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ON EXTREMALITY AND UNIQUE EXTREMALITY OF TEICHMÜLLER MAPPINGS**

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Abstract

Consider the Teichmüller mapping f associated with φ in the unit disc D and the class of all quasiconformal mappings in D with the boundary values of f. For a special holomorphic function φ, the present paper gives the necessary and sufficient condition on φ, such that f is uniquely extremal among the class. Further, for a general holomorphic function φ, the authors suggest the models of the best possible growth conditions on φ, such that f is extremal or uniquely extremal among the class respectively.

Keywords Teichmüller mapping, Extremality, Unique extremality.
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§1. Introduction

Let Ω be a region of the complex z-plane. If f is a quasiconformal mapping of Ω, then we say that f is extremal if f has minimal maximal dilatation among all quasiconformal mappings of Ω in its homotopy class that agree with f on $\partial\Omega$. We say that f is uniquely extremal if it is the only such mapping. The mapping f is called a Teichmüller mapping if its complex dilatation has the form

$$\frac{f_{\bar z}}{f_z} = k\frac{\overline{\varphi(z)}}{|\varphi(z)|},$$

$\varphi(z)$ holomorphic in Ω, with $k, 0 \le k < 1$, a constant.

Let $B(\Omega)$ denote the class of functions $\varphi(z)$, holomorphic in Ω, with

$$\|\varphi\| = \iint_\Omega |\varphi(z)|dxdy < \infty.$$

It is a well known fact that a Teichmüller mapping, with $\varphi \in B(\Omega)$, is uniquely extremal. There are some examples which show that if f is a Teichmller mapping, but $\varphi \notin B(\Omega)$, then, depending on the choice of φ, any of the three possibilities, non-extremality, extremality with or without unique extremality, can occur.

In what follows it will be assumed that $\Omega = D = \{|z| < 1\}$.

In 1968 G. C. Sethares[1] proved

Theorem A. *Let $z_1, \cdots, z_m$ be points of ∂D such that excising an arbitrary neighborhood D_i of each z_i from D results in a region of finite φ-area. Let $\alpha_1, \cdots, \alpha_m$ be non-zero complex*

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numbers and let $t_1, \cdots, t_m$ *be real numbers such that* $-1 < t_i \leq 0$ *for* $i = 1, \cdots, m$. *Then* f *is uniquely extremal if* φ *also satisfies, for each* $i = 1, \cdots, m$, *the growth condition*

$$\Big|\frac{\sqrt{\varphi(z)}}{\log^{t_i}(z_i - z)} - \frac{\alpha_i}{z_i - z}\Big| = O(1), \quad z \to z_i. \tag{1.1}$$

In this case it is easy to see that φ satisfies the growth condition

$$m(\varphi, r) = \frac{1}{2\pi}\int_0^{2\pi} |\varphi(re^{i\theta})| d\theta = O\Big(\frac{1}{1-r}\Big), \quad r \to 1. \tag{1.2}$$

In 1974 E. Reich and K. Strebel[2] proved that (1.2) implied that f was extremal, and that the extremality of f is no longer implied if $O(1/(1-r))$ in the right-hand side of (1.2) is replaced by $O(1/(1-r)^s)$, for any $s > 1$. In 1982 W. K. Hayman and Reich[3] showed that the growth condition (1.2) also implied that f was uniquely extremal. In 1987 Li wei and Lü Yong[4] gave an example: "If $\varphi = \log^2(1-z)/(1-z)^2$, then f is uniquely extremal". This example shows that (1.2) is not best for the growth conditions of unique extremality.

In this paper the growth condition given by Theorem A has been sharpened and we obtain

Theorem 1.1. *Suppose that* φ *satisfies all conditions in Theorem A except the condition for* t_i, *then* f *is uniquely extremal if and only if* $t_i \leq 1/2$, $i = 1, \cdots, m$.

First, Theorem 1.1 contradicts the example given by Li Wei and Lü Yong, so we have to point out a mistake in their proof.

Secondly, φ in Theorem 1.1 satisfies the growth condition

$$m(\varphi, r) = O\Big(\frac{\log(1-r)^{-1}}{1-r}\Big), \quad r \to 1.$$

In view of this it seems that the best possible growth condition for unique extremality may be

$$m(\varphi, r) = o\Big(\frac{\log(1-r)^{-s}}{1-r}\Big), \quad r \to 1$$

for any $s > 1$, and that the best possible growth condition for extremality may be

$$m(\varphi, r) = o\Big(\frac{1}{(1-r)^s}\Big), \quad r \to 1$$

for any $s > 1$.

§2. Some Preliminary Results

Let φ be holomorphic in D, and $\{S_n\}$ denotes a sequence of open subsets of D such that, for every n, the boundary ∂S_n is a union of countably many smooth arcs. For every n define Γ_n to be that part of $D \cap \partial S_n$ remaining after all horizontal arcs are deleted. Write

$$|S_n|_\varphi = \iint_{S_n} |\varphi| dxdy, \quad |\Gamma_n|_\varphi = \int_{\Gamma_n} \sqrt{|\varphi|}|dz|.$$

We further assume that S_n satisfies the following three conditions:

(i) $|S_n|_\varphi < \infty, \quad n = 1, 2, \cdots,$

(ii) $|S_n|_\varphi \to \infty, \quad n \to \infty,$

(iii) $l_n = |\Gamma_n|_\varphi < \infty, \quad n = 1, 2, \cdots.$

In [1] Sethares obtained the following

Theorem B. *Let* f *be a* K*-quasiconformal self mapping of* D *with the complex dilatation*

$$\kappa_f = k\bar{\varphi}/|\varphi|, \; K = (1+k)/(1-k).$$

Let h be a $\widetilde{K}$-quasiconformal self mapping of D with $h|_{\partial D} = f|_{\partial D}$. For every n let

$$T_n = h(S_n)\backslash f(S_n), \quad |T_n|_\psi = \iint_{T_n} |\psi| dudv.$$

If both $l_n B_n$ and $|T_n|_\psi$ are $o(1)$ as $n \to \infty$ and if in addition $K = \widetilde{K}$ and $S_n \to D$, then $f \equiv h$, where ψ is holomorphic in $\widetilde{D} = \{|w| < 1\}$ which is determined by φ (see [1, p.101]) and B_n are constants such that

$$d_\psi(f(z), h(z)) = \inf_\gamma \Big\{ \int_\gamma \sqrt{|\psi(w)|}|dw| \Big\} \le B_n, \quad z \in \Gamma_n,$$

with every curve γ in $\widetilde{D}$ which joins the points $f(z)$ and $h(z)$.

Let R be a region in the z-plane whose boundary is a union of countable many arcs. For $F(z) = Kx + iy$ and every y set $S = F(R)$ and $R_y = \{z \in R | \mathrm{Im} z > y\}$ respectively. Next, let H be a $\widetilde{K}$-quasiconformal mapping of R_{y_0} into S that agrees with F on $\partial R_{y_0} \cap \partial R$ and also satisfies $H(R_{y_1}) \subset F(R_{y_0})$ for some $y_1 \ge y_0$. Finally, for every $y \ge y_0$, let

$$\gamma_y = \{z \in R | \mathrm{Im} z = y\}.$$

We define $M(y)$ and $\delta(y)$ as follows:

$$M(y) = \sup_{y_0 \le y' \le y} |\gamma_{y'}|, \quad \delta(y) = \sup_{z \in \gamma_y} |\mathrm{Im} H(z) - y|.$$

Lemma 2.1.[1] *$\delta(y_0) \le \sqrt{K\widetilde{K}} M(y + \delta(y))$ for every $y \ge y_1$.*

Let

$$\widetilde{R}_y = R_{y_1} \backslash R_y, \quad T_y = \{w \in H(\widetilde{R}_y) | \mathrm{Im} w > y\}.$$

Let $|\gamma_y|$ and $|T_y|$ denote the Euclidean length of γ_y and area of T_y, respectively. We have

Lemma 2.2. *Let $\Delta(y) = |H(\gamma_y)| - |F(\gamma_y)|$, $0 < s \le 1/3$. If there exist positive contants C_1 and C_2 such that $M(y) < C_1 y^s$ and*

$$\int_{y_1}^{y} \Delta(\eta) d\eta \le C_2 \delta(y) y^s$$

for all sufficiently large y, and in addition $K = \widetilde{K}$, then $|T_y| = 0$ for every $y > y_1$.

Proof. Set $y^* = y + \delta(y)$. It follows from Lemma 2.1 and the assumption that there exists $y_2 > y_1$ such that

$$\delta(y) = y^* - y \le KM(y^*) \le C_1 K y^{*s},$$

for every $y > y_2$. Again $0 < s \le 1/3$ implies $y^* \le 2y$ for sufficiently large y. We conclude that

$$M(y^*) = M(y + \delta(y)) \le 2^s C_1 K y^s, \ \delta(y) \le 2^s C_1 K y^s$$

and $\lim\limits_{y \to \infty} \delta(y)/y = 0$.

On the other hand, there exists $y_3 > y_2$ such that

$$\int_{y_1}^{y} \Delta(\eta) d\eta \le C_2 \delta(y) y^s, \quad y > y_3. \tag{2.1}$$

Then

$$\lim_{y \to \infty} \Big(\int_{y_1}^{y} \Delta(\eta) d\eta \Big) / y^{1+s} = 0.$$

Hence

$$\liminf_{y\to\infty}\frac{\Delta(y)}{y^s}=0. \tag{2.2}$$

Since $|\gamma_y|\le M(y)\le C_1y^s$ for $y>y_3$, we have for any fixed $\delta(y)$

$$\begin{aligned}\delta(y)^2&\le\frac{|H(\gamma_y)|^2-|F(\gamma_y)|^2}{4}\\&=\frac{K|\gamma_y|\Delta(y)}{2}+\frac{\Delta(y)^2}{4}\\&\le\frac{C_1Ky^s\Delta(y)}{2}+\frac{\Delta(y)^2}{4}.\end{aligned}$$

Thus inequality (2.1) becomes

$$\int_{y_1}^{y}\Delta(\eta)d\eta\le C_3(y^{3s}\Delta(y))^{1/2}\Big(1+\frac{\Delta(y)}{2C_1Ky^s}\Big)^{1/2},\quad y>y_3$$

with $C_3=C_2(C_1K/2)^{1/2}$. Writing $u(y)=\int_{y_1}^{y}\Delta(\eta)d\eta$, we have

$$u(y)\le C_3(y^{3s}u'(y))^{1/2}\Big(1+\frac{u'(y)}{2C_1Ky^s}\Big)^{1/2},\quad y>y_3. \tag{2.3}$$

By Comparing with the proof in [5], (2.2) and (2.3) correspond to (2.15) and (2.18) in [5], respectively. This shows that the conclusion of [5] still holds, namely, $\delta(y)=0$ for $y>y_1$. But

$$|T_y|\le\delta(y)M(y+\delta(y)),$$

this proves $|T_y|=0$ for $y>y_1$.

§3. Proof of Theorem 1.1

Proof of Sufficiency. We may assume $t_i>-1,\ i=1,2,\cdots,m_0$ and $t_i\le-1, i=m_0+1,\ m_0+2,\cdots,m$. If $m_0=0$, then $\varphi\in B(D)$, and we have known that f is uniquely extremal. If $m_0\ne0$, we need only to prove the sufficiency for the case of $z_i,D_i,\alpha_i,-1<t_i\le1/2, i=1,2,\cdots,m_0$. So we can suppose $t_i>-1, i=1,2,\cdots,m$. Hence we know by Remark in [1, p. 117] that f is extremal. In the following we prove that f is uniquely extremal.

We can take S_n in §2 as $\bigcap_{i=1}^{m}S_{i,n}$, where $S_{i,n}=D\backslash D_{i,n}$, and $\{D_{i,n}\}$ are neighborhoods of z_i with

$$D_i\supset D_{i,1}\supset D_{i,2}\supset\cdots\to z_i.$$

Let $\Gamma_{i,n}$ denote that part of $D\cap\partial S_{i,n}$ remaining after all horizontal arcs are deleted, $T_{i,n}=h(S_{i,n})\backslash f(S_{i,n})$. Then

$$S_n=\bigcup_{i=1}^{m}S_{i,n},\quad \Gamma_n=\bigcup_{i=1}^{m}\Gamma_{i,n},\quad T_n=\bigcap_{i=1}^{m}T_{i,n}.$$

The meaning of the symbols $|S_{i,n}|_\varphi,\ l_{i,n}=|\Gamma_{i,n}|_\varphi$ and $|T_{i,n}|_\psi$ are obvious. We have

1) $l_n\le\sum_{i=1}^{m}l_{i,n}$,

2) $|T_n|_\psi\le\sum_{i=1}^{m}|T_{i,n}|_\psi$.

By Theorem B we remain to prove $l_{i,n}\equiv0$ and $|T_{i,n}|_\psi=o(1),\ n\to\infty,\ i=1,2,\cdots,m$. And it is easy to see that for every fixed $i, l_{i,n}\equiv0$ and $|T_{i,n}|_\psi=o(1),\ n\to\infty$, are not different from $l_n \equiv 0$ and $|T_n|_\psi = o(1),\ n \to \infty$, for the case of $m = 1$. So we need only to discuss the typical case $m = 1$ and $z_1 = 1$. By (1.1) and Lemma 4 in [1] there exists a $\rho > 0$ such that a single valued and schlicht branch $\Phi(z)$ of $\int^z \sqrt{\varphi(z)}dz$ can be chosen in

$$N_\rho = \{z||z-1| < \rho\} \cap D.$$

Moreover, we may write

$$\bar{z} = \Phi(z) = \int^z \sqrt{\varphi(z)}dz = -\frac{\alpha_1}{1+t_1}(\log(1-z))^{1+t_1} + \eta(z),\ z \in N_\rho,$$

where

$$\eta(z) = \int^z \big(\sqrt{\varphi(z)}dz - \frac{\alpha_1(\log(1-z))^{t_1}}{1-z}\big)dz = O(1),\ z \to 1,$$

i.e.,

$$\tilde{z} = \Phi(z) = -\frac{\alpha_1}{1-t_1}(\log(1-z))^{1+t_1} + O(1),\ z \to 1. \tag{3.1}$$

Let $R = \Phi(N_\rho)$. We first consider the boundary behavour of R. Set $\zeta = \zeta(z) = \xi + i\eta = \log(1-z)$. Then $\zeta(z)$ maps N_ρ conformally into a region R' bounded by curves

$$\gamma_1 = \Big\{\xi + i\eta|\xi = \log 2\sin\frac{\theta}{2},\ \eta = -\frac{\pi}{2} + \frac{\theta}{2},\ 0 < \theta < \theta_0\Big\},$$
$$\gamma_2 = \Big\{\xi + i\eta|\xi = \log 2\sin\frac{\theta}{2},\ \eta = -\frac{\pi}{2} - \frac{\theta}{2},\ 0 < \theta < \theta_0\Big\},$$
$$\gamma_3 = \Big\{\xi + i\eta|\xi = \log 2\sin\frac{\theta_0}{2},\ \ -\frac{\pi}{2} + \frac{\theta_0}{2} < \eta < \frac{\pi}{2} - \frac{\theta_0}{2}\Big\},$$

where $2\theta_0$ is equal to the length of arc $\{|z| = 1\} \cap \overline{N_\rho}$.

Let $w = w(\zeta) = -\alpha_1\zeta^{1+t_1}/(1+t_1)$. Write

$$-\alpha_1/(1+t_1) = r_1 e^{i\theta_1},\ \ 1 + t_1 = l,\ \ w = u + iv = re^{i\tilde{\alpha}}\ \text{ and }\ \zeta = \rho e^{i\alpha}.$$

Then

$$r = r_1\rho^l,\ \ \tilde{\alpha} = l\alpha + \theta_1. \tag{3.2}$$

If $\zeta \in \gamma_1$, then

$$\rho = \Big(\Big(\log 2\sin\frac{\theta}{2}\Big)^2 + \Big(-\frac{\pi}{2} + \frac{\theta}{2}\Big)^2\Big)^{1/2},$$
$$\alpha = \pi + \text{arctg}\frac{-\frac{\pi}{2} + \frac{\theta}{2}}{\log 2\sin\frac{\theta}{2}} \equiv \pi + \tilde{\theta}.$$

Thus

$$\rho = -\log\theta + o(1),$$
$$\tilde{\theta} = \text{arctg}\frac{-\frac{\pi}{2} + \frac{\theta}{2}}{\log 2\sin\frac{\theta}{2}}$$
$$= -\frac{\pi}{2\log\theta} + o\Big(-\frac{1}{\log\theta}\Big),$$
$$\sin l\tilde{\theta} = -\frac{l\pi}{2\log\theta} + o\Big(-\frac{1}{\log\theta}\Big),$$
$$\cos l\tilde{\theta} = 1 + o\Big(-\frac{1}{\log\theta}\Big),\ \ \theta \to 0. \tag{3.3}$$

For simplicity, we may assume (see Remark 5 in [1, p. 117])

$$\theta_1 = \frac{\pi}{2} - l\pi. \tag{3.4}$$

From (3.2) and (3.4) we get

$$u = r\cos\tilde{\alpha} = -r_1\rho^l \sin l\tilde{\theta}, \quad v = r\sin\bar{\alpha} = r\rho^l \cos l\tilde{\theta}. \tag{3.5}$$

If we set $-\log\theta = \tau$, it follows from (3.3) and (3.5) that $w(\gamma_1)$ satisfies

$$u = -\frac{\pi l r_1}{2}\tau^{l-1} + o(\tau^{l-1}), \quad v = r_1\tau^l + o(\tau^l), \quad \tau \to \infty.$$

Similar reason shows that $w(\gamma_2)$ satisfies

$$u = -\frac{\pi l r_1}{2}\tau^{l-1} + o(\tau^{l-1}), \quad v = r_1\tau^l + o(\tau^l), \quad \tau \to \infty.$$

Hence, the boundary curves of $R'' = w(R')$ satisfy

$$u = \pm\frac{\pi l r_1}{2}\tau^{l-1} + o(\tau^{l-1}), \quad v = r_1\tau^l + o(\tau^l), \quad \tau \to \infty. \tag{3.6}$$

By (3.1) and (3.6), the boundary curves of R are determined by

$$\tilde{x} = \pm\frac{\pi l r_1}{2}\tau^{l-1} + o(\tau^{l-1}), \quad \tilde{y} = r_1\tau^l + o(\tau^l), \quad \tau \to \infty.$$

Writing

$$C = \frac{r_1}{(\frac{\pi l r_1}{2})^{l/(l-1)}}, \quad \frac{1}{s} = \frac{l}{l-1},$$

we get

$$\tilde{y} = \pm c\tilde{x}^{1/s} + o(\tilde{x}^{1/s}), \quad \tilde{x} \to \infty. \tag{3.7}$$

Next, choose an increasing sequence $\{\tilde{y}_n\}$ such that $\tilde{y}_n \to \infty, n \to \infty$. Then $\gamma_{z_n} = \Phi^{-1}(\gamma_{\tilde{y}_n})$ is a sequence of horizontal arcs in D. We set $S_n = D\backslash\Phi^{-1}(R_{\tilde{y}_n})$. Thus $l_n \equiv 0$. It is therefore sufficient to prove that $|T_n|_\psi = o(1),\ n \to \infty$. Choose $0 < \rho' < \rho$ such that $N_{\rho'}$ and $h(N_{\rho'})$ are free of zeros of φ and ψ respectively. Then h can be lifted to a K-quasiconformal mapping of $\Phi(N_{\rho'})$ into $S = F(R)$. Call this mapping H. We may choose sufficiently large $\tilde{y}_0$ and $\tilde{y}_1$ such that the conditions in §2 are fulfilled. Noticing that $|T_n|_\psi = |T_{\tilde{y}_n}|$, we need only to prove $|T_{\tilde{y}_n}| = o(1), \quad n \to \infty$.

If $0 < t_1 \le 1/2$, then $0 < s \le 1/3$. Following (3.6) we see that there exists n_0 and positive constants C_0, C_1 such that

$$|\gamma_{\tilde{y}}| = C_0\tilde{y}^s + o(\tilde{y}^s), \quad 0 < s \le \frac{1}{3},\ \tilde{y} \to \infty,$$

$$M(\tilde{y}) \le C_1\tilde{y}^s, \quad 0 < s \le \frac{1}{3}, \quad \tilde{y} \ge \tilde{y}_{n_0}. \tag{3.8}$$

Noting $l_n = 0$ and the proof of Theorem 1 in [1], we have

$$\Big(\int_{\beta(n)} |\gamma_{w'}|_\psi d\sigma\Big)^2 \le K^2|S_n|_\varphi^2 + K|S_n|_\varphi|T_n|_\psi, \tag{3.9}$$

where $\beta(n)$ is a union of vertical arcs, $d\sigma$ denotes the differential of arc length $\sqrt{|\varphi(z)|}|dz|$ along $\beta(n), \gamma_{w'} = h(\gamma_z)$ and γ_z are horizontal arcs (see [1, p. 103]). If we set

$$|\gamma_{w'}|_\psi = |h(\gamma_z)|_\psi = K|\gamma_z|_\varphi + \Delta(z),$$

(3.9) becomes

$$K|S_n|_\varphi + \int_{\beta(n)} \Delta(z)d\sigma \le K|S_n|_\varphi\Big(1 + \frac{|T_n|_\psi}{K|S_n|_\varphi}\Big)^{1/2}.$$

Because we always have

$$\Big(1 + \frac{|T_n|_\psi}{K|S_n|_\varphi}\Big)^{1/2} \le 1 + \frac{|T_n|_\psi}{K|S_n|_\varphi},$$

we obtain

$$\int_{\beta(n)} \Delta(z)d\sigma \le |T_n|_\psi.$$

If we set $\tilde\beta_n = \beta(n) \cap \Phi^{-1}(\tilde R_{y_1})$, then

$$\int_{\tilde\beta} \Delta(z)d\sigma \le |T_n|_\psi. \tag{3.10}$$

On the one hand, $z \in \Phi^{-1}(R_{\tilde y_1})$ implies $\Phi(\gamma_z) = \gamma_{\tilde y}$, and we have

$$\begin{aligned}\Delta(z) &= |h(\gamma_z)|_\psi - K|\gamma_z|_\varphi\\ &= |H(\gamma_{\tilde y})| - |F(\gamma_{\tilde y})| \equiv \Delta(\tilde y).\end{aligned}$$

On the other hand, making use of (3.8) and a conclusion in the proof of Lemma 2.2, we se that there exist n_1 and a positive constant C_2 such that

$$\begin{aligned}|T_n|_\psi = |T_{\tilde y_n}| &\le \delta(\tilde y_n)M(\tilde y_n + \delta(\tilde y_n))\\ &\le C_2\delta(\tilde y_n)\tilde y_n^s, \quad \tilde y_n > \tilde y_{n_1}.\end{aligned}$$

Hence (3.10) becomes

$$\int_{\tilde y_1}^{\tilde y_n} \Delta(\eta)d\eta \le |T_{\tilde y_n}| \le C_2\delta(\tilde y_n)\tilde y_n^s, \quad \tilde y_n > y_{n_1}. \tag{3.11}$$

According to the above proof, (3,11) is even true for every $\tilde y > \tilde y_{n_1}$, that is

$$\int_{\tilde y_1}^{\tilde y} \Delta(\zeta)d\zeta \le C_2\delta(\tilde y)\tilde y^s, \quad \tilde y > y_{n_1}. \tag{3.12}$$

In view of Lemma 2.2, (3.8) and (3.12) we know $|T_{\tilde y_n}| = o(1), \quad n \to \infty$.

If $-1 < t_1 \le 0$, by the proof of Theorem A we know also $|T_{\tilde y_n}| = o(1), \quad n \to \infty$.

Proof of Necessity. Assume, as we may, that $t_1 > 1/2, z_1 = 1$. From the proof of sufficiency the boundary curves of $R = \Phi(N_p)$ are still determined by (3.7), but $1 < 1/s < 3$. In view of the conclusions of §4, §5, §6 in [5], it is clear that f is not uniquely extremal. We have thus achieved the desired contradiction.

§4. An Example

Corollary 4.1. *If* $\varphi = \log^s(1-z)/(1-z)^2$, *then* f *is uniquely extremal if and only if* $s \le 1$.

This corollary contradicts the example given by Li Wei and Lü Yong in §1. We now show a mistake in their proof. Their proof (see [4, p.49]) says

$$\begin{aligned}“|y_1' - y_2'| &= |-\xi_1 a - (-\xi_2 b)| = |\xi_2 b - \xi_1 a|\\ &\le a|\xi_2 - \xi_1| \le a\Big|\frac{a^2 - b^2}{\xi_1 + \xi_2}\Big|,\end{aligned}$$

where $a > b > 0$, $\xi_1^2 - \xi_2^2 = a^2 - b^2$. Hence $|y_1' - y_2'| \to 0$ as $\xi_1, \xi_2 \to -\infty$."

The above assertion is not true. Indeed, for sufficiently large $-\xi_2$, the condition $\xi_1^2 - \xi_2^2 = a^2 - b^2$ gives $\xi_1 = -(a^2 - b^2 + \xi_2^2)^{1/2}$. Then

$$|y_1' - y_2'| = |\xi_2 b - \xi_1 a| = |b\xi_2 + a(a^2 - b^2 + \xi_2^2)^{1/2}|$$
$$= \Big|\frac{(a^2 - b^2)(a^2 + \xi_2^2)}{a(a^2 - b^2 + \xi_2^2)^{1/2} - b\xi_2}\Big|.$$

Hence $|y_1' - y_2'| \to +\infty$ as $\xi_2 \to -\infty$.

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