Chin. Ann. of Math.
16B: 3(1995),361-370.

THE GAUSS MAP OF TIMELIKE SURFACES IN R_1^n

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Abstract

Gauss maps of oriented timelike 2-surfaces in R_1^n are characterized, and it is shown that Gauss maps can determine surfaces locally as they do in R^n case. Moreover, some essential differences are discovered between the properties of the Gauss maps of surfaces in R^n and those of the Gauss maps of timelike surfaces in R_1^n. In particular, a counterexample shows that a nonminimal timelike surface in R_1^n cannot be essentially determined by its Gauss map.

Keywords Timelike surface, Gauss map, External surface.
1991 MR Subject Classification 53A, 53C50.

§1. Introduction

Let S be an oriented 2-dimensional timelike surface immersed in Minkowski n-space R_1^n. Let

$$g : S \to G_{2,n}^* \tag{1.1}$$

be the generalized Gauss map, which maps each point p of S to $g(p)$, the tangent plane to S at p, where $G_{2,n}^*$ is the set of oriented timelike 2-planes in R_1^n. It then follows from [4] that $G_{2,n}^*$ is naturally a Lorentzian manifold, and it is called pseudo-Grassmannian. The objective of this paper is to study the properties of the map g, particularly those related to the geometry of S in R_1^n and the conformal structure of S.

The main problems we consider here are:

1. Let S_0 be an oriented 2-dimensional Lorentzian manifold, and

$$X : S_0 \to S \subset R_1^n \tag{1.2}$$

a conformal immersion realizing S. What properties does the map

$$G = g \circ X : S_0 \to G_{2,n}^* \tag{1.3}$$

possess? Here Gauss map g is defined by (1.1).

2. Given a map

$$G : S_0 \to G_{2,n}^* \tag{1.4}$$

defined on an oriented 2-dimensional Loretzian manifold S_0, when does there exist a conformal immersion X of S_0 onto a timelike surface S in R_1^n such that G is of the form (1.3), where g is the Gauss map of S?

3. To what extent is a surface S given by (1.2) determined by its Gauss map g?

Manuscript received January 6, 1993.
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Let us recall some known facts about special class of Gauss maps.

(a) A timelike surface $X : S_0 \to R_1^n$ is a minimal surface if and only if it can be expressed locally as $X(u,v) = F(u) + G(v)$, where $\{u, v\}$ are local null coordinate parameters. The map X is not uniquely determined by its Gauss map G. In fact, T. K. Milnor constructed infinite families of isometric entire, timelike minimal surfaces in R_1^3, no two of which are congruent under similarity transformations of R_1^3, but all of them share the same Gauss map[5].

(b) Wang Hong considered the following problem[6] : Given a real-valued function $H : D \to R$ and a map $G : D \to S^{1,1} = G^*_{2,3}$, where D is a simply connected domain in R_1^2, when does there exist a conformal immersion $X : D \to R_1^3$ with mean curvature H and Gauss map G? She derived an integrable condition which is a single second-order equation involving G and H.

(c) In [2], D. A. Hoffman and R. Osserman have discussed similar problem for surfaces in R^n, some existence and uniqueness theorems are established. They also obtained the following uniqueness result: Let a surface S be defined by a conformal immersion $X : S_0 \to R^n$ of a Riemann surface S_0, then S is determined up to similarity transformations by its Gauss map unless S is minimal. They proved this by virtue of elliptic equation; more precisely, by the unique continuation of the harmonic function on S_0.

In this paper, we obtain a necessary and sufficient condition for a map $G : D \to G^*_{2,n}$ to be a Gauss map of a timelike surface S in R_1^n, where D is a simply connected domain in R_1^2, or a simply connected 2-dimensional Lorentzian manifold. On discussing the uniqueness, we find that the result mentioned in (c) is not always right while considering the timelike surface in R_1^n, because we are now in situation of dealing with hyperbolic equation. First we will give an example to see this, next we will give some uniqueness theorems by the characteristic theory of the hyperbolic equations under the condition that the set of minimal points is not too large. The above example just says that the condition is sharp.

§2. The Gauss Map and its Properties

2.1. The Expression of $G^*_{2,n}$

Let $G^*_{2,n}$ denote the pseudo-Grassmannian of oriented timelike 2-plane in Minkowski n-space R_1^n. The inner product on R_1^n is given by

$$v \cdot w = v \cdot w = v^1 \cdot w^1 + \cdots + v^{n-1} \cdot w^{n-1} - v^n \cdot w^n$$

for any $v = (v^1, v^2, \cdots, v^n), w = (w^1, w^2, \cdots, w^n) \in R_1^n$. Given an oriented timelike 2-plane P in R_1^n, let $\{v, w\}$ be an ordered pair of null vectors spanning P and $v \cdot w > 0$. The order depends on the orientation, and the vectors v and w are null means that $v^2 = v \cdot v = 0$ and $w^2 = w \cdot w = 0$. A different choice of such basic vectors yields an ordered pair of form $\{av, bw\}$, where $a, b \in R, ab > 0$. And if we pass to the real projective space RP^{n-1}, we find that to each plane P corresponds a unique point in $RP^{n-1} \times RP^{n-1}$. The nullity of the vectors v and w implies that the point so obtained must be in $Q^*_{n-2} \times Q^*_{n-2}$, where the quadric $Q^*_{n-2} \subset RP^{n-1}$ is defined by

$$Q^*_{n-2} = \{\ [\xi] \in RP^{n-1} |\ \xi^2 = 0\}. \tag{2.1}$$

In fact, it is easy to verify that the map

$$P \to ([v],[w])$$

of

$$G_{2,n}^* \to Q_{n-2}^* \times Q_{n-2}^* \setminus \text{Diag}$$

is a bijection, where Diag= $\text{diag}\{Q_{n-2}^* \times Q_{n-2}^*\} = \{([\xi],[\xi]) \mid [\xi] \in Q_{n-2}^*\}$, and hence we may identify $Q_{n-2}^* \times Q_{n-2}^* \setminus$ Diag with the Grassmannian $G_{2,n}^*$.

2.2. The Gauss Map

First we recall some facts.

Let S_0 be a 2-dimensional Lorentzian manifold. Then for any point p in S_0, there exists an oriented local null coordinate system $\{U;(u,v)\}$around p such that the metric on S_0 has the form

$$ds_0^2 = 2f du dv \tag{2.2}$$

for some positive function f on U. Furthermore, if $\{\overline{U};(\bar{u},\bar{v})\}$ is another oriented local null coordinate system around p, then

$$\bar{u} = \bar{u}(u), \bar{v} = \bar{v}(v) \tag{2.3}$$

on $U \cap \overline{U}$.

Now given an oriented timelike 2-dimensional surface S in R_1^n, we have the Gauss map

$$g : S \to Q_{n-2}^* \times Q_{n-2}^* \setminus \text{Diag}. \tag{2.4}$$

Locally, if $\{u,v\}$ are oriented null parameters in a neighborhood of a point p on S, so that S is defined near p by a map

$$(u,v) \to X = (x^1, x^2, \cdots, x^n),$$

then the vectors $\dfrac{\partial X}{\partial u}, \dfrac{\partial X}{\partial v}$ are null. It follows that the Gauss map g may be given locally by

$$(u,v) \to \left(\left[\frac{\partial X}{\partial u}\right], \left[\frac{\partial X}{\partial v}\right]\right) \in Q_{n-2}^* \times Q_{n-2}^* \setminus \text{Diag}. \tag{2.5}$$

By the fact mentioned above, the definition is not dependent on the choice of local null parameters.

2.3. Properties of the Gauss Map

We may now formulate our basic questions as follows. Given an oriented 2-dimensional Lorentzian manifold S_0 among all maps of S_0 into $Q_{n-2}^* \times Q_{n-2}^* \setminus$Diag, how can one characterize those which arise in the manner described as Gauss maps of oriented timelike surface in R_1^n?

At first, we will say something on the quadric $Q_{n-2}^* \subset RP^{n-1}$. The quadric Q_{n-2}^* is topologically the $(n-2)$-sphere S^{n-2} by the correspondence

$$(\xi^1, \xi^2, \cdots, \xi^{n-1}, \xi^n) \to \left(\frac{\xi^1}{\xi^n}, \frac{\xi^2}{\xi^n}, \cdots, \frac{\xi^{n-1}}{\xi^n}\right)$$

which is obviously a homeomorphism from Q_{n-2}^* to S^{n-2}.

To answer the above questions, we will start by deriving the necessary condition on such a map.

Given a map G of S_0 into $Q^*_{n-2} \times Q^*_{n-2} \setminus$ Diag, we may represent it locally under the null coordinate system $\{U; (u,v)\}$ in the form $([\Phi], [\Psi])$, where $\Phi(u,v) = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in R^n \setminus \{0\}$ and $\Psi(u,v) = (\psi_1, \psi_2, \cdots, \psi_n) \in R^n \setminus \{0\}$ satisfy

$$\varphi_1^2 + \cdots + \varphi_{n-1}^2 - \varphi_n^2 = \psi_1^2 + \cdots + \psi_{n-1}^2 - \psi_n^2 = 0. \tag{2.6}$$

We then look for $X(u,v) = (x^1, x^2, \cdots x^n)$ such that (2.5) is satisfied. But that means

$$\frac{\partial X}{\partial u} = f\Phi, \quad \frac{\partial X}{\partial v} = g\Psi \tag{2.7}$$

for some functions $f, g : U \to R \setminus \{0\}$. From this we have

$$ds^2 = 2X_u \cdot X_v du dv, \tag{2.8}$$

and the important relationship

$$X_u \cdot X_v H = X_{uv}, \tag{2.9}$$

where H is the mean curvature vector field of immersion X. From (2.7) and (2.9). we have

$$fg\Phi \cdot \Psi H = (X_u)_v = (f\Phi)_v = f_v \Phi + f\Phi_v \tag{2.10}$$

and

$$fg\Phi \cdot \Psi H = (X_v)_u = (g\Psi)_u = g_u \Psi + g\Psi_u. \tag{2.11}$$

Let Π be the tangent plane to X. Denote by C^Π the projection of vector C on Π for any vector C in R_1^n. By (2.6) and (2.7), we get

$$(\Phi_v)^\Pi = \eta_1 \Phi, \quad (\Psi_u)^\Pi = \eta_2 \Psi, \tag{2.12}$$

where

$$\eta_1 = \Phi_v \cdot \Psi / \Phi \cdot \Psi, \quad \eta_2 = \Psi_u \cdot \Phi / \Phi \cdot \Psi. \tag{2.13}$$

Further, we denote by V_1 (resp. V_2) the component of Φ_v (resp. Ψ_u) orthogonal to Π, so that

$$\begin{aligned} V_1 &= \Phi_v - (\Phi_v)^\Pi = \Phi_v - \eta_1 \Phi, \\ V_2 &= \Psi_u - (\Psi_u)^\Pi = \Psi_u - \eta_2 \Psi. \end{aligned} \tag{2.14}$$

Since the mean curvature vector H is orthogonal to the tangent plane Π, we obtain the following equations by taking the tangent and normal components of (2.10) and (2.11):

$$(\log f)_v + \eta_1 = 0, \quad (\log g)_u + \eta_2 = 0, \tag{2.15}$$

and

$$fg\Phi \cdot \Psi H = fV_1 = gV_2. \tag{2.16}$$

We are now in a position to formulate our first result.

Theorem 2.1. *Let S be an oriented timelike surface in R_1^n given locally by a conformal map $X : D \to R_1^n$. Let Φ, Ψ be the Gauss map in the sense of* (2.5) *and* (2.7)*. Form the quantities η_1, η_2 and V_1, V_2 from Φ, Ψ by* (2.13) *and* (2.14)*. Then for any point $(u,v) \in D$, we have*

$$V_1(u,v) = \alpha(u,v) V_2(u,v), \tag{2.17}$$

where $\alpha(u,v)$ is a nonvanishing function on D, and on the set where $V_1(u,v) \neq 0$ (or equivalently $V_2(u,v) \neq 0$) the function $\alpha(u,v)$ is uniquely defined. And on D it satisfies

$$(\log \alpha)_{uv} = (\eta_1)_u - (\eta_2)_v. \tag{2.18}$$

Proof. Setting $\alpha = g/f$, by (2.15) and (2.16) we finish the proof immediately.

Remark 2.1. The vanishing of the V_1 at a point is equivalent to the vanishing of the vector V_2, and by (2.16), is equivalent to the vanishing of the mean curvature vector H. On the other hand, by (2.14) this is also equivalent to the condition that $(\varphi_j/\varphi_n)_v = 0$ or $(\psi_j/\psi_n)_u = 0$ for $j = 1, 2, \cdots, n-1$. The latter two conditions mean that the Gauss map Φ and Ψ may have the form $\Phi = \Phi(u)$ and $\Psi = \Psi(v)$.

We next note the important fact that the conditions (2.17) and (2.18) are purely statement about Gauss map G; that is, they are expressed via (2.13) and (2.14) in terms of the components of a representation $([\Phi], [\Psi])$ of G in homogeneous coordinates, but they are independent of the particular representation. In fact, one has the following lemma.

Lemma 2.1. *Given two maps $\Phi, \Psi : D \to R^n \setminus \{0\}$, set $\widehat{\Phi} = f\Phi$ and $\widehat{\Psi} = g\Psi$ where f and g are two smooth and nonvanishing real functions on D. Use* (2.13) *and* (2.14) *to define the quantities η_1, η_2, V_1, V_2 in terms of Φ, Ψ, and the corresponding quantities $\widehat{\eta}_1, \widehat{\eta}_2, \widehat{V}_1, \widehat{V}_2$ in terms of $\widehat{\Phi}$ and $\widehat{\Psi}$. Then*

$$\widehat{V}_1 = fV_1, \quad \widehat{V}_2 = gV_2, \tag{2.19}$$

and the functions α and $\widehat{\alpha}$ satisfy

$$(\log \widehat{\alpha})_{uv} - (\widehat{\eta}_1)_u + (\widehat{\eta}_2)_v = (\log \alpha)_{uv} - (\eta_1)_u + (\eta_2)_v. \tag{2.20}$$

Proof. By the definitions (2.13) and (2.14), we have

$$\widehat{\eta}_1 = \eta_1 + (\log f)_v, \quad \widehat{\eta}_2 = \eta_2 + (\log g)_u$$

and hence

$$\widehat{V}_1 = \widehat{\Phi}_v - \widehat{\eta}_1 \widehat{\Phi}_v = f_v \Phi + f\Phi_v - \widehat{\eta}_1 f\Phi = fV_1,$$

$$\widehat{V}_2 = \widehat{\Psi}_u - \widehat{\eta}_2 \widehat{\Psi}_u = g_u \Psi + g\Psi_u - \widehat{\eta}_2 g\Psi = gV_2.$$

From (2.17), we can set $\widehat{\alpha} = \alpha f g^{-1}$. This gives the lemma.

Remark 2.2. We may note that Lemma 2.1 can be used to give an alternative proof of Theorem 2.1.

To see Remark 2.2, we proceed as follows. By (2.7), we can let

$$\frac{\partial X}{\partial u} = f\Phi = \widehat{\Phi}, \quad \frac{\partial X}{\partial v} = g\Psi = \widehat{\Psi}.$$

Then by (2.13) and (2.14) we have

$$\widehat{\eta}_1 = \widehat{\eta}_2 = 0, \quad \widehat{V}_1 = \widehat{V}_2.$$

Thus we may choose $\widehat{\alpha} \equiv 1$, so that (2.17) and (2.18) follow trivially for $\widehat{\Phi}$ and $\widehat{\Psi}$. But by (2.19) and (2.20) they must also hold for Φ and Ψ.

§3. Existence and Uniqueness

3.1. Local Existence Theorem

First we will prove that the conditions (2.17) and (2.18) are also sufficient for $([\Phi], [\Psi])$ to be Gauss map of a timelike surface in R_1^n locally. We have

Theorem 3.1. *Let $\{D, 2dudv\}$ be a simply connected domain in R_1^2. For two maps $\Phi, \Psi : D \to R^n \setminus \{0\}$, define η_1, η_2, V_1 and V_2 by* (2.13) *and* (2.14)*. If there exists a nowhere* *vanishing function* $\alpha : D \to R$ *such that* (2.17) *and* (2.18) *hold, then* $([\Phi],[\Psi])$ *can be a Gauss map of a timelike surface given by a conformal map* $X : D \to R_1^n$*.*

Proof. What we need to do is to find two functions $f, g : D \to R$ such that

$$X_u = f\Phi, \quad X_v = g\Psi$$

is completely integrable, or equivalently they satisfy

$$fV_1 = gV_2 \tag{3.1}$$

and

$$(\log f)_v + \eta_1 = 0, \quad (\log g)_u + \eta_2 = 0. \tag{3.2}$$

First we fix a pair of solutions $\tilde{f}, \tilde{g}$ to the equations (3.2), then the unknown functions f and g must have the forms:

$$f(u,v) = \tilde{f}(u,v)F(u), \quad g(u,v) = \tilde{g}(u,v)G(v). \tag{3.3}$$

From (2.18), we have

$$(\log \alpha\tilde{f}\tilde{g}^{-1})_{uv} = (\log\alpha)_{uv} + (\eta_1)_u - (\eta_2)_v = 0.$$

So there exist two functions $F(u)$ and $G(v)$ such that $F(u)^{-1}G(v) = \alpha\tilde{f}\tilde{g}^{-1}$. Then by (2.17)

$$\tilde{f}(u,v)F(u)V_1 = \alpha^{-1}\tilde{g}(u,v)G(v)V_1 = \tilde{g}(u,v)G(v)V_2.$$

This gives the result.

Remark 3.1. In Theorem 3.1, the resulting functions f and g depend only on Φ, Ψ and α. More precisely, functions f and g are independent of the choice of functions $\tilde{f}$ and $\tilde{g}$ satisfying (3.2), and they are determined up to a similar factor by Φ, Ψ and α.

From this remark, we see that if α is uniquely determined by the maps Φ and Ψ, and $([\Phi],[\Psi])$ can be made to be a Gauss map of a timelike surface S in R_1^n, then the resulting surface S is essentially unique. Following this, we have

Theorem 3.2. *Let* S *be a timelike surface defined by a conformal immersion* $X : S_0 \to R_1^n$ *of an oriented connected* 2*-dimensional Lorentzian manifold* S_0 *with the Gauss map* $G : S_0 \to Q^*_{n-2} \times Q^*_{n-2} \setminus Diag$*. If the mean curvature vector of* S *is nowhere vanishing, then* S *is determined up to a similarity transformation of* R_1^n *by its Gauss map* G*; i.e., if* $Y : S_0 \to R_1^n$ *is another conformal immersion inducing the same Gauss map* G*, then there exist a real constant* c *and a constant vector* X_0 *such that* $Y = cX + X_0$*.*

Proof. Given maps $X, Y : S_0 \to R_1^n$, set $\Phi = X_u$ and $\Psi = X_v$ under a local null coordinate system $\{U; (u,v)\}$ on S_0. The fact that X and Y induce the same Gauss map is equivalent to the existence of two nonvanishing functions f and g on U, such that

$$Y_u = f\Phi, \quad Y_v = g\Psi.$$

From (2.13) we have

$$\eta_1 = \eta_2 = 0,$$

and hence by (2.15)

$$f(u,v) = F(u), \quad g(u,v) = G(v).$$

Then the completely integrable condition with respect to the immersion Y gives

$$(F(u) - G(v))X_{uv} = 0. \tag{3.4}$$

Since X has nonvanishing mean curvature, we conclude by (2.9)

$$X_{uv} \not\equiv 0. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$F(u) = G(v) = c, \tag{3.6}$$

where c is a real constant. So by $Y_u = cX_u$ and $Y_v = cX_v$ we must have

$$Y = cX + X_0 \tag{3.7}$$

on U, where X_0 is a constant real vector. Finally note that S_0 is connected, so (3.7) holds on S_0.

3.2. A Global Existence Theorem

We now turn to the question of the extent to which the two necessary conditions (2.17) and (2.18) for map G to be a Gauss map are also sufficient. In local case, we have Theorem 3.1, we also have a theorem on global existence as follows.

Theorem 3.3. *Let S_0 be a simply connected 2-dimensional Lorentzian manifold, and let $G : S_0 \to Q^*_{n-2} \times Q^*_{n-2} \setminus Diag$ be a map that any local representation $([\Phi], [\Psi])$ satisfies* (2.17) *and* (2.18). *If V_1 never vanishes on S_0, then G can be the Gauss map of a timelike surface S in R_1^n given by a conformal map $X : S_0 \to R_1^n$.*

Proof. Theorem 3.1 says that local problem always has a solution by the conditions (2.17) and (2.18). To obtain a global solution, we note first that the condition $V_1 \not\equiv 0$ means that the local surface obtained via Theorem 3.1 has nonvanishing mean curvature vector. It then follows from Theorem 3.2 that the surfaces so obtained are unique up to a similarity transformation of R_1^n. Thus if we fix any point p_0 of S_0, and define a surface S in a neighborhood of p_0 with the prescribed Gauss map, then S can be uniquely continued along every path starting at p_0 by piecing together local solutions. Finally, since S_0 is simply connected, the monodromy theorem guarantees that the resulting surface is independent of path and is globally well defined over S_0. This proves the theorem.

Remark 3.2. Topologically S_0 in Theorem 3.3 must be a plane R^2, since there is no Lorentzian metric on the 2-sphere S^2.

An obvious question which remains is what one can say concerning existence, if the vector V_1 (or V_2) constructed from the map G vanishes somewhere, or identically? To answer, we note first that the local existence always holds. Next, to get a global solution, we have to piece together these local surfaces. Many problems appear when one does this. And here we can say nothing even in the case that V_1 vanishes identically; in this case, one can easily solve it while considering the surface in R^n with the prescribed Gauss map.

3.3. Uniqueness

In the following, we will say something more about uniqueness. We know from Remark 2.1 that if the function α in (2.17) is uniquely determined by the maps Φ and Ψ, we will have essentially one resulting local surface with the Gauss map $([\Phi], [\Psi])$. As to α, from (2.17), on the point where $V_1 \neq 0$, α is determined by Φ and Ψ. So if V_1 never vanishes or V_1 has only isolated zero points, then α is unique and we have a uniqueness theorem (Theorem 3.2). Not in this case, first if $V_1 = 0$ identically, the case corresponding to the minimal timelike surface, then we already know that the surface cannot be essentially determined by its Gauss map. Next, if V_1 vanishes somewhere but not identically, then we will face the situation that on the closure of the set where V_1 does not vanish α is determined, and on other hand the conditions (2.17) and (2.18) only say that α must satisfy equation: $(\log\alpha)_{uv} = (\eta_1)_u - (\eta_2)_v$ under any null coordinate system $\{U;(u,v)\}$. So we should discuss the uniqueness of the following problem. Let $\{D, 2dudv\}$ be a simply connected domain in R_1^n, Ω be an open set of D. α is a smooth function on D such that it is given on $D\setminus\Omega$, and it satisfies the equation: $(\log\alpha)_{uv} = (\eta_1)_u - (\eta_2)_v$ on D for some smooth functions η_1 and η_2 on D. When is α unique?

When one considers the surface in R^n, the above equation is replaced by an elliptic equation. The unique continuation ensures that if $D\setminus\Omega$ is not empty, α is unique. It turns out that every nonminimal surface in R^n is essentially determined by its Gauss map[2]. This is not always right when we consider the timelike surfaces in R_1^n, first we will give a counterexample.

Example. Let $\{R_1^2, 2dudv\}$ be a Minkowski 2-space. Let

$$X(u,v) = (u+v, f(v), u-v) : R_1^2 \to R_1^3$$

be a timelike surface in R_1^3, where

$$f(v) = \begin{cases} 0, & v \le 0, \\ 0 < f' < 1/2, \quad f'' \not\equiv 0, & v > 0 \end{cases}$$

is a smooth function on R.

Let $\{\overline{U};(\bar u,\bar v)\}$ be a null coordinate system around point $(0,0)$ for X with $u(0,0)=0$ and $v(0,0)=0$. Since $X(u,v) = (u+v,0,u-v)$ when $v<0$ and $\{u,v\}$ are also null parameters, by Lemma 2.1 we have

$$\bar u = \bar u(u), \quad \bar v = \bar v(v) \quad \text{when} \quad v < 0.$$

We may also require that $u' > 0$ and $v' > 0$ on $v<0$, and then $\bar u = 0$ (resp. $\bar v = 0$) if and only if $u=0$ (resp. $v=0$). On $\overline{U}$, we have under the new parameters $\bar u$ and $\bar v$

$$X(\bar u,\bar v) = \begin{cases} u(\bar u)(1,0,1) + v(\bar v)(1,0,-1), & \bar v \le 0, \\ \overline{X}(\bar u,\bar v), & \bar v > 0. \end{cases}$$

Now we define

$$Y(\bar u,\bar v) = \begin{cases} u(\bar u)(1,0,1) + g(\bar v)v(\bar v)(1,0,-1), & \bar v \le 0, \\ \overline{X}(\bar u,\bar v), & \bar v > 0, \end{cases}$$

where $g(\bar v)$ is a smooth function on $[0,+\infty)$ and satisfies $g(0)=1$ and $g^{(k)}(0)=0$. Then it is easily checked that the new local surface Y has the same Gauss map induced by X, and it cannot be obtained from X by similarity transformations of R_1^n.

To obtain local or global unique theorem, we must exclude the case appearing in the above example. Following this, we have

Theorem 3.4. *Let $\Phi,\Psi : D \to R^n\setminus\{0\}$ be two maps satisfying* (2.17) *and* (2.18) *where D is a connected domain in R_1^2. Denote $D' = \mathrm{int}\{p\in D | V_1(p) = 0\}$. If for any point $p=(u_0,v_0)\in D'$, both the lines $u=u_0$ and $v=v_0$ in D' have at least one endpoint in $D\setminus D'$, then up to a similarity transformation of R_1^n, there is only one surface S given by a conformal immersion $X : D\to R_1^n$ whose Gauss map is given by* $([\Phi],[\Psi])$.

Proof. By Theorem 3.1, it remains to prove that the function α is uniquely determined by Φ and Ψ. Since $D \setminus D'$ is not empty, $\alpha|_{D\setminus D'}$ is already determined. In D we have

$$(\log \alpha)_{uv} = (\eta_1)_u - (\eta_2)_v. \tag{3.8}$$

If $\tilde{\alpha}$ is another smooth function on D satisfying (3.8) and $\alpha = \tilde{\alpha}$ on $D \setminus D'$, for any point $p = (u_0, v_0)$ in D', (3.8) implies

$$(\log \alpha - \log \tilde{\alpha})_u(u_0, v) = \text{constant}.$$

By the assumption, the constant must be zero because $\alpha = \tilde{\alpha}$ on $D \setminus D'$. Similarly we have

$$(\log \alpha - \log \tilde{\alpha})_v(u, v_0) = 0,$$

that is to say,

$$d(\log \alpha - \log \tilde{\alpha})(u_0, v_0) = 0$$

for any point $p = (u_0, v_0) \in D'$. So we conclude that $\log \alpha - \log \tilde{\alpha} = 0$ on D, this finishes the proof.

We will end this section with a global unique theorem.

Theorem 3.5. *Let S_0 be a connected oriented 2-dimensional Lorentzian manifold, which is null geodesic complete. Let a timelike surface S be defined by a conformal immersion $X : S_0 \to R_1^n$. Denote $S_0' = \text{int}\{p \in S_0 | H(p) = 0\}$, where H is the mean curvature vector of S. If $S_0 \setminus S_0'$ is connected and for any point $p \in S_0'$ each null geodesic passing through p in S_0' has at least one endpoint in $S_0 \setminus S_0'$, then X is determined up to a similarity transformation of R_1^n by its Gauss map G.*

Proof. Let $Y : S_0 \to R_1^n$ be another conformal immersion with the Gauss map G. Then we have $X = cY + X_0$ on $S_0 \setminus S_0'$, where c is a real constant and X_0 is a constant vector in R_1^n.

For $p \in S_0'$, let γ be a null geodesic passing through p, and $q \in \partial S_0'$ be one of the first points at which γ intersects S_0', and N be a null vector along γ such that $N(t) \cdot \gamma' > 0$. Assume that $\gamma(0) = p, \gamma(l) = q$ and $\{U_i; (u_i, v_i)\}_{i=1,2\cdots,m}$ is a finite local null coordinate system which covers $\gamma([0, l])$ and $q \in U_0$. Denote $V_i = S_0 \cap U_i$ for $i = 1, 2, \cdots, m$. We write $\gamma_i = \gamma|_{V_i} = (u_i(t), v_i^0), ds_i^2 = ds^2|_{V_i} = 2f_i du_i dv_i, N_i = N|_{V_i} = g_i \dfrac{\partial}{\partial v_i}$, where v_i^0 is a real constant and $u_i(t), f_i$ and g_i are functions on V_i, for $i = 1, 2, \cdots, m$.

Calculating on V_i, we get

$$(N_i X) \circ \gamma_i = g_i(t) X_{v_i}(u_i(t), v_i^0)$$

and

$$\frac{d}{dt}(N_i X) \circ \gamma_i = g_i'(t) X_{v_i}(u_i(t), v_i^0) = (\log g_i)'(N_i X) \circ \gamma_i \tag{3.9}$$

by the fact that $X_{u_i v_i} = 0$ on V_i. Also we have

$$\frac{d}{dt}(N_i Y) \circ \gamma_i = (\log g_i)'(N_i Y) \circ \gamma_i. \tag{3.10}$$

Combining these gives

$$\frac{d}{dt}(N_i(X - cY)) \circ \gamma_i = (\log g_i)'(N_i(X - cY)) \circ \gamma_i.$$

So

$$N_i(X - cY) \circ \gamma_i = g_i Z_i$$

for some constant vectors Z_i.

But by the choice of point q we have $N(X - cY)(q) = 0$, and from this we see that the vector Z_0 must be zero vector, and hence $N(X - cY) \circ \gamma|_{V_0} = 0$. By induction, we conclude that $Z_i = 0$ for all i and finally reach to $N(X - cY)(p) = 0$.

On the other hand, we can prove by the same method that $N'(X - cY)(p) = 0$, where N' is another null vector such that $N \cdot N' > 0$.

From above, we have

$$d(X - cY) = 0 \qquad \text{on} \qquad S_0'.$$

But we already know $X - cY = X_0$ on $S_0 \setminus S_0'$, so $X = cY + X_0$ should hold on S_0.

Acknowledgement. The author wishes to express his gratitude to Professors C. H. Gu, H. S. Hu and Y. L. Xin for their helpful advice and encouragement.

References

[1] Akutagawa, K. & Nishikawa, S., The Gauss map and spacelike surface with prescribed mean curvature in Minkowski 3-space, *Tohoku Mat. J.,* **42** (1990), 67–82.

[2] Hoffman, D. A. & Osserman. R., The geometry of the generalized Gauss map, *Mem. Amer. Math. Soc.,* **236** (1980).

[3] Hoffman, D. A. & Osserman, R., The Gauss map of surface in R^n, *J. D. Geom.,* **18** (1983), 733–754.

[4] Ishihara, T., The harmonic Guass maps in a generalized sense, *J. London Math. Soc.,* **26:**2 (1982), 104–112.

[5] Milnor, T. K., Entire timelike minimal surface in $E^{3,1}$, *Michigan Math. J.,* **37** (1990), 163–177.

[6] Wang, H., Some geometric problems on the mean curvature of surfaces in 3-dimensional Minikowski space, Ph.D. Thesis, Fudan Univeristy, 1992 (in Chinese).