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EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTION OF NONLINEAR PARABOLIC EQUATION WITH NONLINEAR BOUNDARY CONDITION

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Abstract

This paper deals with the existence and nonexistence of global positive solution of the following equation:

$$\begin{cases} u_t = \nabla(u^{q-1}\nabla u) - \alpha u^m, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = u^p, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \bar{\Omega}, \end{cases}$$

where p, q, m, α are parameters with $q \geq 1,\ m, p > 0, \alpha \geq 0$. $\Omega \subset R^N$ is a bounded domain with $\partial\Omega$ smooth enough, $N \geq 1$. The necessary and sufficient conditions for the global existence of solution are obtained.

Keywords Nonlinear parabolic equiation, Nonlinear boundary condition, Existence, Blow-up, Subsolution and supersolution.
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§1. Introduction

In this paper, we study the following problem:

$$\begin{cases} u_t = \nabla(u^{q-1}\nabla u) - \alpha u^m, & x \in \Omega \times (0,T), & (1.1) \\ \dfrac{\partial u}{\partial n} = u^p, & (x,t) \in S_T = \partial\Omega \times (0,T), & (1.2) \\ u(x,0) = u_0(x) > 0, & x \in \bar{\Omega}, & (1.3) \end{cases}$$

where p, q, m, α are parameters with $p, m > 0,\ q \geq 1,\ \alpha \in R,\ n$ is the outward normal vector. When $\alpha = 0$, equation (1.1) is the well-known porous medium equation, and it has a variety of applications in physical, biological and engineering problems. Many mathematicians devote themselves to the study of (1.1) with Dirichlet, Neumann or Rabin boundary conditions, and some beautiful results have been obtained (see [1] and the references there). However, just as Professor H. A. Levine pointed out (in [2]), few efforts are made when the nonlinearity occurs in the boundary, and the results are far from complete. To our knowledge, [3] first studies (1.1)−(1.3) with $\alpha = 0,\ q = 1$ and $\dfrac{\partial u}{\partial n} = b(u)$, and obtains that if $b(u) = |u|^{1+\epsilon}h(u)$ with $\epsilon > 0,\ h(u)$ is increasing, then (1.1)−(1.3) has no global solution.

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Later, [5] deals with the generalized nonlinear boundary condition and obtains that if $b(u)$ and $b'(u)$ are continuous, positive and increasing, then the solution of (1.1)$-$(1.3), with $\alpha = 0,\ q = 1$ and $\frac{\partial u}{\partial n} = b(u)$, blows up in finite time provided that

$$\int^{\infty} \frac{ds}{b(s)b'(s)} < \infty,$$

and it exists globally provided that $\int^{\infty} \frac{ds}{b(s)b'(s)} = \infty$. [8] considers the problem from another point of view, i.e., the nonlinear diffusion

$$\begin{cases} u_t = \nabla(a(u)\nabla u), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = 1, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \bar{\Omega}. \end{cases} \tag{1.4}$$

Let $a(u), a'(u)$ be positive, continuous and

$$\lim_{u \to \infty} \frac{a'(u)}{a(u)} \leq M.$$

[8] proves that if $\int^{\infty} \frac{ds}{a(s)} < \infty$, then the solution of (1.4) blows up in finite time; if $\int^{\infty} \frac{ds}{a(s)} = \infty$, then the solution of (1.4) exists globally. Recently [7] deals with the case with both nonlinear diffusion and nonlinear boundary condition,

$$\begin{cases} u_t = \nabla(a(u)\nabla u), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = b(u), & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \bar{\Omega}. \end{cases} \tag{1.5}$$

Under the hypothesis that $a(s)$ and $b(s)$ are positive nondecreasing c^1 functions for $s > 0$, if $\int^{\infty} \frac{ds}{b(s)} < \infty$, then the solution of (1.5) blows up in finite time; if $\int^{\infty} \frac{ds}{b(s)} = \infty$, and $a(s) + a(s)b'(s) + a'(s)b(s)$ is nondecreasing, then there are two subcases: if

$$\int^{\infty} \frac{ds}{(a(s) + a(s)b'(s) + a'(s)b(s))b(s)} < \infty,$$

then the solution blows up in finite time; if

$$\int^{\infty} \frac{ds}{(a(s) + a(s)b'(s) + a'(s)b(s))b(s)} = \infty,$$

then the solution of (1.5) exists globally.

In this paper, we put three kinds of nonlinearity together, i.e., nonlinear diffusion, nonlinear reaction and nonlinear boundary condition. In this paper we assume that $\alpha > 0$. (For the case of $\alpha \leq 0$, i.e., the reaction term is a source, we can deal with it in the same way as that of [7] and the results are trivial, so we omit it). Here the reaction term is absorption. It is very important to realize the interactional relations of the three nonlinear terms.

In the case of $p + q \leq 2$ or $p + q > 2$ and $m \leq 1$, we can prove in the same way as that of [7] that the solution of (1.1)$-$(1.3) exists globally for $p + q \leq 2$, and it (with large initial data) blows up in finite time for $p + q > 2$ and $m \leq 1$. So here we focus our attention on the case of $p + q > 2$ and $m > 1$.

§2. Main Results and Preliminaries

A solution $u(x,t)$ of (1.1)−(1.3) means that

$$u(x,t) \in C^{2,1}(\Omega \times (0,T)) \cap C(\bar{\Omega} \times [0,T)),$$

and $u(x,t)$ satisfies (1.1)−(1.3), where $T \le \infty$. If $T = \infty$, we say that $u(x,t)$ exists globally; if $T < \infty$ and there exists $t_n \nearrow T^-$ such that

$$\lim_{n\to\infty} \sup_{x\in\bar{\Omega}} |u(x,t_n)| = \infty,$$

we say that $u(x,t)$ blows up in finite time.

It is easy to prove that then $u(x,t)$ blows up in finite time if $T < +\infty$.

Our main results are as follows.

Theorem 2.1. *Suppose* $p > 1$, $p + q > 2$ *and* $m > 1$. *Then the solution of* (1.1)-(1.3) *exists globally if* $2p + q < m + 2$, *and the solution blows up in finite time for large initial data if* $2p + q \ge m + 2$.

Theorem 2.2. *Suppose* $0 < p \le 1$, $p + q > 2$, *and* $m > 1$. *If* $p + q < m + 1$, *then the solution of* (1.1)-(1.3) *exists globally; if* $p + q > m + 1$, *then the solution of* (1.1)-(1.3) *blows up in finite time for large initial data. For the case of* $p + q = m + 1$, *the situation is very subtle, and the geometry of the domain* Ω *becomes very important. In this case, if* $\dfrac{|\partial\Omega|}{|\Omega|} < 1$, *then the solution of* (1.1)-(1.3) *exists globally; if* $\dfrac{|\partial\Omega|}{|\Omega|} > 1$, *then the solution of* (1.1)-(1.3) *blows up in finite time.*

Our approach to the proof of the theorems is based upon the subsolution and supersolution method. $\bar{u}(x,t)$ is called a supersolution of (1.1)-(1.3) if $\bar{u}(x,t) \in C^{2,1}(\Omega \times (0,T)) \cap C(\bar{\Omega} \times [0,T))$ and satisfies

$$\begin{cases} \bar{u}_t \ge \nabla(\bar{u}^{q-1}\nabla\bar{u}) - \alpha\bar{u}^m, & x \in \Omega,\ t > 0, \quad (2.1)\\ \dfrac{\partial\bar{u}}{\partial n} \ge \bar{u}^p, & x \in \partial\Omega,\ t > 0, \quad (2.2)\\ \bar{u}(x,0) \ge u_0(x), & x \in \bar{\Omega}. \quad (2.3) \end{cases}$$

A subsolution $\underline{u}$ of (1.1)−(1.3) is defined in the same way but with each "$\ge$" replaced by "$\le$".

For subsolution and supersolution, we have the following comparison theorem.

Lemma 2.1. *Let* $\underline{u}$ *and* $\bar{u}$ *be a subsolution and a supersolution of* (1.1)-(1.3) *respectively. Then* $\underline{u}(x,t) \le \bar{u}(x,t)$ *for all* $x \in \bar{\Omega}$ *and* $t \in (0, T_{\max})$, *where* $T_{\max}$ *is the maximal time of existence of* $\bar{u}$. *Moreover, if one of the inequalities* (2.1)-(2.3) *is strict, then* $\underline{u}(x,t) < \bar{u}(x,t)$ *for all* $x \in \Omega$ *and* $t \in (0, T_{\max})$.

Under the previous assumptions, problem (1.1)-(1.3) has a unique, local solution $u(x,t)$. Furthermore, if $u_0(x) > 0$ on $\bar{\Omega}$, then $u(x,t) > 0$ for $t > 0$ and $x \in \bar{\Omega}$. Here, we restrict our attention to the positive solution of (1.1)-(1.3). By the standard monotone method we know that if we can find a supersolution $\bar{u}$ with $u_0(x) \le \bar{u}(x,0)$, and $\bar{u}$ exists globally, then problem (1.1)-(1.3) admits a unique global solution $u(x,t)$ satisfying

$$0 < u(x,t) \le \bar{u}(x,t).$$

If there exists a subsolution $\underline{u}$ of (1.1)−(1.3) such that $\underline{u}(x,0) \le u_0(x)$ and $\underline{u}(x,t)$ blows up in finite time, then $u(x,t)$ must blow up in finite time. Therefore, in the proofs of the main theorems, our main work is looking for the appropriate subsolution or supersolution. Without loss of generality, we assume that $\alpha = 1$ in the following.

§3. Proof of Theorem 2.1

In the case of $2p+q < m+2$, we just need to find a positive global supersolution $\bar{u}(x,t)$. Denote by $h(x)$ the eigenfunction corresponding to the first eigenvalue λ_1 of $-\triangle$ with Dirichlet boundary condition, i.e.,

$$\begin{cases} -\triangle h = \lambda_1 h, & x \in \Omega, \\ h(x) = 0, & x \in \partial\Omega. \end{cases}$$

Then $h(x) > 0$ in Ω. Assume that $\max\limits_{x\in\Omega} h(x) \le \dfrac{1}{2}$ and $\max\limits_{x\in\bar\Omega} |\nabla h(x)| = C_1$. By the strong maximal principle we know that $\dfrac{\partial h}{\partial n} < 0$ on $\partial\Omega$. Thus, there is a positive constant C_2 such that $|\dfrac{\partial h}{\partial n}| > C_2$ on $\partial\Omega$.

We construct the supersolution $\bar{u}(x,t)$ of the form

$$\bar{u}(x,t) = [2\epsilon - \epsilon(1-h(x))^{\frac{k}{\epsilon}}]^{\frac{1}{1-p}},$$

where $k = (p-1)/C_2$ and

$$\epsilon < \frac{1}{2}\min\{1, [(2p+q-2)k^2C_1^2/(p-1)^2]^{\frac{1-p}{m+2-2p-q}}\}.$$

A straight forward but routine computation shows that

$$\begin{cases} -\nabla(\bar{u}^{q-1}\nabla\bar{u}) \ge -\bar{u}^m, & x \in \Omega, \\ \dfrac{\partial \bar{u}}{\partial n} \ge \bar{u}^p, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Note that ϵ can be chosen arbitrarily small, so that for any $u_0(x) \in C(\bar\Omega)$ and $u_0(x) > 0$ on $\bar\Omega$ we can find $\epsilon > 0$ such that

$$u_0(x) \le \bar{u}(x). \tag{3.2}$$

This, together with (3.1), means that $\bar{u}$ is indeed a supersolution of problem (1.1)−(1.3), and by a standard argument we see that $u(x,t)$ exists globally.

For $2p+q = m+2$, we construct a subsolution $\underline{u}$ of the form

$$\underline{u}(x,t) = [A - (p-1)(\delta t + x_1)]^{\frac{1}{1-p}},$$

where $A > px \in \overline{\Omega} \to \max\limits_{x\in\overline{\Omega}} |x_1|$ and

$$\delta < (p+q-1)(2A)^{\frac{p+q-2}{1-p}}.$$

Direct computations show that

$$\underline{u}_t - \nabla(\underline{u}^{q-1}\nabla\underline{u}) < -\underline{u}^m,$$

$$\frac{\partial \underline{u}}{\partial n} = \underline{u}^p \cos(n, x_1) \le \underline{u}^p.$$

For any $u_0(x) \in C(\overline{\Omega})$ and $u_0(x) > 0$ on $\overline{\Omega}$, we can always find an A large enough such that

$$\underline{u}(x,0) \le u_0(x) \qquad \text{on } \overline{\Omega}.$$

Hence, $\underline{u}(x,t)$ is a subsolution of (1.1)−(1.3) and blows up in finite time, which, in turn , means that $u(x,t)$ must blow up in finite time.

For the case of $2p+q>m+2$, we have two subcases, $q>m$ and $q\le m$. If $q>m$, let

$$\underline{u}(x,t)=C[1-(p-1)(\delta t+\frac{x_1}{A})]^{\frac{1}{1-p}},$$

where $A=2(p-1)\overline{\Omega}\to \max|x_1|$,

$$C\ge \max\{A^{-\frac{1}{p-1}},p^{-\frac{1}{q-m}}A^{\frac{2}{q-m}}\}$$

and

$$\delta\le (q-1)A^{-2}C^{q-1}.$$

It is easy to verify that

$$\begin{cases}\underline{u}_t-\nabla(\underline{u}^{q-1}\nabla\underline{u})\le -\underline{u}^m,\\ \dfrac{\partial \underline{u}}{\partial n}<\underline{u}^p.\end{cases}$$

Hence, if $u_0(x)>2^{\frac{1}{p-1}}C$, then $\underline{u}(x,t)$ is a subsolution of (1.1)-(1.3) and it blows up in finite time, and so does $u(x,t)$.

If $q\le m$, we only deal with the one dimensional case. Without loss of generality, we assume $\Omega=(-1,1)$ and construct a subsolution as follows:

$$\underline{u}(x,t)=[\epsilon(2+x)^{\frac{p-1}{\epsilon}}-\delta t]^{\frac{1}{1-p}}. \tag{3.3}$$

Denote

$$y=\epsilon(2+x)^{\frac{p-1}{\epsilon}}-\delta t.$$

Then

$$\underline{u}_t-\nabla(\underline{u}^{q-1}\nabla\underline{u})\le -\underline{u}^m, \tag{3.4}$$

iff

$$\begin{aligned}&(p+q-1)(2+x)^{2(\frac{p-1}{\epsilon}-1)}y^{\frac{2p}{1-p}}+(2+x)^{\frac{p-1}{\epsilon}-2}y^{\frac{p+1}{1-p}}\\ \ge\ & y^{\frac{m-q+2}{1-p}}+\frac{\delta}{p-1}y^{\frac{p+2-q}{1-p}}+\frac{p-1}{\epsilon}(2+x)^{\frac{p-1}{\epsilon}-2}y^{\frac{1+p}{1-p}}.\end{aligned} \tag{3.5}$$

Since $y\le \epsilon(2+x)^{\frac{p-1}{\epsilon}}$, we have

$$\frac{1}{\epsilon}\le (2+x)^{\frac{p-1}{\epsilon}}y^{-1}.$$

Thus, to ensure (3.5) holds we need only

$$\begin{aligned}&(p+q-1)(2+x)^{2(\frac{p-1}{\epsilon}-1)}y^{\frac{2p}{1-p}}+(2+x)^{\frac{p-1}{\epsilon}-2}y^{\frac{p+1}{1-p}}\\ \ge\ & y^{\frac{m-q+2}{1-p}}+(p-1)(2+x)^{2(\frac{p-1}{\epsilon}-1)}y^{\frac{2p}{1-p}}+\frac{\delta}{p-1}y^{\frac{p+2-q}{1-p}}.\end{aligned} \tag{3.6}$$

It is obvious that (3.6) holds if

$$q(2+x)^{2(\frac{p-1}{\epsilon}-1)}y^{\frac{2p}{1-p}}\ge y^{\frac{m-q+2}{1-p}}, \tag{3.7}$$

$$(2+x)^{\frac{p-1}{\epsilon}-2}>\frac{\delta}{p-1}y^{\frac{q-1}{p-1}}. \tag{3.8}$$

It is easy to see that if

$$\begin{cases} \epsilon < \min\{1, \dfrac{p-1}{2}, (q/10)^{\frac{p-1}{2p+q-m-2}}\}, \\ \delta < 3^{\frac{p-q}{\epsilon}-2}, \end{cases} \tag{3.9}$$

then (3.7), (3.8) hold and in turn (3.4) holds. On the other hand

$$\frac{\partial \underline{u}}{\partial n}(-1) = -\frac{\partial \underline{u}}{\partial x}(-1) = y^{\frac{p}{1-p}} = \underline{u}^p,$$
$$\frac{\partial \underline{u}}{\partial n}(1) = \frac{\partial \underline{u}}{\partial x}(1) = -3^{\frac{p-1}{\epsilon}-1}\underline{u}^p < \underline{u}^p.$$

Thus, if

$$u_0(x) \geq [\epsilon(2+x)^{\frac{p-1}{\epsilon}}]^{\frac{1}{1-p}}$$

and (3.9) holds, then $\underline{u}(x,t)$ defined by (3.3) is a subsolution of (1.1)−(1.3), and it blows up in finite time. Therefore, $u(x,t)$ blows up in finite time.

Remark 3.1. For the dimension $N \geq 2$, we can only prove that if the domain is sufficiently narrow, then $u(x,t)$ blows up in finite time. But we conjecture that for general case the same result is also true. This problem is open.

§4. Proof of Theorem 2.2

In this section, we shall prove Theorem 2.2. To do this, we divide the proof into two subcases.

(1) $p+q > m+1$ or $p+q = m+1$ and $|\partial\Omega|/|\Omega| > 1$. We show that the solution of (1.1)−(1.3) $u(x,t)$ blows up in finite time for large initial data.

Denote by $h(x)$ the solution of

$$\begin{cases} \triangle h(x) = \dfrac{|\partial\Omega|}{|\Omega|}\triangle \rightarrow =K, & x \in \Omega, \\ \dfrac{\partial h}{\partial n} = 1, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $|\partial\Omega|$ and $|\Omega|$ are the volume of $\partial\Omega$ in $(N-1)$-dimension and the volume of Ω in N dimension respectively. Since if $h(x)$ is a solution of (4.1) , so are $h(x)+C$ for any $C \in R$, we may assume that $h(x) > 0$ on $\bar{\Omega}$. Denote

$$L = \max_{x\in\bar{\Omega}} h(x), \qquad C_4 = \max_{x\in\bar{\Omega}} |\nabla h|.$$

Let

$$\psi(t) = \begin{cases} \exp t, & \text{if } \ p = 1, \\ [1+(1-p)t]^{\frac{1}{1-p}}, & \text{if } \ 0 < p < 1. \end{cases} \tag{4.2}$$

It is easy to check that $\psi'(t) = \psi^p(t)$. Now, we construct a subsolution $\underline{u}(x,t)$ of (1.1)−(1.3) as follows:

$$\underline{u}(x,t) = \psi(g(t)+h(x)), \tag{4.3}$$

where $g(t)$ satisfies

$$\begin{cases} g'(t) = K\psi^{q-1}(g(t)) - \psi^{m-p}(g(t)), \\ g(0) = g_0, \end{cases} \tag{4.4}$$

and $g_0 > 0$ satisfies

$$K\psi^{q-1}(g_0) - \psi^{m-p}(g_0) > 0. \tag{4.5}$$

Since $p + q \geq m + 1,\ p + q > 2$ and $K > 1$ if $p + q = m + 1$, from (4.4) and (4.5) we have that $g(t) > g_0,\ g'(t) > 0$ and $g(t)$ blows up in finite time. Moreover,

$$K\psi^{q-1}(g(t) + h(x)) - \psi^{m-p}(g(t) + h(x)) \geq K\psi^{q-1}(g(t)) - \psi^{m-p}(g(t)),$$

which implies that

$$g'(t) \leq K\psi^{q-1}(g(t) + h(x)) - \psi^{m-p}(g(t) + h(x)). \tag{4.6}$$

By (4.1) and (4.6), it is easy to verify that

$$\begin{cases} \underline{u}_t - \nabla(\underline{u}^{q-1}\nabla\underline{u}) \leq -\underline{u}^m, \\ \dfrac{\partial\underline{u}}{\partial n} = \underline{u}^p. \end{cases}$$

If we choose

$$u_0(x) \geq \psi(g_0 + h(x)) = \underline{u}(x, 0),$$

then $\underline{u}(x,t)$, defined by (4.3), is a subsolution of (1.1)−(1.3), and $\underline{u}(x,t)$ blows up in finite time, which, in turn, means that $u(x,t)$ must blow up in finite time.

(2) $p + q < m + 1$, or $p + q = m + 1$ and $|\partial\Omega|/|\Omega| < 1$. We show that the solution of (1.1)−(1.3) exists globally. In this case, we construct a supersolution $\bar{u}(x,t)$ of the form:

$$\bar{u}(x,t) = \psi(g(t) + h(x)), \tag{4.7}$$

where $\psi(s)$ is defined by (4.2) and $g(t)$ satisfies

$$\begin{cases} g'(t) = K\psi^{q-1}(g(t) + L) + (p + q - 1)C_4^2\psi^{p+q-2}(g(t) + L) - \psi^{m-p}(g(t)), \\ g(0) = g_0, \end{cases} \tag{4.8}$$

g_0 satisfies

$$\begin{cases} K\psi^{q-1}(g_0 + L) + (p + q - 1)C_4^2\psi^{p+q-2}(g_0 + L) < \psi^{m-p}(g_0), \\ g_0 > \bar{g}, \end{cases} \tag{4.9}$$

where $\bar{g}$ is the maximal positive equilibrium point of (4.8).

Noting that $2 < p+q < m+1$, or $2 < p+q = m+1$ and $K < 1$, we can always find a g_0 satisfying (4.9), and from classical ordinary differential equation theory we know that $g(t)$ exists all the time and $\bar{g} < g(t) < g_0,\ g'(t) < 0$ for all $t > 0$.

Now, by means of $h(x) \leq L$, we can obtain easily

$$\begin{cases} \bar{u}_t - \nabla(\bar{u}^{q-1}\nabla\bar{u}) \geq -\bar{u}^m, \\ \dfrac{\partial\bar{u}}{\partial n} = \bar{u}^p. \end{cases}$$

For any $u_0(x) \in C(\bar{\Omega})$ and $u_0(x) > 0$ on $\bar{\Omega}$, we can always find a g_0 sufficiently large such that

$$u_0(x) < \psi(g_0) \quad \text{for} \quad x \in \bar{\Omega}.$$

Hence, $\bar{u}(x,t)$ defined by (4.7) is indeed a supersolution of (1.1)−(1.3), and it exists globally. Therefore, $u(x,t)$ exists globally, which ends our proof.

Remark 4.1. For the case $0 < p \leq 1,\ p + q = m + 1$ and $|\partial\Omega|/|\Omega| = 1$, it remains open whether the solution of (1.1)−(1.3) exists globally or blows up in finite time for large initial data. We conjecture that it will blow up in finite time for large initial values.

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