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A NEW REGULARITY CLASS FOR THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^n$

H. Beirão da Veiga*

Abstract

Consider the Navier-Stokes equations in $\mathbb{R}^n\times(0,T)$, for $n\geq 3$. Let $1<\alpha\leq\min\{2,n/(n-2)\}$ and define β by $(2/\alpha)+(n/\beta)=2$. Set $\alpha'=\alpha/(\alpha-1)$. It is proved that Dv belongs to $C(0,T;L^{\alpha'})\cap L^{\alpha'}(0,T;L^{2\beta/(n-2)})$ whenever $Dv\in L^\alpha(0,T;L^\beta)$. In particular, v is a regular solution. This results is the natural extension to $\alpha\in(1,2]$ of the classical sufficient condition that establishes that $L^\alpha(0,T;L^\gamma)$ is a regularity class if $(2/\alpha)+(n/\gamma)=1$. Even the borderline case $\alpha=2$ is significant. In fact, this result states that $L^2(0,T;W^{1,n})$ is a regularity class if $n\leq 4$. Since $W^{1,n}\hookrightarrow L^\infty$ is false, this result does not follow from the classical one that states that $L^2(0,T;L^\infty)$ is a regularity class.

Keywords Navies-Stokes equation, Regularity of solution, Extension.
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§1. Introduction

In this paper we shall consider the initial value problem for the Navier-Stokes equations in $\mathbb{R}^n\times(0,T),\ n\geq 3$,

$$\begin{cases}\partial_t v+(v\cdot\nabla)v-\Delta v=\nabla\pi,\\ \mathrm{div} v=0,\\ v(x,0)=v_0(x).\end{cases}\tag{1.1}$$

We assume, for simplicity, that the external forces vanish, although it is an easy exercise to include non-zero external forces. We are interested in the classical problem of finding, in the framework of Sobolev spaces, sufficient conditions for the existence of a regular (unique) solution.

If $\gamma\in[1,+\infty]$, we denote the space $L^\gamma(\mathbb{R}^n)$ simply by $L^\gamma$and the canonical norm in this space by $\|\cdot\|_\gamma$. We use the same symbol to denote functional spaces consisting of scalar functions or consisting of vector functions. For instance, we denote the space $L^\gamma\times\cdots\times L^\gamma$ (n times) simply by L^γ. This convention also applies to other symbols as, for instance, norms.

Many authors proved that uniqueness and regularity for solutions of the Navier-Stokes equations hold under the assumption that v belongs to $L^\alpha(0,T;L^\gamma)$ where

$$\frac{2}{\alpha}+\frac{n}{\gamma}=1,\tag{1.2}$$

$\gamma>n$. See, for instance, the classical references [11, 13] (for $n=2$, [10, 7, 12]); see also [7, 9] and the more recent developments in [3, 4, 6, 14, 16, 15]. More precisely, under the

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*Department of Mathematics, Pisa University, Pisa, Italy.

above assumption (1.2), the uniqueness of the solution was proved by Prodi in reference [11] for $n = 3$ and by Sather and Serrin (see [13]) for $n \geq 3$. In [13] regularity is also shown if $n \leq 4$ and $(2/\alpha) + (n/\gamma) < 1$. Sohr[14] succeeded in proving that the above class (1.2) is even a regularity class. This last result was also proved (independently) by Giga[4]. For $n = 3$, a simplified version of the proof is given in [17]. It is also known that $C(0, T; L^n)$ is a regularity class (see [16]) and that $L^\infty(0, T; L^n)$ is a uniqueness class (see [14]). We are interested in obtaining results in this same spirit.

Let $1 < \alpha \leq \min\{2, n/(n-2)\}$ and define β by

$$\frac{2}{\alpha} + \frac{n}{\beta} = 2. \tag{1.3}$$

We prove that if

$$Dv \in L^\alpha(0, T; L^\beta), \tag{1.4}$$

then $Dv \in C(0, T; L^{\alpha'}) \cap L^{\alpha'}(0, T; L^{2\beta/(n-2)})$. In particular v is a regular solution. Moreover, the sharp estimate (2.6) holds. See Theorem 2.2 below, where $\alpha = p'$ and $\beta = pn/2$ (the assumption $p \geq \max\{2, n/2\}$ is equivalent to the above assumption on α).

Let us show that our result is the natural extension of the above classical result to values $\alpha \leq 2$. For convenience let us denote by $W^{1,\beta}$ the completion of $C_0^\infty(I\!\!R^n)$ with respect to the norm $\|Dv\|_\beta$. Note that, in the classical condition, $\alpha \geq 2$ and $\gamma \leq n$. In our condition, $\alpha \leq 2$ and $\beta \geq n$. Nevertheless, in order to compare with the classical result, let us overlap both situations by assuming $\alpha \geq 2$ in our theorem (in fact our theorem holds also for $\alpha > 2$). Since $\beta < n$, the Sobolev embedding theorem $W^{1,\beta} \hookrightarrow L^{\beta^*}$ holds, where $\beta^* = n\beta/(n-\beta)$. Consequently, our assumption (1.4) yields (exactly) $v \in L^\alpha(0, T; L^{\beta'})$. But this is just the classical assumption, since the pair (α, β^*) satisfies (1.2). This argument shows that our result is just the natural extension of the classical one to values $\alpha \leq 2$. In this last case, less regularity in time is balanced by additional regularity in space. In the classical situation the regularity assumption in space, L^γ, reaches its maximum $\gamma = \infty$ for $\alpha = 2$. Hence, if $\alpha \leq 2$, one has to go beyond L^∞. In our Sobolev spaces framework, this means starting to use $W^{1,\beta}$ spaces. For $\alpha = 2$ (common to both conditions) our condition (1.3) gives $\beta = n$. This borderline case is particularly interesting. Our result shows that (if $n \leq 4$) $L^2(0, T; W^{1,n})$ is a regularity class. This does not follow from the classical result, that states that $L^2(0, T; L^\infty)$ is a regularity class, since $W^{1,n} \hookrightarrow L^\infty$ is false (if $n \geq 2$).

Next, consider the case $a \in (1, 2)$. Now the value of the classical index $2/\alpha + n/\gamma$, applied to our regularity class $L^\alpha(0, T; W^{1,\beta})$, is $2/\alpha$ (since $\gamma = \infty$). Since $2/\alpha$ is larger than 1, the classical theorem does not apply. On the other hand, our result shows that, in this new situation, the significant index is $(2/\alpha) + (n/\beta^*)$, which is equal to one if the assumption (1.3) holds. Here $\beta^* = n\beta/(n-\beta)$, independently of the fact that the Sobolev's embedding theorem $W^{1,\beta} \hookrightarrow L^{\beta^*}$ is true or false (we could also consider fractionary Sobolev spaces).

Curious enough, for $\alpha = 1$ one gets $L^1(0, T; W^{1,\infty})$, which is a regularity class for the Euler equations. In fact, it is the sole (among the above classes (1.4)) to be a regularity class for the Euler equations (according to what is known at present). In this regard, note that in equation (1.4) one can replace Dv by curl v.

§2. Proofs

Let us introduce some notation. We set $\partial_i = \partial/\partial x_i,\ i = 1, 2, \cdots, n$, and $\partial_t = \partial/\partial t$. The symbol ∂ denotes indifferently ∂x_i, for any i, or ∂_t. Moreover Dv denotes the tensor $\partial_i v_j,\ i, j = 1, \cdots, n$, and

$$|Dv(x)|^2 = \sum_{i,j=1}^{n} |\partial_i v_j(x)|^2,$$

where $v = (v_1, \cdots, v_n)$ is a vector field over $I\!R^n$. We define

$$\|D^k v\|_r = \Big(\sum_{|\alpha|=k} \sum_{i=1}^{n} \|\partial^k v_j/\partial x^\alpha\|_r^r \Big)^{1/r},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index. A similar definition holds for scalar fields. We denote by $C(0, T; X)$ the Banach space of bounded continuous functions on $[0, T]$ with values in a Banach space X. Finally, if $p \in (1, \infty)$, we denote by p' its dual exponent $p' = p/(p-1)$ and, if $p \in [1, n)$, by p^* the Sobolev embedding exponent $p^* = pn/(n-p)$.

In the sequel we prove the following a priori estimate.

Theorem 2.1. *Let $p \in [2, \infty)$. Assume that v is a regular solution of problem (1.1) in some interval $[0, T)$. Then, if*

$$Dv \in L^{p'}(0, T; L^{pn/2}), \tag{2.1}$$

one has

$$Dv \in C(0, T; L^p) \cap L^p(0, T; L^{pn/(n-2)}). \tag{2.2}$$

Moreover,

$$\sup_{0 \le t < T} \|Dv(t)\|_p^p + \int_0^T \|Dv(\tau)\|^p_{\frac{pn}{n-2}}\, d\tau$$
$$\le c\|Dv(0)\|_p^p \Big[1 + \exp\Big(c \int_0^T \|Dv(\tau)\|^{p'}_{\frac{np}{2}}\, d\tau \Big) \Big]. \tag{2.3}$$

Here, and in the sequel, we denote by c (or by $c_0, c_1, \cdots$) positive constants that depend, at most, on n and p. The symbol c may be used, even in the same equation, to denote distinct constants.

Remark. It is already known that (2.1) is a regularity class if $p \in (1, 2)$, since in this case one has $2/p' + n/(pn/2) = 1$. For that reason, we assume here that $p \ge 2$.

In order to avoid argumentations of secondary importance in our context, we shall state the following application of the above a priori estimate in the framework of the classical Leray-Hopf solutions [8, 5] (defined as in [3], section 5).

Theorem 2.2. *Suppose $v_0 \in L^2$ and is divergence free. Assume, moreover, that $Dv_0 \in L^p$ for some $p \ge \max\{2, n/2\}$. Suppose v is a Leray-Hopf solution of problem (1.1) in $[0, T)$. If*

$$Dv \in L^{p'}(0, T; L^{pn/2}), \tag{2.4}$$

Then

$$Dv \in C(0, T; L^p) \cap L^p(0, T; L^{pn/(n-2)}), \tag{2.5}$$

Moreover,

$$\sup_{0\le t<T}\|Dv(t)\|_p^p+\int_0^T\|Dv(t)\|^p_{\frac{pn}{n-2}}dt$$
$$\le c\|Dv(0)\|_p^p\Big[1+\exp\Big(c\int_0^T\|Dv(\tau)\|^{p'}_{\frac{np}{2}}d\tau\Big)\Big]. \tag{2.6}$$

In particular v is a regular (unique) solution in $[0,T]$.

Proof of Theorem 2.2. Since $v_0\in L^2$ and $Dv_0\in L^p$ with $p\ge n/2$ it follows (by Sobolev embedding theorems) that $v_0\in L^q$ for some $q\ge n$. Hence, the solution v is regular and unique (for instance, in the Hopf-Leray class) on $[0,T_1]$, for some $T_1>0$. See [3, 6, 16, 14, 4]. By the a priori estimate in Theorem 2.1, together with the assumption (2.4), it follows that (2.6) holds in $[0,T_1]$ (together with the energy inequality, etc.). This argument shows that as long as (2.4) holds (i.e., until the time T) the regular solution v satisfies (2.6), and can be extended by a continuation argument.

Let us show, in a more direct way, that (2.5) is a regularity class. If $p>n/2$ it follows that $v\in L^\infty(0,T;L^q)$ for some $q>n$, since $Dv\in L^\infty(0,T;L^p)$. Since $2/\infty+n/q<1$, the result follows. If $p=n/2$ (hence $n\ge 4$) and if, moreover, $n>4$, then $pn/(n-2)<n$. By a Sobolev's embedding theorem $v\in L^p(0,T;L^q)$, where $q=[pn/(n-2)]^*$. Since $2/p+n/q=1$, the result follows. Finally, if $p=n/2$ and if $n=4$, one has $Dv\in L^\infty(0,T;L^2)\cap L^2(0,T;L^4)$. Consider any θ-interpolation space, $\theta\in(0,1)$, between $L^\infty(0,T;L^2)$ and $L^2(0,T;L^4)$. Choose, for instance, $\theta=1/3$. Then

$$\|Dv\|_3\le\|Dv\|_2^{1/3}\|Dv\|_4^{2/3}.$$

Hence $Dv\in L^3(0,T;L^3)$. In particular $v\in L^3(0,T;L^{12})$, which is a regularity class since $2/3+4/12=1$. Note that we use the classical regularity result under the simplified condition $2/\alpha+n/\gamma<1$ (except when $n=4$ and $p=2$).

Proof of Theorem 2.1. The following identities will be usefull in the sequel.

$$\partial(|f|^{p-2}f)=(p-1)|f|^{p-2}\partial f, \tag{2.7}$$

$$\nabla f\cdot\nabla(|f|^{p-2}f)=(p-1)|f|^{p-2}|\nabla f|^2, \tag{2.8}$$

$$\nabla(|f|^{\frac p2-1}f)=\frac p2|f|^{\frac p2-1}\nabla f. \tag{2.9}$$

From (2.8) and (2.9) one gets

$$\nabla f\cdot\nabla(|f|^{p-2}f)=\frac{4(p-1)}{p^2}|\nabla(|f|^{\frac p2-1}f)|^2. \tag{2.10}$$

Apply ∂_k to both sides of equation $(1.1)_1$, multiply by $|\partial_k v_j|^{p-2}\partial_k v_j$ and integrate over $\mathbb{R}^n$. By taking into account that v is divergence free and by doing suitable integrations by parts one easily gets

$$\frac1p\frac{d}{dt}\|\partial_k v_j\|_p^p+\int\nabla(\partial_k v_j)\cdot\nabla(|\partial_k v_j|^{p-2}\partial_k v_j)dx$$
$$\le c\int|\nabla\partial_k\pi||Dv|^{p-1}dx+c\int|Dv|^{p+1}dx, \tag{2.11}$$

where integrals are over $I\!R^n$. By using (2.10) we show that

$$\frac{1}{p}\frac{d}{dt}\|\partial_k v_j\|_p^p + \frac{4(p-1)}{p^2}\int |\nabla(|\partial_k v_j|^{\frac{p}{2}-1}\partial_k v_j)|^2 dx$$
$$\leq c\|D^2\pi\|_p\|Dv\|_p^{p-1} + \|Dv\|_{p+1}^{p+1},$$

where Hölder's inequality has been used in order to estimate the first integral on the right hand side of (2.11).

Next, we apply the Sobolev embedding theorem

$$\int |\nabla f|^2 dx \geq c\Big(\int |f|^{2^*} dx\Big)^{2/2^*}$$

in order to estimate from below the integral that appears in the last equation. This yields

$$\frac{1}{p}\frac{d}{dt}\|\partial_k v_j\|_p^p + c\|\partial_k v_j\|_{\frac{2^*p}{2}}^p \leq c\|Dv\|_{p+1}^{p+1} + c\|D^2\pi\|_p\|Dv\|_p^{p-1}.$$

By adding with respect to k and j we show that

$$\frac{1}{p}\frac{d}{dt}\|Dv\|_p^p + c_1\|Dv\|_{\frac{2^*p}{2}}^p \leq c_2\|Dv\|_{p+1}^{p+1} + c_3\|D^2\pi\|_p\|Dv\|_p^{p+1}. \tag{2.12}$$

Next, by applying Hölder's inequality (with exponents $2^*p/2, p'$ and $pn/2$) to the integral on the right hand side of the identity

$$\|Dv\|_{p+1}^{p+1} = \int |Dv||Dv|^{p/p'}|Dv|dx,$$

one proves that

$$\|Dv\|_{p+1}^{p+1} \leq \|Dv\|_{\frac{2^*p}{2}}\|Dv\|_p^{p/p'}\|Dv\|_{\frac{np}{2}}.$$

Hence, by Young's inequality,

$$c_2\|Dv\|_{p+1}^{p+1} \leq (c_1/4)\|Dv\|_{\frac{2^*p}{2}}^p + c\|Dv\|_{\frac{pn}{p}}^{p'}\|Dv\|_p^p. \tag{2.13}$$

On the other hand, since v is divergence free,

$$\Delta\pi = \sum_{i,j}(\partial_i v_j)(\partial_j v_i).$$

Hence, by Calderon-Zygmund inequality[1,2] it follows that

$$\|D^2\pi\|_p \leq c\|Dv\|_{2p}^2. \tag{2.14}$$

Next, note that

$$\frac{1}{2p} = \frac{1/2}{2^*p/2} + \frac{1/2}{pn/2}.$$

Hence, by interpolation, one shows that

$$\|Dv\|_{2p} \leq \|Dv\|_{\frac{2^*p}{2}}^{1/2}\|Dv\|_{\frac{pn}{2}}^{1/2}. \tag{2.15}$$

From (2.14) and (2.15) it follows that

$$\|D^2\pi\|_p\|Dv\|_p^{p-1} \leq c\|Dv\|_{\frac{2^*p}{2}}\|Dv\|_{\frac{pn}{2}}\|Dv\|_p^{p/p'}.$$

By Young's inequality

$$c_3\|D^2\pi\|_p\|Dv\|_p^{p-1} \leq (c_1/4)\|Dv\|_{\frac{2^*p}{2}}^p + c\|Dv\|_{\frac{pn}{2}}^{p'}\|Dv\|_p^p. \tag{2.16}$$

From (2.12), (2.13) and (2.16) it readily follows that

$$\frac{1}{p}\frac{d}{dt}\|Dv\|_p^p + \frac{1}{2}\|Dv\|_{\frac{2^*p}{2}}^p \leq c\|Dv\|_{\frac{np}{2}}^{p'}\|Dv\|_p^p. \tag{1.17}$$

This shows (2.3), since $\frac{2^*p}{2} = \frac{pn}{n-2}$.

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