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AVERAGE σ–K WIDTH OF CLASS OF $L_p(\mathrm{R}^n)$ IN $L_q(\mathrm{R}^n)$**

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Abstract

The average $\sigma - K$ width of the Sobolev-Wiener class $W^r_{pq}(\mathbf{R}^n)$ in $L_q(\mathbf{R}^n)$ is studied for $1 \le q \le p \le \infty$, and the asymptotic behaviour of this quantity is determined. The exact value of average $\sigma - K$ width of some class of smooth functions in $L_2(\mathbf{R}^n)$ is obtained.

Keywords Average $\sigma - K$ width, Optimal subspace, Sobolev-Wiener class, Riesz potential.

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§1. Introduction

1.1. The Amalgams of L_p and l^q

Let $1 \le q, p \le \infty$, $n \in \mathbf{Z}_+ =: \{1, 2, \cdots\}$. Denote by $L_{pq}(\mathbf{R}^n)$ the normed linear space of functions defined on the Euclidean space $\mathbf{R}^n$ ($\mathbf{R}^1 = \mathbf{R}$) in which each function f is locally L_p-integrable and satisfies $\|f\|_{pq} < \infty$. Here the norm $\|\cdot\|_{pq}$ is defined by

$$\|f\|_{pq} = \begin{cases} \Big\{ \sum\limits_{v\in\mathbf{Z}^n} \|f(\cdot + v)\|^q_{L_p([0,1]^n)} \Big\}^{\frac{1}{q}}, & \text{for } 1 \le q \le \infty, \\ \sup\limits_{v\in\mathbf{Z}^n} \|f(\cdot + v)\|_{L_p([0,1]^n)}, & \text{for } q = \infty, \end{cases}$$

where $\mathbf{Z}^n$ denotes the set of all points in $\mathbf{R}^n$ having integral coordinates, and $\|\cdot\|_{L_p(D)}$ the usual L_p-norm on the subset D of $\mathbf{R}^n$. $L_{pq}(\mathbf{R}^n)$ is a Banach space with norm $\|\cdot\|_{pq}$. When $p = q$, $L_{qq}(\mathbf{R}^n) = L_q(\mathbf{R}^n)$ is the usual $L_q(\mathbf{R}^n)$-space. When $n = 1$, these notions may be seen in [3]. For convenience, we write $\|\cdot\|_p$ instead of $\|\cdot\|_{pp}$.

1.2. Definition of Average σ–K Width

Let $\mathbf{X}$ be a normed linear space with norm $\|\cdot\|_X$ and L any subspace of $\mathbf{X}$. For each $f \in \mathbf{X}$,

$$E(f, L, \mathbf{X}) =: \inf\{\|f - g\|_X : g \in L\}$$

is the distance of the subspace L from f. For a subset $\mathfrak{M}$ of $\mathbf{X}$, the quantity

$$E(\mathfrak{M}, L, \mathbf{X}) =: \sup\{E(f, L, \mathbf{X}) : f \in \mathfrak{M}\}$$

is the deviation of $\mathfrak{M}$ from L.

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Let L be a subspace $L_q(\mathbf{R}^n)$. For any $a>0$ and $\varepsilon>0$, set $I_a^n =: [-a,a]^n$,

$$\begin{aligned} B(L) &=: \{f \in L : \|f\|_q \le 1\}, \\ (BL)(I_a^n) &=: \{f|_{I_a^n} : f \in B(L)\}, \\ k_\varepsilon(a, L, L_q(\mathbf{R}^n)) &=: \min\{m : \text{there exists a linear subspace B of dimension } m \\ &\qquad \text{of } L_q(I_a^n) \text{ such that } E((BL)(I_a^n), B, L_q(I_a^n)) < \varepsilon\}, \end{aligned}$$

where $f|_{I_a^n}(x) = f(x)$ or 0 according as whether $x \in I_a^n$ or not. It is easy to see that $k_\varepsilon(a, L, L_q(\mathbf{R}^n))$ is non-decreasing in a and non-increasing in ε.

Let σ be a positive number. A linear subspace L of $L_q(\mathbf{R}^n)$ is said to be of average dimension σ if

$$\overline{\dim}(L, L_q(\mathbf{R}^n)) =: \lim_{\varepsilon\to 0^+} \varliminf_{a\to\infty} \frac{k_\varepsilon(a, L, L_q(\mathbf{R}^n))}{(2a)^n} = \sigma.$$

Let $\mathfrak{M}$ be a centrally symmetric subset of $L_q(\mathbf{R}^n)$. The quantity

$$\overline{d_\sigma}(\mathfrak{M}, L_q(\mathbf{R}^n)) =: \inf\{E(\mathfrak{M}, L, L_q(\mathbf{R}^n)) : \overline{\dim}(L, L_q(\mathbf{R}^n)) \le \sigma\}$$

is called the average σ-width of $\mathfrak{M}$ in $L_q(\mathbf{R}^n)$ in the sense of Kolmogorov (shortly, average $\sigma - K$ width). A subspace L_σ^* of $L_q(\mathbf{R}^n)$ of average dimension at most σ for which

$$\overline{d_\sigma}(\mathfrak{M}, L_q(\mathbf{R}^n)) = E(\mathfrak{M}, L_\sigma^*, L_q(\mathbf{R}^n))$$

is called an optimal subspace for $\overline{d_\sigma}(\mathfrak{M}, L_q(\mathbf{R}^n))$. When $n=1$, the notion of average width was first proposed by Tikhomirov[12] in order to consider problems of optimal approximation methods on a non-compact Sobolev class $W_p^r(R)$.

1.3. Classes of Smooth Functions

Let $\mathbf{S} = \mathbf{S}(\mathbf{R}^n)$ (see [9]) be the space of rapidly decreasing functions on $\mathbf{R}^n$. The Fourier transform of a function $\varphi \in \mathbf{S}$ will be denoted as follows

$$(F\varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \varphi(t) e^{-itx} dt, tx = \sum_{j=1}^n t_j x_j,$$

and the transformation inverse to it as follows

$$\big(F^{-1}\varphi\big)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \varphi(t) e^{itx} dt.$$

The set of all generalized (over $\mathbf{S}$) functionals is denoted by $\mathbf{S}' = \mathbf{S}'(\mathbf{R}^n)$. The Fourier transforms (direct and inverse) for $f \in \mathbf{S}'$ are defined respectively by the equations

$$(Ff, \varphi) = (f, F\varphi) \text{ and } (F^{-1}f, \varphi) = (f, F^{-1}\varphi),$$

for any $\varphi \in \mathbf{S}$, where $(g,\varphi) = \int_{\mathbf{R}^n} g(x)\varphi(x)dx, g \in \mathbf{S}', \varphi \in \mathbf{S}$.

For any $\alpha \in R, K_\alpha : S' \to S'$ denotes the operator $(K_\alpha f)(x) = (1+|x|^2)^{\alpha/2} f(x)$, $x = (x_1, \cdots, x_n) \in \mathbf{R}^n,\ |x|^2 = x_1^2 + \cdots + x_n^2$. Define $I_\alpha =: F^{-1}K_\alpha F$. Set

$$\mathcal{K}_q^\alpha(\mathbf{R}^n) = \{f \in S' \in (\mathbf{R}^n) : I_\alpha \in L_q(\mathbf{R}^n)\},$$

which is a Banach space (of Bessel petentials) with norm $\|f\|_{\mathcal{K}_q^\alpha(\mathbf{R}^n)} =: \|I_\alpha f\|_q$ for any $f \in \mathcal{K}_q^\alpha(\mathbf{R}^n)$.

When $1<q<\infty,\ \alpha = r \in Z_+$, denote the Sobolev space by

$$L_q^r(\mathbf{R}^n) = \{f \in L_q(\mathbf{R}^n) :\ D^\alpha f \in L_q(\mathbf{R}^n), |\alpha|_1 \le r\},$$

where $D^\alpha f = \frac{\partial^{|\alpha|_1} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, $\alpha_j \in Z_+ \cup \{0\}$, $j = 1, 2, \cdots, n$, $|\alpha|_1 = \alpha_1 + \cdots + \alpha_n$ (see [11]). For $1 \le p, q \le \infty$, the set

$$W^r_{pq}(\mathbf{R}^n) = \Big\{ f \in L_q(\mathbf{R}^n) : \sum_{|\alpha|_1 = r} ||D^\alpha f||_{pq} \le 1 \Big\}$$

is called the Sobolev -Wiener class. When $p = q, W_{qq}(\mathbf{R}^n)$ is the usual Sobolev class . When $1 \le q \le p \le \infty$, it is easy to verify that $W^r_{pq}(\mathbf{R}^n) \subset W^r_q(\mathbf{R}^n)$.

For $n = 1$, the exact values of the average $\sigma - K$ width $\bar{d}_\sigma(W^r_{pq}(\mathbf{R}), L_q(\mathbf{R}))$ have been obtained for the case $1 \le q < p \le \infty$ and $\sigma \in Z_+$ (see [2, 5]), and the case $1 \le p = q \le \infty$ and $\sigma \in (0, \infty)$ (see [6]).

Let $\mathcal{R}^\alpha_2$ ($\alpha > 0$) denote the space of Riesz-Potentials, defined by

$$\mathcal{R}^\alpha_2 = \{f \in L_2(\mathbf{R}^n) : |y|^\alpha (Ff)(y) \in L_2(\mathbf{R}^n)\},$$
$$B(\mathcal{R}^\alpha_2) = \{f \in \mathcal{R}^\alpha_2 : \||\cdot|^\alpha Ff\|_2 \le 1\}.$$

Here $|x|^2 = x_1^2 + \cdots + x_n^2$, for any $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$.

Denote by $SB^p_\sigma(\mathbf{R}^n)$ the collection of all functions of the spherical exponential type $\sigma \ge 0$ (see [7]).

1.4. Our Main Results

In this paper, we obtain the following results.

Theorem 1.1. *Let $1 \le q \le p \le \infty, r \in Z_+$. Then*

(1) $\bar{d}_\sigma(W^r_{pq}(\mathbf{R}^n), L_q(\mathbf{R}^n)) \asymp \sigma^{-\frac{r}{n}}, \sigma \longrightarrow \infty$, *where the notation "$f(\sigma) \asymp g(\sigma)$" means that there exist $c_1 > 0$ and $c_2 > 0$ $(c_1 < c_2)$ such that the inequalities $c_1|g(\sigma)| \le |f(\sigma)| \le c_2|g(\sigma)|$ hold for sufficiently large σ.*

(2) $SB^q_{\rho(\sigma)}(\mathbf{R}^n)$ *is a weakly asymptotic optimal subspace of average dimenstion σ for $\bar{d}_\sigma(W^r_{pq}(\mathbf{R}^n), L_q(\mathbf{R}^n))$, where $\rho(\sigma) \ge 0$ is defined by the equation $\rho^n \mathrm{mes} B_n(0,1) = (2\pi)^n\sigma$, while $B_n(0,1) = \{x \in \mathbf{R}^n : |x| \le 1\}$ is the unit ball of the Euclidean space $\mathbf{R}^n$.*

Theorem 1.2. *Let $\alpha > 0$. Then*

$$\bar{d}_\sigma(B(\mathcal{R}^\alpha_2), L_2(\mathbf{R}^n)) = E(B(\mathcal{R}^\alpha_2), SB^2_{\rho(\sigma)}(\mathbf{R}^n), L_2(\mathbf{R}^n)) = C_{n,\alpha}\sigma^{-\frac{\alpha}{n}},$$

where $\rho(\sigma)$ is defined as in Theorem 1.1 *and $C_{n,\alpha} =: \{\mathrm{mes} B_n(0,1)\}^{\alpha/n}(2\pi)^{-\alpha}$.*

§2. Proof of Theorem 1.1

To prove the Theorem 1.1, we first give some lemmas as follows.

Lemma 2.1 (cf. [1]). *Let $\rho > 0$. Then*

$$\overline{\dim}(SB^p_\rho(\mathbf{R}^n), L_p(\mathbf{R}^n)) = \frac{\rho^n \mathrm{mes} B_n(0,1)}{(2\pi)^n}.$$

Lemma 2.2. *Let $1 \le q \le \infty$. Then*

$$E(W^r_q(\mathbf{R}^n), SB^q_{\rho(\sigma)}(\mathbf{R}^n), L_q(\mathbf{R}^n)) \le C\sigma^{-\frac{r}{n}}, \tag{2.1}$$

where C is a constant independent of σ.

Proof. For any real number $\rho > 0$, let

$$k_{\rho,s}(t) = \Big\{\frac{\sin t\rho}{t}\Big\}^{2s} \quad (t \in \mathbf{R}, \quad 2s > n) \tag{2.2}$$

be an even entire function of one variable of exponential type $2s\rho$. Then, by [7], the function $k_{\rho,s}(|x|), x=(x_1,\cdots,x_n)\in\mathbf{R}^n$, is an entire function of n variable of spherical exponential type $2s\rho$. We write

$$K_{\rho,s}(x)=\lambda_{\rho,s}^{-1}k_{\rho,s}(|x|),\quad x\in\mathbf{R}^n,$$

where

$$\lambda_{\rho,s}=\int_{\mathbf{R}^n}k_{\rho,s}(|x|)dx\asymp\rho^{2s-n},\quad \rho\to\infty.$$

If $\alpha>0$ and $2s>n+\alpha$, then it is easy to verify that

$$\int_{\mathbf{R}^n}K_{\rho,s}(x)|x|^{\alpha}dx\asymp\rho^{-\alpha},\quad \rho\to\infty.$$

For each $f\in L_q(\mathbf{R}^n)$, set

$$(\Delta_t^r f)(x)=\sum_{j=0}^{r}\binom{r}{j}(-1)^j f(x-jt), \tag{2.3}$$

$$(T_\rho f)(x)=-\int_{\mathbf{R}^n}\sum_{j=1}^{r}\binom{r}{j}(-1)^j f(x-jt)K_{\rho,s}(t)dt. \tag{2.4}$$

By [7], $T_\rho f\in SB^q_{2s\rho}(\mathbf{R}^n)$. For $1\le q\le\infty$, if $f\in W_q^r(\mathbf{R}^n)$, then by a proper calculation we have

$$\|\Delta_t^r f\|_q\le|t|^r\sum_{|\alpha|_1=r}\|D^\alpha f\|_q\le|t|^r, t\in\mathbf{R}^n. \tag{2.5}$$

Hence, when $1\le q<\infty$ and $2s>n+r$, by (2.5) and Minkowski's inequality for integral, we have

$$\begin{aligned}\|f-T_\rho f\|_q&=\Big\{\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^n}(\Delta_t^r f)(x)K_{\rho,s}(t)dt\Big|^q dx\Big\}^{\frac1q}\\&\le\int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n}|(\Delta_t^r)(x)|^q dx\Big)^{\frac1q}K_{\rho,s}(t)dt\\&\le\int_{\mathbf{R}^n}|t|^rK_{\rho,s}(t)dt\\&\le C\rho^{-r},\end{aligned} \tag{2.6}$$

where the constant C is only dependent on r,q,s, and n.

Similarly, we have

$$\|f-T_\sigma f\|_\infty\le C\rho^{-r}, \tag{2.7}$$

for any $f\in W_\infty^r(\mathbf{R}^n)$.

Let $\rho=\rho(\sigma)/2s$ and $2s>n+r$ in (2.2). Then (2.1) follows (2.6) and (2.7).

Let $B(l_p^N)$ denote the unit ball of the space l_p^N.

Lemma 2.3 (see [8]). *If $1\le q\le p\le\infty, 1\le k<N$, then*

$$d_k(B(l_p^N),l_q^N)=(N-k)^{\frac1q-\frac1p}, \tag{2.8}$$

where $d_k(A,X)$ denotes the usual $k-K$ width of A in X, while X is a normed linear space and A one of its subsets.

Proof of Theorem 1.1. First, by Lemma 2.1 and Lemma 2.2 , we see

$$\begin{aligned}\overline{d}_\sigma(W^r_{pq}(\mathbf{R}^n), L_q(\mathbf{R}^n)) &\le E(W^r_{pq}(\mathbf{R}^n), SB^q_{\rho(\sigma)}(\mathbf{R}^n), L_q(\mathbf{R}^n))\\ &\le E(W^r_q(\mathbf{R}^n), SB^q_{\rho(\sigma)}(\mathbf{R}^n), L_q(\mathbf{R}^n))\\ &\le C\sigma^{-\frac{r}{n}}. \end{aligned} \tag{2.9}$$

Next, we shall prove that

$$\overline{d}_\sigma(W^r_{pq}(\mathbf{R}^n), L_q(\mathbf{R}^n)) \ge C\sigma^{-\frac{r}{n}} \tag{2.10}$$

holds for $1 \le q \le p \le \infty, \sigma > 0$, where C is a constant only dependent on r, n and q.

The proof of (2.10) is divided into the following two steps.

(I) Locallization. Let L be a linear subspace of average dimension σ of $L_q(\mathbf{R}^n)$. By definition, for any $a > 0$ there exists a linear subspace $M = M(\varepsilon, a, L)$ of finite dimension of $L_q(I^n_a), I^n_a = [-a, a]^n$, such that

$$\dim(M) = k_\varepsilon(a, L, L_q(\mathbf{R}^n)),$$

and

$$E((BL)(I^n_a), M, L_q(I^n_a)) < \varepsilon.$$

For any $a > 0$, set

$$W^{r,0}_p(I^n_a) =: \{f \in W^r_p(\mathbf{R}^n) : \mathrm{supp} f \subseteq I^n_a\}.$$

If $f \in W^{r,0}_p(I^n_a)$, then for each $g \in L$ we have

$$\begin{aligned}E(f, M, L_q(I^n_a)) &\le \|f - g\|_q + \varepsilon\|g\|_q\\ &\le (1 + \varepsilon)\|f - g\|_q + \varepsilon\|f\|_q.\end{aligned} \tag{2.11}$$

For any $N \in Z_+$, it is easy to verify that

$$W^{r,0}_p(I^n_N) \subseteq (2N)^{n(\frac{1}{q}-\frac{1}{p})} W^r_{pq}(\mathbf{R}^n), \quad 1 \le q \le p \le \infty. \tag{2.12}$$

In fact, for any $f \in W^{r,0}_p(I^n_N)$, since supp $f \subseteq I^n_N$, by Hölder inequality for integral we have

$$\begin{aligned}\|D^\alpha f\|_{pq} &= \Big\{\sum_{j_1=-N}^{N-1} \cdots \sum_{j_n=-N}^{N-1} \|D^\alpha f(\cdot + j)\|^q_{L_p([0,1]^n)}\Big\}^{\frac{1}{q}}\\ &\le (2N)^{\frac{n}{q}-\frac{n}{p}} \|D^\alpha f\|_{L_p(I^n_a)}\end{aligned}$$

for $1 \le q \le p < \infty$, where $j = (j_1, j_2, \cdots, j_n) \in Z^n$.

Similarly, for the case $1 \le q \le p = \infty$, (2.12) also is valid.

Hence, by (2.11) and (2.12), we have

$$\begin{aligned}&(1 + \varepsilon)E(W^r_{pq}(\mathbf{R}^n), L, L_q(\mathbf{R}^n))\\ &\ge (1 + \varepsilon)(2N)^{n(\frac{1}{p}-\frac{1}{q})} E(W^{r,0}_p(I^n_N), L, L_q(\mathbf{R}^n))\\ &\ge (2N)^{n(\frac{1}{p}-\frac{1}{q})}\{E(f, M, L_q(I^n_N)) - \varepsilon\|f\|_q\}\end{aligned} \tag{2.13}$$

for any $f \in W^{r,0}_p(I^n_N)$.

For any $f \in L_q(\mathbf{R}^n)$, set

$$(\delta_N f)(x) =: (2N)^{-r+\frac{n}{p}} f(2Nx_1 - N, .., 2Nx_n - N).$$

Then, for $f \in W_p^{r,0}(I_N^n)$, we have

$$\sum_{|\alpha|_1=r} \|D^\alpha(\delta_N f)\|_{L_p([0,1]^n)} = \sum_{|\alpha|_1=r} \|D^\alpha f\|_{L_p(I_N^n)} \le 1,$$

which implies $\delta_N f \in W_p^{r,0}([0,1]^n)$. Thus, by (2.13) and a change of scale, we have

$$\begin{aligned}&(1+\varepsilon)E(W_{pq}^r(\mathbf{R}^n), L, L_q(\mathbf{R}^n))\\ &\ge (2N)^r\{E(f, \delta_N(M), L_q([0,1]^n)) - \varepsilon\|f\|_{L_q([0,1]^n)}\}\end{aligned} \tag{2.14}$$

for any $f \in W_p^{r,0}([0,1]^n)$.

(II) Discretization. Let $m \in Z_+$ such that

$$m^n > k_\varepsilon(N) =: k_\varepsilon(N, L, L_q(\mathbf{R}^n))$$

and ϕ be any non-zero function in $C^\infty(\mathbf{R})$ with support in $[0,1]$, i.e., $\sup(\phi) \subseteq [0,1]$. Set

$$\phi_k(x) =: \phi(xm-(k-1)), \quad k = 1, 2, \cdots, m.$$

Thus,

$$\operatorname{supp}(\phi_k) \subseteq [\frac{k-1}{m}, \frac{k}{m}] =: \Delta_k, k = 1, 2, \cdots, m.$$

Set

$$\phi_{k_1,k_2,\cdots,k_n}(x) =: \phi_{k_1}(x_1)\phi_{k_2}(x_2)\cdots\phi_{k_n}(x_n), \quad x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n.$$

It is easy to see that these $\phi_{k_1,k_2,\cdots,k_n}$ have the following properties: for $1 \le s \le \infty$,

(i) $\operatorname{supp}(\phi_{k_1,\cdots,k_n}) \subseteq \Delta_{k_1} \times \cdots \times \Delta_{k_n} =: \Delta_{k_1,\cdots,k_n}$; (2.15)

(ii) $\Big\|\sum\limits_{k_1=1}^m \cdots \sum\limits_{k_n=1}^m a_{k_1,\cdots,k_n}\phi_{k_1,\cdots,k_n}\Big\|_s = m^{-\frac{n}{s}}\|a\|_{l_s^{m(n)}}\|\phi\|_{L_s(\mathbf{R})}^n$, (2.16)

where $\|\cdot\|_{l_s^{m(n)}}$ is the usual $l_s^{m(n)}$-norm, while $m(n) =: m^n$ and $a =: (a_{k_1,\cdots,k_n})_{k_1=1,\cdots,k_n=1}^{m\;\;\;\;\;\;\;m}$.

(iii) $\sum\limits_{|\alpha|_1=r}\Big\|\sum\limits_{k_1=1}^m \cdots \sum\limits_{k_n=1}^m a_{k_1,\cdots,k_n}D^\alpha\phi_{k_1,\cdots,k_n}\Big\|_s = m^{r-\frac{n}{s}}\|a\|_{l_s^{m(n)}}C_s(\phi)$, (2.17)

where

$$C_s(\phi) =: \sum_{|\alpha|_1=r}\|\phi^{(\alpha_1)}\|_{L_s(\mathbf{R})}\cdots\|\phi^{(\alpha_n)}\|_{L_s(\mathbf{R})}, \quad \alpha = (\alpha_1, \cdots, \alpha_n).$$

For $m \in Z_+(m^n > k_\varepsilon(N))$, set

$$\begin{aligned}Q_p(m) = \Big\{\sum_{k_1=1}^m \cdots \sum_{k_n=1}^m a_{k_1,\cdots,k_n}\phi_{k_1,\cdots,k_n} : \|a\|_{l_p^{m(n)}}\\ \le C_p^{-1}(\phi)m^{-r+\frac{n}{p}}\Big\},\end{aligned} \tag{2.18}$$

where $C_p(\phi)$ is defined as in (2.17) for $s = p$.

By (2.17), we have $Q_p(m) \subseteq W_p^{r,0}([0,1]^n)$. For any $f \in Q_p(m)$, by the inequality

$$\|a\|_{l_q^{m(n)}} \le m^{n(\frac{1}{q}-\frac{1}{p})}\|a\|_{l_p^{m(n)}}$$

for $1 \le q \le p \le \infty$, we have

$$\begin{aligned} \|f\|_{L_q([0,1]^n)} &= m^{-\frac{n}{q}} \|a\|_{l_q^{m(n)}} \|\phi\|^n_{L_q(\mathbf{R})} \\ &\le m^{-\frac{n}{p}} \|a\|_{l_p^{m(n)}} \|\phi\|^n_{L_q(\mathbf{R})} \\ &\le C_p^{-1}(\phi)\|\phi\|^n_{L_q(\mathbf{R}^n)} m^{-r} \\ &=: C^* m^{-r}. \end{aligned} \tag{2.19}$$

Hence, by (2.19), we have

$$\begin{aligned} &\sup_{f\in Q_p(m)} \{E(f, \delta_N(M), L_q(\mathbf{R}^n)) - \varepsilon\|f\|_{L_q([0,1]^n)}\} \\ &\ge E(Q_p(m), \delta_N(M), L_q([0,1]^n)) - C^*\varepsilon m^{-r}. \end{aligned} \tag{2.20}$$

Similar to the case of one variable (see [8]), by a proper calculation, we have

$$\begin{aligned} &E(Q_p(m), \delta_N(M), L_q([0,1]^n)) \\ &\ge d_{k(N)}(Q_p(m), L_q([0,1]^n)) \\ &\ge C_0 d_{k(N)}(B(l_p^{m(n)}), l_q^{m(n)}) m^{-r+n(\frac{1}{p}-\frac{1}{q})}, \end{aligned} \tag{2.21}$$

where $k(N) =: \dim(\delta_N(M)) = k_\varepsilon(N)$ and

$$C_0 =: (C_p(\phi)\|\phi\|^n_{L_{q'}(\mathbf{R})})^{-1}\|\phi\|^{2n}_{L_2(\mathbf{R})}, \quad \frac{1}{q}+\frac{1}{q'}=1.$$

Since $Q_p(m) \subseteq W_p^{r,0}([0,1]^n)$, by (2.14), (2.20) and (2.21) we have

$$\begin{aligned} &(1+\varepsilon)\overline{d_\sigma}(W^r_{pq}(\mathbf{R}^n), L_q(\mathbf{R}^n) \\ &\ge (2N)^r \Big\{ d_{k(N)}(Q_p(m), L_q([0,1]^n)) - C^*\varepsilon M^{-r} \Big\} \\ &\ge (2N)^r \Big\{ m^{-r+n(\frac{1}{p}-\frac{1}{q})} C_0 d_{k(N)}(Bl_p^{m(n)}, l_q^{m(n)}) - C^*\varepsilon m^{-r} \Big\} \\ &= (2N)^r m^{-r} \Big\{ C_0 m^{n(\frac{1}{p}-\frac{1}{q})} (m^n - k(N))^{\frac{1}{q}-\frac{1}{p}} - C^*\varepsilon \Big\} \\ &= \Big(\frac{2N}{m}\Big)^r \Big\{ C_0 \Big(1-\frac{k(N)}{m^n}\Big)^{\frac{1}{q}-\frac{1}{p}} - C^*\varepsilon \Big\}. \end{aligned} \tag{2.22}$$

Since $k_\varepsilon(a, L, L_q(\mathbf{R}^n))$ is non-decreasing in $a > 0$ and $k(N) = k_\varepsilon(N, L, L_q(\mathbf{R}^n))$, it is easy to see that

$$\lim_{\varepsilon\to 0^+} \lim_{N\to\infty} \frac{k(N)}{(2N)^n} = \lim_{\varepsilon\to 0^+} \lim_{a\to\infty} \frac{k_\varepsilon(a, L, L_q(\mathbf{R}^n))}{(2a)^n} = \sigma.$$

Let $\{m_N\}_{N=1}^\infty$ be a sequence such that

$$\lim_{N\to\infty} \Big(\frac{2N}{m_N}\Big)^n = \frac{r}{r+n}\frac{1}{\sigma}. \tag{2.23}$$

Then, by (2.22), we have

$$(1+\varepsilon)\overline{d_\sigma}(W^r_{pq}(\mathbf{R}^n), L_q(\mathbf{R}^n)) \ge \Big(\frac{r}{\sigma(r+n)}\Big)^{\frac{r}{n}} \Big\{ C_0 (1-\frac{r}{r+n})^{\frac{1}{q}-\frac{1}{p}} - C^*\varepsilon \Big\}. \tag{2.24}$$

Letting $\varepsilon \to 0^+$ in (2.24), we have

$$\overline{d_\sigma}(W^r_{pq}(\mathbf{R}^n), L_q(\mathbf{R}^n)) \ge \sigma^{-\frac{r}{n}} C_0 \Big(\frac{r}{r+n}\Big)^{\frac{r}{n}} \Big(\frac{n}{r+n}\Big)^{\frac{1}{q}-\frac{1}{p}}, \tag{2.25}$$

which is (2.9). Theorem 1.1 follows (2.8) and (2.25).

§3. Proof of Theorem 1.2

To prove Theorem 1.2, we first give

Lemma 3.1 (cf. [8]). *Let X_{k+1} be any $(k+1)$-dimensional subspace of a normed linear space X, and let $B(X_{k+1})$ denote the unit ball of X_{k+1}. Then*

$$d_j(B(X_{k+1}), X) = 1, \quad j = 0, 1, \cdots, k.$$

Lemma 3.2. *Let $\rho > \rho(\sigma)$. Then*

$$\overline{d_\sigma}(SB_\rho^q(\mathbf{R}^n) \cap BL_q(\mathbf{R}^n), L_q(\mathbf{R}^n)) = 1, \quad 1 \le q \le \infty, \tag{3.1}$$

where $BL_q(\mathbf{R}^n)$ is the unit ball of $L_q(\mathbf{R}^n)$.

Proof. First, it is obvious that

$$\overline{d_\sigma}(SB_\rho^q(\mathbf{R}^n) \cap BL_q(\mathbf{R}^n), L_q(\mathbf{R}^n)) \le 1 \tag{3.2}$$

for any $\rho > \rho(\sigma)$.

Next, we shall prove that

$$\overline{d_\sigma}(SB_\rho^q(\mathbf{R}^n) \cap BL_q(\mathbf{R}^n), L_q(\mathbf{R}^n)) \ge 1, \quad \forall \rho > \rho(\sigma). \tag{3.3}$$

For any $\rho > \rho(\sigma)$, there exist N elements $\xi_j \in \mathbf{R}^n$ and a set

$$\Delta_\delta =: \{t = (t_1, \cdots, t_n) \in \mathbf{R}^n : |t_j| \le \delta, j = 1, \cdots, n\} (\delta > 0)$$

such that

$$\mathrm{int}(\xi_i + \Delta_\delta) \cap \mathrm{int}(\xi_j + \Delta_\delta) = \emptyset, \quad i \ne j, \quad \bigcup_{s=1}^N (\xi_s + \Delta_\delta) \subset B_n(0, \rho),$$

and

$$\Big\{\mathrm{mes}\Big(\bigcup_{s=1}^N (\xi_s + \Delta_\delta)\Big)\Big\}\Big/(2\pi)^n = \{N(2\delta)^n\}/(2\pi)^n > \sigma.$$

Let $\eta_1 \in (0, \delta)$ satisfy $N(2(\delta - \eta_1))^n > (2\pi)^n \sigma$. For any $\eta \in (0, \eta_1)$, $\quad k = (k_1, \cdots, k_n) \in \mathbf{Z}^n$, set

$$\phi_{k,\delta}(t_1, \cdots, t_n) =: \prod_{j=1}^n \frac{\sin(\delta - \eta)(t_j - \frac{k_j \pi}{\delta - \eta}) \sin \eta (t_j - \frac{k_j \pi}{\delta - \eta})}{\eta(\delta - \eta)\big(t_j - \frac{k_j \pi}{\delta - \eta}\big)^2}.$$

It is easy to verify that $(F\phi_{k,\delta})(y) = 0$ for any $y \notin \Delta_\delta$.

For any $a > 0$, set

$$Q(a) = \mathrm{span}\Big\{\phi_{k,\delta}(t) e^{i\xi_s t} : |k_j| \le \Big[\frac{a(\delta - \eta_1)}{\pi}\Big], \quad s = 1, \cdots, N, \quad j = 1, \cdots, n\Big\}.$$

If $\phi \in Q(a)$, then $(F\phi)(y) = 0$, a.e. $y \in \mathbf{R}^n \setminus \bigcup_{s=1}^N (\xi_s + \Delta_\delta)$. This shows that $Q(a) \subseteq SB_\rho^q(\mathbf{R}^n)$. By [1], when $1 < q < \infty$, there exists $a_0 > 0$ such that the inequality

$$\|f\|_q \le \eta(a) \|f\|_{L_q(I_a^n)} \tag{3.4}$$

holds for any $a \ge a_0$ and any $f \in Q(a)$. It is easy to see that (3.4) is also valid for $q = 1$ and $q = \infty$.

Set $S(a) =: \{f|_{I_a^n} : f \in Q(a)\}$. If $f \in Q(a)$ such that $f|_{I_a^n} \in S(a) \cap \frac{1}{\eta(a)} BL_q(I_a^n)$, then $\|f\|_q \le 1$. Let L be a subspace of average dimension $\le \sigma$ of $L_q(\mathbf{R})$. By an argument similar to Theorem 1.1 (see (2.12)), we have

$$\begin{aligned}&(1+\varepsilon)E(SB_\rho^q(\mathbf{R}^n)\cap BL_q(\mathbf{R}^n),L,L_q(\mathbf{R}^n))\\ &\geq \frac{1}{\eta(a)}E(S(a)\cap BL_q(I_a^n),M,L_q(I_a^n))-\varepsilon,\end{aligned} \tag{3.5}$$

where $M =: M(a,\varepsilon,L)$ is a subspace of $L_q(I_a^n)$ of dimension $k_\varepsilon(a,L,L_q(\mathbf{R}^n))$ such that $E(BL(I_a^n),M,L_q(I_a^n))<\varepsilon$.

Let $\{a_s\}_{s=1}^\infty$ be a sequence such that

$$\underline{\lim}_{a\to\infty}\frac{k_\varepsilon(a,L,L_q(\mathbf{R}^n))}{(2a)^n}=\lim_{s\to\infty}\frac{k_\varepsilon(a_s,L,L_q(\mathbf{R}^n))}{(2a_s)^n}=\sigma. \tag{3.6}$$

Consider that

$$\dim(S(a))=\dim(Q(a))=N(2[\frac{a(\delta-\eta_1)}{\pi}]+1)^n$$

and

$$\lim_{a\to\infty}\frac{\dim(S(a))}{(2a)^n}=\frac{N(2(\delta-\eta_1))^n}{(2\pi)^n}=v_1>\sigma.$$

Then, for some $\delta_1\in(0,\frac{v_1-\delta}{2})$, there exists a natural number s_0 such that

$$k_\varepsilon(a_s,L,L_q(\mathbf{R}^n))\leq(\sigma+\delta_1)(2a_s)^n$$

and $\dim(S(a_s))\geq(v_1-\delta_1)(2a_s)^n, s\geq s_0$, i.e.,

$$\dim M(a,\varepsilon,L)\leq k_\varepsilon(a_s,L,L_q(\mathbf{R}^n))<\dim(S(a_s)).$$

Hence, by Lemma 3.1 we have

$$d_k(S(a_s)\cap BL_q(I_{a_s}^n),L_q(I_{a_s}^n))=1,\quad s\geq s_0, \tag{3.7}$$

where $k =: \dim(M(a_s,\varepsilon,L))$.

Thus, by (3.5) and (3.7), we have

$$\overline{d_\sigma}(SB_\rho^q(\mathbf{R}^n)\cap BL_q(\mathbf{R}^n),L_q(\mathbf{R}^n))\geq 1, \tag{3.8}$$

which is (3.3). We complete the proof of Lemma 3.2.

Proof of Theorem 1.2.

The upper bound. For any $f\in B(\mathcal{R}_2^\alpha)$, let $g\in SB_\rho^2(\mathbf{R}^n)$ be defined by $(Fg)(y)=(Ff)(y)$ or 0 according as whether $|y|<\rho$ or not. Then, by Plancherel's theorem, we have

$$\begin{aligned}\|f-g\|_2^2&=\int_{|y|\geq\rho}|(Ff)(y)|^2dy\\ &\leq\rho^{-2\alpha}\int_{|y|\geq\rho}|y|^{2\alpha}|(Ff)(y)|^2dy\leq\rho^{-2\alpha}.\end{aligned}$$

Hence, we have

$$E(B(\mathcal{R}_2^\alpha),\ SB_\rho^2(\mathbf{R}^n),L_2(\mathbf{R}^n))\leq\rho^{-\alpha}. \tag{3.9}$$

Therefore, when $\rho=\rho(\sigma)$ as in Theorem 1.1, we get

$$\begin{aligned}\overline{d_\sigma}(B(\mathcal{R}_2^\alpha),L_2(\mathbf{R}^n))&\leq E(B(\mathcal{R}_2^\alpha),SB_{\rho(\sigma)}^2(\mathbf{R}^n),L_2(\mathbf{R}^n))\\ &\leq(\rho(\sigma))^{-\alpha}=C_{n,\alpha}\sigma^{-\frac{\alpha}{n}}.\end{aligned} \tag{3.10}$$

The lower bound. We first prove the following Bernstein-type inequality

$$\||\cdot|^\alpha Ff\|_2\leq\rho^\alpha\|f\|_2 \tag{3.11}$$

for any $f \in SB^2_\rho(\mathbf{R}^n)$. In fact, by Plancherel's Theorem, we have

$$\begin{aligned} \||\cdot|^\alpha Ff\|_2^2 &= \int_{\mathbf{R}^n} |y|^{2\alpha}|(Ff)(y)|^2 dy \\ &= \int_{|y|\le\rho} |y|^{2\alpha}|(Ff)(y)|^2 dy \\ &\le \rho^{2\alpha}\int_{|y|\le\rho} |(Ff)(y)|^2 dy \\ &= \rho^{2\alpha}\|f\|_2^2. \end{aligned}$$

Hence, we obtain

$$SB^2_\rho(\mathbf{R}^n) \cap \rho^{-\alpha} BL_2(\mathbf{R}^n) \subseteq B(\mathcal{R}^\alpha_2). \tag{3.12}$$

Thus, by Lemma 3.2, we have

$$\overline{d_\sigma}(B(\mathcal{R}^\alpha_2), L_2(\mathbf{R}^n)) \ge \rho^{-\alpha}\overline{d_\sigma}(SB^2_\rho(\mathbf{R}^n) \cap BL_2(\mathbf{R}^n), L_2(\mathbf{R}^n)) \ge \rho^{-\alpha}. \tag{3.13}$$

Letting $\rho \to \rho(\sigma)^+$ in (3.13), we get

$$\overline{d_\sigma}(B(\mathcal{R}^\alpha_2), L_2(\mathbf{R}^n)) \ge (\rho(\sigma))^{-\alpha} = C_{n,\alpha}\sigma^{-\frac{\alpha}{n}}. \tag{3.14}$$

Thus, Theorem 1.2 follows (3.10) and (3.14).

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