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NONLINEAR GALERKIN METHODS FOR SOLVING TWO DIMENSIONAL NEWTON-BOUSSINESQ EQUATIONS

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Abstract

The nonlinear Galerkin methods for solving two-dimensional Newton-Boussinesq equations are proposed. The existence and uniqueness of global generalized solution of these equations, and the convergence of approximate solutions are also obtained.

Keywords Nonlinear Galerkin method, Newton-Boussnesq equations,
A uniformly prior estimate.
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§1. Introduction

In many problems on the integration of evolution equations for large intervals of time, for example, bifurcations, global attractors, inertial manifolds, chaos for dynamical system, the usual numerical methods are irrelevant for large time T, because a large number of existing numerical integration algorithms lead to erro estimates of the form $C(h)\exp(T)$, where $C(h)$ is an appropriate constant that is small for h and $[0, T]$ is the interval of time under consideration. Now a new method of integrating evolution differential equations—the nonlinear Galerkin method—is presented and studied in [1,2,3]. In order to study the chaos phenomenon and global attractors in the problems of Benard flow[4−9], the nonlinear Galerkin method is adapted in this paper.

Newton-Boussinesq equations describing the Benard flow are as follows[7]:

$$\partial_t \xi + u\partial_x \xi + v\partial_y \xi = \Delta\xi - \frac{R_a}{p_r}\partial_x \theta, \tag{1.1}$$

$$\Delta\psi = \xi,\ u = \psi_y,\ v = -\psi_x, \tag{1.2}$$

$$\partial_t \theta + u\partial_x \theta + v\partial_y \theta = \frac{1}{p_r}\Delta\theta, \tag{1.3}$$

where $\mathbf{u} = (u, v)$ is the velocity vector, θ is the temperature, ψ is the flow function, ξ is the vertex, $p_r > 0$ is the Prandtl number, and $R_a > 0$ is the Rayleigh number. The equations (1.1)-(1.3) can be rewritten as

$$\frac{\partial}{\partial t}\Delta\psi + J(\psi, \Delta\psi) = \Delta^2\psi - \frac{R_a}{p_r}\frac{\partial\theta}{\partial x}, \tag{1.4}$$

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$$\frac{\partial \theta}{\partial t} + J(\psi, \theta) = \frac{1}{p_r}\Delta\theta, \tag{1.5}$$

where

$$J(u, v) = u_y v_x - u_x v_y.$$

We consider the periodic value problem of equations (1.4) and (1.5)

$$\begin{aligned} &\psi(x+2D, y, t) = \psi(x, y, t), \quad \psi(x, y+2D, t) = \psi(x, y, t), \\ &\theta(x+2D, y, t) = \theta(x, y, t), \quad \theta(x, y+2D, t) = \theta(x, y, t), \end{aligned} \tag{1.6}$$

$$\psi(x, y, 0) = \psi_0(x, y), \quad \theta(x, y, 0) = \theta_0(x, y), \tag{1.7}$$

where $\psi_0(x, y)$ and $\theta_0(x, y)$ are given functions with period $2D$.

Let $\{w_j(x, y)\}(j = 1, 2, \cdots)$ be the periodic eigenvectors of the operator $A = -\Delta$, which satisfies

$$-\Delta w_j = \lambda_j w_j, \ \ j = 1, 2, \cdots, \ \ \lambda_1 < \lambda_2 < \cdots. \tag{1.8}$$

For every integer m, we are looking for an approximate solution of problem (1.4)-(1.7) of the form

$$\psi_m(t) = \sum_{j=1}^{m} \alpha_{jm}(t) w_j, \quad \theta_m(t) = \sum_{j=1}^{m} \beta_{jm}(t) w_j, \tag{1.9}$$

$\psi_m(t), \theta_m(t) : R^+ \to W_m =$ the space spanned by $w_1, \cdots, w_m$. The functions ψ_m, θ_m are determined by the resolution of a system involving other unknown functions $\xi_m(t)$ and $\eta_m(t)$ respectively, where

$$\xi_m(t) = \sum_{j=m+1}^{2m} \delta_{jm}(t) w_j, \quad \eta_m(t) = \sum_{j=m+1}^{2m} \gamma_{jm}(t) w_j, \tag{1.10}$$

$\xi_m(t), \eta_m(t) : R^+ \to W_m =$ the space spanned by $w_{m+1}, \cdots, w_{2m}$.

The pairs (ψ_m, ξ_m) and (θ_m, η_m) satisfy

$$\begin{aligned} &\frac{d}{dt}(-\Delta\psi_m, v) + r(\psi, \Delta\psi_m, v) + r(\xi_m, \Delta\psi_m, v) \\ &+ r(\psi_m, \Delta\xi_m, v) + (\Delta\psi_m, \Delta v) + \frac{R_a}{p_r}(\theta_{mx}, v) = 0, \quad v \in W_m, \end{aligned} \tag{1.11}$$

$$(\Delta\xi_m, \Delta\ v + r(\psi_m, \Delta\psi_m,\ v) + \frac{R_a}{p_r}(\xi_{mx},\ v) = 0, \quad v \in W_m, \tag{1.12}$$

$$\begin{aligned} &\frac{d}{dt}(\theta_m, v) + r(\psi_m, \theta_m, v) + r(\xi_m, \theta_m, v) + r(\psi_m, \eta_m, v) \\ &= \frac{1}{p_r}(\Delta\theta_m, v), \ v \in W_m, \end{aligned} \tag{1.13}$$

$$-\frac{1}{p_r}(\Delta\eta_m,\ v) + r(\psi_m, \theta_m,\ v) = 0, \quad v \in W_m, \tag{1.14}$$

$$\psi_m(0, x, y) = P_m \psi_0(x, y), \tag{1.15}$$

$$\theta_m(0, x, y) = P_m \theta_0(x, y), \tag{1.16}$$

where P_m is the orthogonal projector from L_2 to W_m,

$$r(u,v,w)=\int\!\!\int(u_yv_x-u_xv_y)w\,dxdy,\quad \Omega=[0,2D]\times[0,2D].$$

§2. A Uniformly Prior Estimate of Approximate Solution

Lemma 2.1. *Suppose that $\nabla\psi_0(x,y)\in L_2(\Omega),\theta_0(x,y)\in L_2(\Omega)$. Then for the solutions $\{\psi_m,\xi_m\}$ and $\{\theta_m,\eta_m\}$ of problem (1.11)-(1.16), we have the following estimates*

$$\|\nabla\psi_m(\cdot,\cdot)\|^2\le 2\|\nabla\psi_m(\cdot,0)\|^2\exp\Big(-\frac{1}{C_0}t\Big)+C_0^2\Big(\frac{R_a}{p_r}\Big)^2\|\theta_m(\cdot,0)\|^2\le E_0, \tag{2.1}$$

$$\int_0^\infty(\|\nabla\theta_m\|^2+\|\nabla\eta_m\|^2)dt+\int_0^T\|\Delta\psi_m\|^2dt\le E_1, \tag{2.2}$$

$$\|\theta_m(\cdot,t)\|\le\|\theta_m(\cdot,0)\|,\quad t\ge 0, \tag{2.3}$$

where E_0 and E_1 are constants independent of m, and C_0 is the smallest constant which satisfies the following Poincare inequality

$$\|u\|\le C_0\|\nabla u\|, \tag{2.4}$$

where $\int\!\int u\,dxdy=0$.

Proof. Let us take $v=\psi_m$ in (1.11), $v=\xi_m$ in (1.12), $v=\theta_m$ in (1.13) and $v=\eta_m$ in (1.14). We obtain

$$\begin{aligned}&(-\Delta\psi_{mt},\psi_m)+r(\psi_m,\Delta\psi_m,\psi_m)+r(\xi_m,\Delta\psi_m,\psi_m)\\&+r(\psi_m,\Delta\xi_m,\psi_m)+(\Delta\psi_m,\Delta\psi_m)+\frac{R_a}{p_r}(\theta_{mx},\psi_m)=0,\end{aligned} \tag{2.5}$$

$$(\Delta\xi_m,\Delta\xi_m)+r(\psi_m,\Delta\psi_m,\xi_m)+\frac{R_a}{p_r}(\xi_{mx},\xi_m)=0, \tag{2.6}$$

$$\begin{aligned}&(\theta_{mt},\theta_m)+r(\psi_m,\eta_m,\theta_m)+r(\xi_m,\theta_m,\theta_m)+r(\psi_m,\eta_m,\theta_m)\\&=\frac{1}{p_r}(\Delta\theta_m,\theta_m),\end{aligned} \tag{2.7}$$

$$-\frac{1}{p_r}(\Delta\eta_m,\eta_m)+r(\psi_m,\theta_m,\eta_m)=0, \tag{2.8}$$

where

$$\begin{aligned}r(\psi_m,\Delta\psi_m,\psi_m)&=\int\!\!\int[\psi_{my}(\Delta\psi_m)_x-\psi_{mx}(\Delta\psi_m)_y]\psi_m dxdy\\&=\frac12\int\!\!\int[(\psi_m^2)_y(\Delta\psi_m)_x-(\psi_m^2)_x(\Delta\psi_m)_y]dxdy=0,\end{aligned}$$

$$\begin{aligned}r(\xi_m,\Delta\psi_m,\psi_m)&=\int\!\!\int[\xi_{my}(\Delta\psi_m)_x-\xi_{mx}(\Delta\psi_m)_y]\psi_m dxdy\\&=-\int\!\!\int\xi_m[(\Delta\psi_m)_{xy}\psi_m+(\Delta\psi_m)_x\psi_{my}]dxdy\\&\quad+\int\!\!\int\xi_m[(\Delta\psi_m)_{xy}\psi_m+(\Delta\psi_m)_y\psi_{mx}]dxdy\\&=-\int\!\!\int\xi_m[\psi_{my}(\Delta\psi_m)_x-\psi_{mx}(\Delta\psi_m)_y]dxdy\\&=-r(\psi_m,\Delta\psi_m,\xi_m).\end{aligned}$$

Similarly,

$$r(\psi_m,\Delta\xi_m,\psi_m)=0,\quad r(\psi_m,\theta_m,\theta_m)=0,\quad r(\xi_m,\theta_m,\theta_m)=0,$$

$$r(\psi_m,\eta_m,\theta_m)=-r(\psi_m,\theta_m,\eta_m).$$

Thus adding the equations (2.5) and (2.6), (2.7) and (2.8) respectively, we have

$$(-\Delta\psi_{mt},\psi_m)+(\Delta\psi_m,\Delta\psi_m)+\frac{R_a}{p_r}(\theta_{mx},\psi_m)+(\Delta\xi_m,\Delta\xi_m)=0, \tag{2.9}$$

$$(\theta_{mt},\theta_m)+\frac{1}{p_r}[(\nabla\theta_m,\nabla\theta_m)+(\nabla\eta_m,\nabla\eta_m)]=0. \tag{2.10}$$

From (2.9), it follows that

$$\frac{1}{2}\frac{d}{dt}\|\theta_m\|^2+\frac{1}{p_r}(\|\nabla\theta_m\|^2+\|\nabla\eta_m\|^2)=0, \tag{2.11}$$

$$\|\theta_m(.,t)\|^2\le\|\theta_m(.,0)\|^2, \tag{2.12}$$

$$\int_0^\infty(\|\nabla\theta_m\|^2+\|\nabla\eta_m\|^2)dt\le\frac{p_r}{2}\|\theta_m(.,0)\|^2<\infty. \tag{2.13}$$

From (2.9), we can get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\psi_m\|^2+\|\Delta\psi_m\|^2+\|\Delta\xi_m\|^2\\ &\le\frac{1}{2C_0}\|\psi_{mx}\|^2+\frac{C_0}{2}\Big(\frac{R_a}{p_r}\Big)^2\|\theta_m\|^2\\ &\le\frac{1}{2}\|\Delta\psi_m\|^2+\frac{C_0}{2}\Big(\frac{R_a}{p_r}\Big)^2\|\theta_m\|^2,\end{aligned} \tag{2.14}$$

where

$$\|\psi_{mx}\|^2\le\|\nabla\psi_m\|^2\le C_0\|\Delta\psi_m\|^2,$$

in view of

$$\int\int\nabla\psi_m dxdy=0.$$

Hence from (2.14) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi_m\|^2+\frac{1}{2C_0}\|\nabla\psi_m\|^2\le\frac{C_0}{2}\Big(\frac{R_a}{p_r}\Big)^2\|\theta_m(\cdot,0)\|^2. \tag{2.15}$$

By using Gronwall's inequality, it follows that

$$\|\nabla\psi_m(\cdot,t)\|^2\le\|\nabla\psi_m(\cdot,0)\|\exp\Big(-\frac{1}{C_0}t\Big)+C_0^2\Big(1-\exp\Big(-\frac{1}{C_0}t\Big)\Big)\Big(\frac{R_a}{p_r}\Big)^2\|\theta_m(\cdot,0)\|^2.$$

Lemma 2.2 (The uniform Gronwall lemma[9]). *Let g,h,y be three positive locally integrable functions on $[t_0,+\infty)$, such that y' is locally integrable on $[t_0,+\infty)$ and g,h,y satisfy*

$$\frac{dy}{dt}\le gy+h,\quad t\ge t_0, \tag{2.16}$$

$$\int_t^{t+r}g(s)ds\le a_1,\quad \int_t^{t+r}h(s)ds\le a_2,\quad \int_t^{t+r}y(s)ds\le a_3,\quad t\ge t_0, \tag{2.17}$$

where a_1,a_2,a_3 are positive constants. Then

$$y(t+r)\le\Big(\frac{a_3}{r}+a_2\Big)\exp(a_1),\ t\ge t_0. \tag{2.18}$$

Lemma 2.3. *Suppose that the conditions in Lemma* 2.1 *are satisfied. Assume that* $\Delta\psi_m(\cdot,0)\in L_2(\Omega)$, $\nabla\theta_m(\cdot,0)\in L_2(\Omega)$. *Then, for the solutions* $\{\psi_m,\xi_m\}$ *and* $\{\theta_m,\eta_m\}$ *of problem* (1.11)-(1.16), *we have the following estimates*

$$\|\nabla\theta_m(.,t)\|^2+\|\Delta\psi_m(.,t)\|^2\le E_2,\quad t\ge 0, \tag{2.19}$$

$$\int_0^T(\|\nabla\Delta\psi_m\|^2+\|\nabla\Delta\xi_m\|^2+\|\Delta\theta_m\|^2)dt\le E_3,\quad t\ge 0, \tag{2.20}$$

where the constants E_2 *and* E_3 *are independent of* m.

Proof. Let us take $v=-\Delta\psi_m$ in (1.11), $v=-\Delta\xi_m$ in (1.12), $v=-\Delta\theta_m$ in (1.13) and $v=-\Delta\eta_m$ in (1.14). We find

$$\begin{aligned}&(-\Delta\psi_{mt},-\Delta\psi_m)+r(\psi_m,\Delta\psi_m,-\Delta\psi_m)\\&+r(\xi_m,\Delta\psi_m,-\Delta\psi_m)+r(\psi_m,\Delta\xi_m,-\Delta\psi_m)\\&+(\Delta\psi_m,-\Delta^2\psi_m)+\frac{R_a}{p_r}(\theta_{mx},-\Delta\psi_m)=0,\end{aligned} \tag{2.21}$$

$$(\Delta\xi_m,-\Delta^2\xi_m)+r(\psi_m,\Delta\psi_m,-\Delta\xi_m)+\frac{R_a}{p_r}(\xi_{mx},\Delta\xi_m)=0, \tag{2.22}$$

$$\begin{aligned}&(\theta_{mt},-\Delta\theta_m)+r(\psi_m,\theta_m,-\Delta\theta_m)+r(\xi_m,\theta_m,-\Delta\theta_m)+r(\psi_m,\eta_m,-\Delta\theta_m)\\&=\frac{1}{p_r}(\Delta\theta_m,-\Delta\theta_m),\end{aligned} \tag{2.23}$$

$$-\frac{1}{p_r}(\Delta\eta_m,-\Delta\eta_m)+r(\psi_m,\theta_m,-\Delta\eta_m)=0, \tag{2.24}$$

where

$$\begin{aligned}(-\psi_{mt},\psi_m)&=\frac{1}{2}\frac{d}{dt}\|\Delta\psi_m\|^2,\quad(\Delta\psi_m,-\Delta^2\psi_m)=\|\nabla\Delta\psi_m\|^2,\\(\theta_{mt},-\Delta\theta_m)&=\frac{1}{2}\frac{d}{dt}\|\nabla\theta_m\|^2,\quad-\frac{1}{p_r}(\Delta\eta_m,-\Delta\eta_m)=\frac{1}{p_r}\|\Delta\eta_m\|^2,\\r(\psi_m,\Delta\psi_m,-\Delta\psi_m)&=0,\quad r(\xi_m,\Delta\psi_m,-\Delta\psi_m)=0,\end{aligned}$$

$$\begin{aligned}r(\psi_m,\Delta\xi_m,-\Delta\psi_m)&=r(\psi_m,\Delta\psi_m,\Delta\psi_m),\\|r(\psi_m,\theta_m,-\Delta\theta_m)|&\le(\|\psi_{mx}\|_\infty+\|\psi_{my}\|_\infty)\|\nabla\theta_m\|\|\Delta\theta_m\|,\\\|\psi_{mx}\|_\infty&\le C\|\psi_{mx}\|^{\frac12}\|\Delta\psi_{mx}\|^{\frac12}\le C_1\|\nabla\Delta\psi_m\|^{\frac12},\\\|\psi_{my}\|_\infty&\le C\|\psi_{my}\|^{\frac12}\|\Delta\psi_{my}\|^{\frac12}\le C_1\|\nabla\Delta\psi_m\|^{\frac12},\\|r(\psi_m,\theta_m,-\Delta\theta_m)|&\le\frac{1}{4p_r}\|\Delta\theta_m\|^2+2p_rC_1^2\|\nabla\Delta\psi_m\|\|\nabla\theta_m\|^2\\&\le\frac{1}{4p_r}\|\Delta\theta_m\|^2+\frac{1}{3}\|\nabla\Delta\psi_m\|^2+3p_r^2C_1^4\|\nabla\theta_m\|^4,\end{aligned} \tag{2.25}$$

$$|r(\xi_m,\theta_m,-\Delta\theta_m)|\le\frac{1}{4p_r}\|\Delta\theta_m\|^2+\frac{1}{3}\|\nabla\Delta\xi_m\|^2+3p_r^2C_1^4\|\nabla\theta_m\|^4, \tag{2.26}$$

$$
\begin{aligned}
|r(\psi_m,\eta_m,-\Delta\theta_m)| &\le (\|\psi_{my}\|_\infty+\|\psi_{mx}\|_\infty)\|\nabla\eta_m\|\|\Delta\theta_m\| \\
&\le 2C_1\|\nabla\Delta\psi_m\|^{\frac12}\|\nabla\eta_m\|\|\Delta\theta_m\| \\
&\le \frac{1}{4p_r}\|\Delta\theta_m\|^2+\frac13\|\nabla\Delta\psi_m\|^2+3p_r^2C_1^2\|\nabla\eta_m\|^4 \\
&\le \frac{1}{4p_r}\|\Delta\theta_m\|^2+\frac13\|\nabla\Delta\psi_m\|^2+3p_r^2C_1^2\|\eta_m\|^2\|\Delta\eta_m\|^2.
\end{aligned} \tag{2.27}
$$

In order to estimate $r(\psi_m,\eta_m,-\Delta\theta_m)$, one needs to estimate $\|\eta_m\|^2$. In fact, from (2.8) it follows that

$$\frac{1}{p_r}\|\nabla\eta_m\|^2+r(\psi_m,\theta_m,\eta_m)=0,$$

where

$$
\begin{aligned}
|r(\psi_m,\theta_m,\eta_m)| &\le (\|\psi_{my}\|_\infty+\|\psi_{mx}\|_\infty)\|\nabla\theta_m\|\|\eta_m\| \\
&\le 2C_1\|\nabla\Delta\psi_m\|^{\frac12}\|\nabla\theta_m\|\|\eta_m\| \\
&\le 2C_1\lambda_m^{\frac12}\|\nabla\psi_m\|\lambda_m^{\frac12}\|\theta_m\|\|\eta_m\| \\
&\le 2C_1\lambda_m E_0\|\theta_m(.,0)\|\|\eta_m\|.
\end{aligned}
$$

Since

$$\|\nabla\eta_m\|^2\ge\lambda_{m+1}\|\eta_m\|^2,$$

we have

$$\|\eta_m\|\le 2C_1E_0\Big(\frac{\lambda_m}{\lambda_{m+1}}\Big)\|\theta_m(\cdot,0)\|\le 2C_1E_0\|\theta_m(\cdot,0)\|. \tag{2.28}$$

Put (2.28) into (2.27). Then we have

$$
\begin{aligned}
|r(\psi_m,\eta_m,-\Delta\theta_m)| &\le \frac{1}{4p_r}\|\Delta\theta_m\|^2+\frac13\|\nabla\Delta\psi_m\|^2 \\
&\quad+12p_r^2C_1^4E_0^2\|\theta_m(.,0)\|^2\|\Delta\eta_m\|^2 \\
&\le \frac{1}{4p_r}\|\Delta\theta_m\|^2+\frac13\|\nabla\Delta\psi_m\|^2+\frac{C_2}{p_r}\|\Delta\eta_m\|^2.
\end{aligned} \tag{2.29}
$$

On the other hand,

$$
\begin{aligned}
|r(\psi_m,\theta_m,-\Delta\eta_m)| &\le (\|\psi_{my}\|_\infty+\|\psi_{mx}\|_\infty)\|\nabla\theta_m\|\|\Delta\eta_m\| \\
&\le \frac{1}{2p_r(C_2+1)}\|\Delta\eta_m\|^2+\frac14\|\nabla\Delta\psi_m\|^2 \\
&\quad+p_r^2(C_2+1)^2C_1^4\|\nabla\theta_m\|^4.
\end{aligned} \tag{2.30}
$$

Thus from (2.21)-(2.24),(2.29) and (2.30) we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\Delta\psi_m\|^2+\|\nabla\Delta\psi_m\|^2+\|\nabla\Delta\xi_m\|^2+\frac{1}{p_r}\|\Delta\theta_m\|^2 \\
&+\frac12\frac{d}{dt}\|\nabla\theta_m\|^2+\frac{(C_2+1)}{p_r}\|\Delta\eta_m\|^2 \\
\le\ &\frac{R_a}{p_r}\|\nabla\theta_m\|\|\Delta\psi_m\|+\frac{3}{4p_r}\|\Delta\theta_m\|^2+\frac{1}{2p_r}\|\Delta\eta_m\|^2 \\
&+p_r^2(C_2+1)^3C_1^4\|\nabla\theta_m\|^4+\frac{C_2}{p_r}\|\Delta\eta_m\|^2+\frac{11}{12}\|\nabla\Delta\psi_m\|^2 \\
&+(6p_r^2C_1^4+p_r^2(C_2+1)^2C_1^4)\|\nabla\theta_m\|^4,
\end{aligned} \tag{2.31}
$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|\Delta\psi_m\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla\theta_m\|^2+\frac{1}{12}\|\nabla\Delta\psi_m\|^2+\|\nabla\Delta\xi_m\|^2$$
$$+\frac{1}{2p_r}\|\Delta\eta_m\|^2+\frac{1}{4p_r}\|\Delta\theta_m\|^2$$
$$\leq\frac{R_a}{p_r}\|\nabla\theta_m\|\|\Delta\psi_m\|+C_3\|\nabla\theta_m\|^4,\tag{2.32}$$

where

$$C_3=p_r^2(C_2+1)^3C_1^4+6p_r^2C_1^4+p_r^2(C_2+1)^2C_1^4.$$

This gives the differential inequality in particular

$$\frac{dy_m}{dt}\leq g_my_m+h_m,\tag{2.33}$$

where we have set

$$y_m(t)=\frac{1}{2}(\|\Delta\psi_m\|^2+\|\nabla\theta_m\|^2),$$
$$g_m(t)=C_3\|\nabla\theta_m\|^2,$$
$$h_m(t)=\frac{R_a}{2p_r}[\|\nabla\theta_m\|^2+\|\Delta\psi_m\|^2].\tag{2.34}$$

By integrating (2.11) and (2.15) between t and $t+1$, we find that $\int_t^{t+1}\|\nabla\theta_m\|^2ds$ and $\int_t^{t+1}\|\Delta\psi_m\|^2ds$ are bounded for all $t\geq0$ by a constant independent of m. Consequently, the functions $y_m(t)$, $h_m(t)$ and $g_m(t)$ given by (2.34) satisfy (2.17) with constants a_i independent of m, and we infer from (2.18) that

$$y_m(t)=\frac{1}{2}(\|\Delta\psi_m\|^2+\|\nabla\theta_m\|^2)\leq C_4,\ t\geq1,\tag{2.35}$$

where the constant C_4 is independent of m. By using usual Gronwall's inequality from (2.33), $y_m(t)$ is uniformly bounded for $0\leq t\leq1$. The inequality (2.19) has been obtained. By integrating (2.32) between 0 and T, we get (2.20).

Lemma 2.4. *Under the conditions of Lemma* 2.3, *we have the following estimates:*

$$\|\frac{d}{dt}\Delta\psi_m\|_{L_2(0,T,H^{-1})}+\|\frac{d}{dt}\Delta\psi_m\|_{L_\infty(R^+,H^{-2})}\leq E_4,\tag{2.36}$$

$$\|\frac{d\theta_m}{dt}\|_{L_\infty(R^+,H^{-1}(\Omega))}\leq E_5,\tag{2.37}$$

where the constants E_4 and E_5 are independent of m.

Proof. From (1.11), we have the estimate

$$\Big|\frac{d}{dt}(\Delta\psi_m,v)\Big|\leq|r(\psi_m,\Delta\psi_m,v)|+|(\Delta\psi_m,\Delta v)|+\frac{R_a}{p_r}|(\theta_{mx},v)|$$
$$+|r(\xi_m,\Delta\psi_m,v)|+|r(\psi_m,\Delta\xi_m,v)|,\ v\in W_m,\tag{2.38}$$

where

$$|(\Delta\psi_m,\Delta v)|\leq\|\nabla(\Delta\psi_m)\|\|\nabla v\|,\tag{2.39}$$

$$\Big|\frac{R_a}{p_r}(\theta_{mx},v)\Big|\leq\frac{R_a}{p_r}\|\nabla\theta_m\|\|v\|,\tag{2.40}$$

$$
\begin{aligned}
|r(\psi_m,\Delta\psi_m,v)| &= \Big|\int\!\!\int(\psi_{my}v_x-\psi_{mx}v_y)\Delta\psi_m dxdy\Big| \\
&\le (\|\psi_{my}\|_{L_4}+\|\psi_{mx}\|_{L_4})\|\Delta\psi_m\|_{L_4}\|\nabla v\|, \\
\|\psi_{my}\|_{L_4} &\le C\|\psi_{my}\|^{\frac12}\|\psi_{my}\|^{\frac12}_{H^1}\le C\|\Delta\psi_m\|^{\frac12}\le\text{const.}, \\
\|\psi_{mx}\|_{L_4} &\le \text{const.}, \\
\|\Delta\psi_m\|_{L_4} &\le C\|\Delta\psi_m\|^{\frac12}\|\Delta\psi_m\|^{\frac12}_{H^1}\le C\|\nabla(\Delta\psi_m)\|^{\frac12}, \\
|r(\psi_m,\Delta\psi_m,v)| &\le C\|\nabla(\Delta\psi_m)\|^{\frac12}\|v\|_{H^1},
\end{aligned}
\tag{2.41}
$$

i.e.,

$$\frac{r(\xi_m,\Delta\psi_m,v)}{\|v\|_{H^1}}\in L_4(0,T).$$

The fourth and fifth terms in the right hand side of inequality (2.38) can be estimated as follows respectively

$$
\begin{aligned}
|r(\xi_m,\Delta\psi_m,v)| &= \Big|\int\!\!\int(\xi_{my}v_x-\xi_{mx}v_y)\Delta\psi_m dxdy\Big| \\
&\le (\|\xi_{my}\|_\infty+\|\xi_{mx}\|_\infty)\|\Delta\psi_m\|\|v\|_{H^1} \\
&\le C\|\nabla\Delta\xi_m\|^{\frac12}\|v\|_{H^1},
\end{aligned}
\tag{2.42}
$$

i.e.,

$$\frac{r(\xi_m,\Delta\psi_m,v)}{\|v\|_{H^1}}\in L_4(0,T),$$

$$
\begin{aligned}
|r(\psi_m,\Delta\xi_m,v)| &= \Big|\int\!\!\int(\psi_{my}v_x-\psi_{mx}v_y)\Delta\xi_m dxdy\Big| \\
&\le (\|\psi_{my}\|_{L_4}+\|\psi_{mx}\|_{L_4})\|\Delta\xi_m\|_{L_4}\|v\|_{H^1} \\
&\le C\|\nabla\Delta\xi_m\|^{\frac12}\|v\|_{H^1}.
\end{aligned}
\tag{2.43}
$$

Because $v\in W_m$ in $v\in H^1$ is dense, from (2.38)-(2.42) we have

$$\|\Delta\psi_{mt}\|_{L_2(0,T,H^{-1})}\le E_4, \tag{2.44}$$

where the constant E_4 is independent of m.

On the other hand, from (2.38), we have

$$|(\Delta\psi_m,\Delta v)|\le\|\Delta\psi_m\|\|\Delta v\|\le C\|v\|_{H^2}, \tag{2.45}$$

$$\Big|\frac{R_a}{p_r}(\theta_{mx},v)\Big|\le\frac{R_a}{p_r}\|\nabla\theta_m\|\|v\|\le\|v\|_{H^2}, \tag{2.46}$$

$$
\begin{aligned}
|r(\psi_m,\Delta\psi_m,v)| &= \Big|\int\!\!\int(\psi_{my}v_x-\psi_{mx}v_y)\Delta\psi_m dxdy\Big| \\
&\le (\|\psi_{my}\|_{L_4}\|v_x\|_{L_4}+\|\psi_{mx}\|_{L_4}\|v_y\|_{L_4})\|\Delta\psi_m\|\le C\|v\|_{H^2},
\end{aligned}
\tag{2.47}
$$

$$
\begin{aligned}
|r(\xi_m,\Delta\psi_m,v)| &= \Big|\int\!\!\int(\xi_{my}v_x-\xi_{mx}v_y)\Delta\psi_m dxdy\Big| \\
&\le (\|\xi_{my}\|_{L_4}\|v_x\|_{L_4}+\|\xi_{mx}\|_{L_4}\|v_y\|_{L_4})\|\Delta\psi_m\| \\
&\le C\|\nabla\xi_m\|^{\frac12}\|\nabla\xi_m\|^{\frac12}\|v\|_{H^2},
\end{aligned}
\tag{2.48}
$$

$$\begin{aligned}|r(\psi_m,\Delta\xi_m,v)| &= \Big|\int\int(\psi_{my}v_x-\psi_{mx}v_y)\Delta\xi_m dxdy\Big| \\ &\le (\|\psi_{my}\|_{L_4}\|v_x\|_{L_4}+\|\psi_{mx}\|_{L_4}\|v_y\|_{L_4})\|\Delta\xi_m\| \\ &\le C\|\Delta\xi_m\|\|v\|_{H^2}. \end{aligned} \tag{2.49}$$

Now we need to estimate $\|\nabla\xi_m\|$ and $\|\Delta\xi_m\|$. From (2.6), it follows that

$$\|\Delta\xi_m\|^2 + r(\psi_m,\Delta\psi_m,\xi_m)=0,$$

where

$$\begin{aligned}|r(\psi_m,\Delta\psi_m,\xi_m)| &= \Big|\int\int(\psi_{my}(\Delta\psi_m)_x-\psi_{mx}(\Delta\psi_m)_y)\xi_m dxdy\Big| \\ &\le C(\|\psi_{my}\|_\infty+\|\psi_{mx}\|_\infty)\|\nabla\Delta\psi_m\|\|\xi_m\| \\ &\le C\lambda_m^{\frac34}\|\xi_m\|, \\ \lambda_{m+1}^2\|\xi_m\|^2 &\le |\Delta\xi_m|^2 \le C\lambda_m^{\frac34}\|\xi_m\|, \\ \lambda_{m+1}^{\frac54}\|\xi_m\| &\le C\Big(\frac{\lambda_m}{\lambda_{m+1}}\Big)^{\frac34}\le \text{const}. \end{aligned} \tag{2.50}$$

On the other hand,

$$\begin{aligned}|r(\psi_m,\Delta\psi_m,\xi_m)| &= \Big|\int\int(\psi_{my}\Delta\psi_{mx}-\psi_{mx}\Delta\psi_{my})\xi_m dxdy\Big| \\ &= \Big|\int\int(\psi_{my}\xi_{mx}-\psi_{mx}\xi_{my})\Delta\psi_m dxdy\Big| \\ &\le (\|\psi_{my}\|_\infty+\|\psi_{mx}\|_\infty)\|\nabla\xi_m\|\|\Delta\psi_m\| \\ &\le C\|\nabla\Delta\psi_m\|^{\frac12}\|\nabla\xi_m\|\le C\lambda_m^{\frac14}\|\nabla\xi_m\|, \\ \lambda_{m+1}\|\nabla\xi_m\|^2 &\le \|\Delta\xi_m\|^2\le C\lambda_m^{\frac14}\|\nabla\xi_m\|^2, \\ \lambda_{m+1}^{\frac34}\|\nabla\xi_m\| &\le C(\frac{\lambda_m}{\lambda_{m+1}})^{\frac14}\le\text{const}. \end{aligned} \tag{2.51}$$

From (2.16), there is

$$\|\nabla\Delta\xi_m\|^2 + r(\psi_m,\Delta\psi_m,-\Delta\xi_m)=0, \tag{2.52}$$

where

$$\begin{aligned}|r(\psi_m,\Delta\psi_m,-\Delta\xi_m)| &= \Big|\int\int(\psi_{my}\Delta\psi_{mx}-\psi_{mx}\Delta\psi_{my})\Delta\xi_m dxdy\Big| \\ &\le (\|\psi_{my}\|_\infty+\|\psi_{mx}\|_\infty)\|\nabla\Delta\psi_m\|\|\Delta\xi_m\| \\ &\le C\|\nabla\Delta\psi_m\|^{\frac32}\|\Delta\xi_m\|\le C\lambda_m^{\frac34}\|\Delta\xi_m\|, \\ \lambda_{m+1}\|\Delta\xi_m\|^2 &\le \|\nabla\Delta\xi_m\|^2\le C\lambda_m^{\frac34}\|\Delta\xi_m\|, \\ \lambda_m^{\frac14}\|\Delta\xi_m\| &\le C\Big(\frac{\lambda_m}{\lambda_{m+1}}\Big)^{\frac34}\le\text{const}. \end{aligned} \tag{2.53}$$

Therefore, from (2.51),(2.53),(2.48) and (2.49), it follows that

$$|r(\xi_m,\Delta\psi_m,v)|\le C\|v\|_{H^2}, \tag{2.54}$$

$$|r(\psi_m,\Delta\xi_m,v)|\le C\|v\|_{H^2}. \tag{2.55}$$

From (2.40)-(2.45),(2.54) and (2.55) we get

$$\|\Delta\psi_{mt}\|_{L_\infty(R^+,H^{-2}(\Omega))}\le E_4, \tag{2.56}$$

where the constant E_4 is independent of m.

From (1.13), there is

$$|\frac{d}{dt}(\theta_m,v)|\le|r(\psi_m,\theta_m,v)|+|r(\xi_m,\theta_m,v)|+|r(\psi_m,\eta_m,v)|+\frac{1}{p_r}|(\Delta\theta_m,v)|, \tag{2.57}$$

where

$$\frac{1}{p_r}|(\Delta\theta_m,v)|\le\frac{1}{p_r}\|\nabla\theta_m\|\|\nabla v\|\le\|v\|_{H^1}, \tag{2.58}$$

$$\begin{aligned}|r(\psi_m,\theta_m,v)|&=\Big|\int\int(\psi_{my}\theta_{mx}-\psi_{mx}\theta_{my})vdxdy\Big|\\&\le(\|\psi_{my}\|_{L_4}\|\theta_{mx}\|+\|\psi_{mx}\|_{L_4}\|\theta_{my}\|)\|v\|_{L_4}\\&\le C\|\Delta\psi_m\|\|\nabla\theta_m\|\|v\|_{H^1}\le C\|v\|_{H^1},\end{aligned} \tag{2.59}$$

$$\begin{aligned}|r(\xi_m,\theta_m,v)|&=\Big|\int\int(\xi_{my}\theta_{mx}-\xi_{mx}\theta_{my})vdxdy\Big|\\&\le(\|\xi_{my}\|_{L_4}\|\nabla\theta_m\|+\|\xi_{mx}\|_{L_4}\|\nabla\theta_m\|)\|v\|_{L_4}\\&\le C\|\nabla\xi_m\|^{\frac{1}{2}}\|\Delta\xi_m\|^{\frac{1}{2}}\|\nabla\theta_m\|\|v\|_{H^1}\le C\|v\|_{H^1},\end{aligned} \tag{2.60}$$

$$\begin{aligned}|r(\psi_m,\eta_m,v)|&=\Big|\int\int(\psi_{my}\eta_{mx}-\psi_{mx}\eta_{my})vdxdy\Big|\\&\le(\|\psi_{my}\|_{L_4}\|\nabla\eta_m\|+\|\psi_{mx}\|_{L_4}\|\nabla\eta_m\|)\|v\|_{L_4}\\&\le C\|\nabla\eta_m\|\|v\|_{H^1}.\end{aligned} \tag{2.61}$$

One needs to estimate $\|\nabla\eta_m\|$. In fact, from (2.18), it follows that

$$\frac{1}{p_r}\|\Delta\eta_m\|^2+r(\psi_m,\theta_m,-\Delta\eta_m)=0, \tag{2.62}$$

where

$$\begin{aligned}|r(\psi_m,\theta_m,-\Delta\eta_m)|&=\Big|\int\int(\psi_{my}\theta_{mx}-\psi_{mx}\theta_{my})\Delta\eta_m dxdy\Big|\\&\le(\|\psi_{my}\|_\infty+\|\psi_{mx}\|_\infty)\|\nabla\theta_m\|\|\Delta\eta_m\|\\&\le C\|\nabla\Delta\psi_m\|^{\frac{1}{2}}\|\Delta\eta_m\|\le C\lambda_m^{\frac{1}{4}}\|\Delta\eta_m\|.\end{aligned}$$

Equality (2.62) implies

$$\|\Delta\eta_m\|^2\le Cp_r\lambda_m^{\frac{1}{4}}\|\Delta\eta_m\|,$$

i.e.,

$$\begin{aligned}&\lambda_{m+1}^{\frac{1}{2}}\|\nabla\eta_m\|\le\|\Delta\eta_m\|\le Cp_r\lambda_m^{\frac{1}{4}},\\&\lambda_{m+1}^{\frac{1}{4}}\|\nabla\eta_m\|\le Cp_r\Big(\frac{\lambda_m}{\lambda_{m+1}}\Big)^{\frac{1}{4}}\le\text{const}.\end{aligned} \tag{2.63}$$

Thus from (2.61) we get

$$|r(\psi_m,\eta_m,v)|\le C\|v\|_{H^1}. \tag{2.64}$$

From (2.57),(2.59),(2.60) and (2.64), we obtain

$$\|\frac{d}{dt}\theta_m\|_{L_\infty(R^+,H^{-1}(\Omega))}\le E_4, \tag{2.65}$$

where the constant E_4 is independent of m.

This gives the proof of Lemma 2.4 completely.

§3. The Convergence of Approximate Solutions and Existence of Global Generalized Solution

Definition 3.1. *The unknown functions* $\psi(x,y,t)$, $\theta(x,y,t)$ *is called the global generalized solution of problem* (1.4)-(1.7), *if the following conditions are satisfied:*

(1) $\psi(x,y,t)\in L_\infty(R^+;H^2(\Omega)), \quad \Delta\psi(x,y,t)\in L_2(0,T;H^1(\Omega)),$

$$\Delta\psi_t(x,y,t)\in L_\infty(R^+;H^{-2}(\Omega))\cap L_2(0,T;H^{-1}(\Omega)),$$

$$\begin{aligned}&\theta(x,y,t)\in L_\infty(R^+;H^1(\Omega)),\\&\Delta\theta(x,y,t)\in L_2(0,T;L_2(\Omega)),\\&\theta_t(x,y,t)\in L_\infty(R^+;H^{-1}(\Omega)),\\&\Omega=[0,2D]\times[0,2D].\end{aligned}$$

(2) *The following integral equalities hold in* $L_1(0,T)$ *for all* $T>0$*:*

$$\frac{d}{dt}(\Delta\psi,v)+r(\psi,\Delta\psi,v)+(\Delta\psi,\Delta v)+\frac{R_a}{p_r}(\theta_x,v)=0,\ v\in H^2(\Omega), \tag{3.1}$$

$$\frac{d}{dt}(\theta,v_1)+r(\psi,\theta,v_1)-\frac{1}{p_r}(\nabla\theta,\nabla v_1)=0,\ v_1\in H^1(\Omega), \tag{3.2}$$

where

$$r(u,v,w)=\int\int(u_yv_x-u_xv_y)wdxdy,$$

(3) $\psi(x+2D,y,t)=\psi(x,y,t), \quad \psi(x,y+2D,t)=\psi(x,y,t),$

$$\begin{aligned}&\theta(x+2D,y,t)=\theta(x,y,t),\\&\theta(x,y+2D,t)=\theta(x,y,t),\\&(x,y)\in R^2,\quad t\geq0,\end{aligned} \tag{3.3}$$

(4) $\psi(x,y,0)=\psi_0(x,y),\ \theta(x,y,0)=\theta_0(x,y),$

where $\psi_0(x,y)\in H^2(\Omega)$, $\theta_0(x,y)\in H^1(\Omega)$, *and they are periodic functions for variables* x,y *with period* $2D$.

Theorem 3.1. *Assume that* $\psi_0(x,y)\in H^2(\Omega)$, $\theta_0(x,y)\in H^1(\Omega)$, *and they are periodic functions for variables* x,y. *Then the approximate solution* $\{\psi_m,\theta_m\}$ *of problem* (1.12)-(1.17) *converges to the generalized solution* $\{\psi,\theta\}$ *of problem* (1.4)-(1.7) *as* $m\to\infty$, *where* $\Omega=(0,2D)\times(0,2D)$.

Lemma 3.1. *Assume that the conditions of Theorem* 3.1 *are satisfied. Then the generalized solution* $\theta(x,y,t)\in L_\infty(R^+;L_\infty(\Omega))$ *for the problem* (1.4), (1.5), (1.7) *and* (1.8).

Theorem 3.2. *Under the conditions of Theorem* 3.1, *the generalized solution of the problem* (1.4), (1.5), (1.7) *and* (1.8) *is unique.*

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