \le \chi \le 1$, $\mathrm{supp}\chi \subset [-1, 1]$, and set

$$\bar{\phi}(x) = \frac{\int\limits_M \chi(\rho_x^2(y))\phi(y)dy}{\int\limits_M \chi(\rho_x^2(y))dy},$$

where $\rho_x(y) = \mathrm{dist}(x, y)$. Then we have

$$\begin{aligned} |\bar{\phi}(x) - \phi(x)| &= \frac{\left|\int_{B_x(1)} \chi(\rho_x^2(y))(\phi(y) - \phi(x))dy\right|}{\int_{B_x(1)} \chi(\rho_x^2(y))dy} \\ &\le \sup_{B_x(1)} |\phi(y) - \phi(x)| \\ &= osc_{B_x(1)}\phi = O(\rho^{-k}(x)), \end{aligned}$$

while

$$\triangle\bar{\phi}(x_0) = \triangle(\bar{\phi}(x) - \phi(x_0))|_{x=x_0} = \int\limits_M \triangle\left(\frac{\chi(\rho_y^2(x))}{\int\limits_M \chi(\rho_y^2(x))dy}\right)(\phi(y) - \phi(x_0))dy|_{x=x_0}.$$

A direct computation shows

$$\begin{aligned} &\triangle\left(\frac{u}{v}\right) = \frac{v\triangle u - 2\nabla u\nabla v - u\triangle v}{v^2} + \frac{2u}{v^3}|\nabla v|^2, \\ &\nabla u = 2\chi'(\rho^2)\rho\nabla\rho, \\ &\triangle u = 4\rho^2\chi''(\rho^2)|\nabla\rho|^2 + 2\chi'(\rho^2)|\nabla\rho|^2 + 2\rho\chi'(\rho^2)\triangle\rho, \end{aligned}$$

where $u = \chi(\rho_y^2(x))$, $v = \int_M \chi(\rho_y^2(x))dy$.

Since $\mathrm{Ricc}M \ge -K^2$, by the standard Laplace comparison theorem, when $\rho = \rho_y(x) \le 1$, $\rho\triangle\rho \le C$ for some constant $C > 0$, and the other terms in ∇u and $\triangle u$ are finite when $\rho \le 1$. The same reason shows the above conclusion is valid for $\nabla v, \triangle v$. On the other hand, by the volume comparison theorem, when $Riem^M \le 0$, we have $\mathrm{Vol}B_x(1) \ge C$ for some constant $C \ge 0$. Therefore we have

$$v(x) = \int\limits_M \chi(\rho_x^2(y))dy = \int\limits_{B_x(1)} \chi(\rho_x^2(y))dy \ge C > 0.$$

Combining the above facts, we see that $|\triangle(\frac{u}{v})|$ is bounded. Thus

$$|\triangle\bar{\phi}(x)| \le Cosc_{B_x(1)}\phi = O(\rho^{-k}(x)).$$

By (5.2) we have

$$\int\limits_M |\triangle\bar{\phi}(x)|^{(\frac{m}{2}+\varepsilon)}dV_M < \infty. \tag{5.3}$$

Thus, Theorem 3.1 in [8] implies that there exists a harmonic function h such that

$$|h(x) - \bar{\phi}(x)| \to 0 \qquad \text{as} \quad x \to \infty, \tag{5.4}$$

i.e., $h(x)$ reaches $\phi(x)$ on the geometric boundary. The proof of Lemma 5.1 is completed.

Proof of Theorem 5.1. An approximation process can make us assume ϕ in Theorem 5.1 to be smooth. Then by using Lemma 5.1, the proof of Theorem 5.1 is completely similar to that of Theorem 3.1. We outline the proof as follows. Fix a normal coordinate around p. Under this coordinate ϕ can be written as $(\phi^1(x), \cdots, \phi^n(x))$, $x \in S^{m-1}$, so there exist n harmonic functions $h^1(x), \cdots, h^n(x)$ defined in M, such that $h^i|_{S^{m-1}} = \phi^i$. Under the above coordinate $(h^1(x), \cdots, h^n(x))$ defines a map from M into $B_p(\tau)$, which is denoted by h. Consider the following Dirichlet problem

$$\begin{cases} u_\mu : \Omega_\mu \to B_p(\tau), \\ u_\mu|_{\partial\Omega_\mu} = h|_{\partial\Omega_\mu}, \end{cases}$$

where Ω_μ's are exhaustion domains of M. Theorem 4 in [9] implies that u_μ's converge on any compact subset in M, the limitation being denoted by u. From Lemma 2.1, we know $u|_{S^{m-1}} = \phi$. The proof is completed.

Acknowlegements. I want to take this opportunity to thank my advicers Profs. Hu Hesheng and Xin Yuanlong for their constant help and encouragement. I am also grateful to Prof. P. Aviles for kindly sending me his paper [2] and Prof. Hong Jiaxing for his valuable discussion.

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