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KIRCHHOFF TYPE EQUATIONS DEPENDING ON A SMALL PARAMETER

P. D'ancona* S. Spagnolo*

Abstract

The authors prove some global existence results for equations of Kirchhoff type, i.e., nonlinear stretched string with nonlocal terms, depending on a parameter. This general setting includes the known results on the Kirchhoff equation with small data. Moreover, the authors can also handle some cases of degeneracy, which escaped earlier methods.

Keywords Nonlinear hyperbolic equations, Nonlocal equations, Kirchhoff equations.
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§1. Introduction

The Cauchy problem

$$u_{tt} - m\left(\int_{\mathbf{R}^n} |\nabla u|^2 dx\right) \Delta u = 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad u_0, u_1 \in C_0^\infty(\mathbf{R}^n), \tag{1.2}$$

where $m(r)$ is a positive C^1 function, has been extensively studied in the last fifty years, starting with the pioneering work of S. Bernstein[3] (for more extensive references, see [4, 5]; see also [8]). Most investigations have been centered on the *local existence* in Sobolev spaces, while only a few results of global existence are known. Roughly speaking, the global solvability of (1.1), (1.2) has been proved in two different cases:

i) for analytic initial data;

ii) for small initial data decaying as $|x| \to \infty$.

Here we are interested in the case (ii), whereas we refer to [3, 5], for the first kind of problem. The first result of type (ii) was obtained by J. M. Greenberg and S. C. Hu in [7], for the one dimensional Kirchhoff equation

$$u_{tt} - \left(1 + \lambda \int |u_x|^2 dx\right) u_{xx} = 0,$$

where the global existence was proved for small λ or, equivalently, for fixed λ and small initial data ϕ, ψ. Such a result was later extended in [4] to several space dimensions, and to any problem of the form (1.1), (1.2).

In this paper, we propose a comprehensive setting for these results, by giving a general condition that ensures the global solvability for (1.1),(1.2). More precisely, we find a positive functional H, depending on the functions $m(r), u_0(x), u_1(x)$, with the property that (1.1),(1.2) can be globally solved as soon as $H(m, u_0, u_1)$ is small enough.

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*Department of Mathematics, Pisa University, Pisa, Italy.

Introducing the notation

$$N(f)=\sum_{\substack{|\alpha|\le 2\\ |\beta|\le 1}}\int_{\mathbf{R}^n}|x^\alpha D^\beta f|^2dx$$

we have (see Lemma A in the Appendix; see also [6])

$$c_n\equiv 1+\operatorname*{Sup}_{\substack{\tau\in\mathbf{R}\\ f_1,f_2\in C_0^\infty(\mathbf{R}^n)}} N(f_1)^{-1/2}N(f_2)^{-1/2}(1+|\tau|)^2\left|\int e^{i\tau|\xi|}\widehat{f}_1(\xi)\widehat{f}_2(\xi)|\xi|d\xi\right|<+\infty \quad (1.3)$$

(where $\widehat{f}_j$ is the Fourier transform of the function f_j). If we assume that $m(0)>0$ and we consider the quantity

$$\nu_0=256c_n\left(N(\nabla u_0)+\frac{N(u_1)}{m(0)}\right), \quad (1.4)$$

where we write for brevity $N(\nabla u_0)=\sum_{j=1}^{n}N(D_ju_0)$, we can define the functional $H(m,u_0,u_1)$ as

$$H=\nu_0\cdot\frac{\|m'\|_{L^\infty(0,\nu_0)}}{m(0)} \quad (1.5)$$

and we have

Theorem 1.1. *Consider Problem* (1.1), (1.2) *where* $m(r)$ *is a* C^1 *function in a right neighbourhood of* $r=0$ *such that*

$$m(0)>0 \quad (1.6)$$

and u_0,u_1 *are smooth functions on* $\mathbf{R}^n$ *with* $N(\nabla u_0),N(u_1)<\infty$.

Then a global solution $u(t,x)$ *exists on* $\mathbf{R}^+\times\mathbf{R}^n$, *as soon as*

$$H(m,u_0,u_1)<\frac{1}{2}. \quad (1.7)$$

We notice that H cannot be made small by any rescaling of the form

$$u(t,x)\mapsto\widetilde{u}(t,x)=\lambda\cdot u(\mu t,x). \quad (1.8)$$

Indeed, if u solves (1.1),(1.2), then $\widetilde{u}$ solves a similar problem with

$$\widetilde{m}(r)=\mu^2m(\lambda^{-2}r),\quad \widetilde{u}_0=\lambda u_0,\quad \widetilde{u}_1=\lambda\mu u_1,$$

and we have

$$H(\widetilde{m},\widetilde{u}_0,\widetilde{u}_1)=H(m,u_0,u_1).$$

On the other hand, by a suitable choice of μ in (1.8), we can always reduce ourselves to the case $m(0)=1$, for which (1.5) reads

$$H=256c_n(N(\nabla u_0)+N(u_1))\cdot\operatorname{Sup}\{|m'(r)|\ :\ 0\le r\le 256c_n(N(\nabla u_0)+N(u_1))\}.$$

Thus, condition (1.7) means, roughly speaking, that either the initial data of (1.1),(1.2) are small in the N-norms, or $m(r)$ is close to the constant $1\equiv m(0)$. In both cases, we can say that (1.1),(1.2) is close to a problem with global solution, namely the same equation with $u_0=u_1=0$ in the first case or the linear problem $\Box u=0$ in the second case. Finally, we emphasize that assumptions (1.6)-(1.7) imply that $m(r)>0$ on the interval $0\le r\le\nu_0$ with ν_0 given by (1.4).

Theorem 1.1 can be better appreciated when applied to a family of problems (1.1),(1.2) depending on a small parameter $\epsilon > 0$. More precisely, if we consider the problems

$$u_{tt} - m_\epsilon \left(\int_{\mathbf{R}^n} |\nabla u|^2 dx \right) \Delta u = 0 \tag{1.9}$$

$$u(0,x) = u_0^\epsilon(x), \quad u_t(0,x) = u_1^\epsilon(x), \tag{1.10}$$

with $m_\epsilon(0) > 0$, Theorem 1 gives the global existence for $\epsilon < \overline{\epsilon}$ provided that

$$H(m_\epsilon, u_0^\epsilon, u_1^\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{1.11}$$

In some special cases, condition (1.11) can be simplified. Indeed, if we assume that

$$m_\epsilon(0) \geq \lambda_0 > 0,$$

and also that

$$\nu_\epsilon \equiv 256 c_n (N(\nabla u_0^\epsilon) + \lambda_0^{-1} N(u_1^\epsilon)] \leq \overline{\nu} < \infty,$$

with $\overline{\nu}, \lambda_0$ independent of ϵ (for instance when the initial data are fixed or bounded in the N-norms), then (1.11) is implied by the condition

$$\nu_\epsilon \frac{m'_\epsilon(r)}{m_\epsilon(0)} \to 0 \quad \text{in} \quad L^\infty([0,\overline{\nu}]) \quad (\epsilon \to 0),$$

and hence, a fortiori, by

$$m'_\epsilon(r) \to 0 \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbf{R}^+) \quad (\epsilon \to 0). \tag{1.12}$$

In order to illustrate these results, we list now some classes of problems of type (1.9),(1.10), for which Theorem 1.1 ensures the global existence for small $\epsilon > 0$.

Corollary 1.1. *The Cauchy problem*

$$u_{tt} - m \left(\int_{\mathbf{R}^n} |\nabla u|^2 dx \right) \Delta u = 0,$$

$$u(0,x) = u_0^\epsilon(x), \quad u_t(0,x) = u_1^\epsilon(x),$$

where $N(\nabla u_0^\epsilon) + N(u_1^\epsilon) \to 0$ *as* $\epsilon \to 0$, *and* $m(r)$ *is a* C^1 *function defined in a right neighbourhood of* $r = 0$ *with* $m(0) > 0$, *admits a global solution for* ϵ *small enough.*

Corollary 1.2. *The Cauchy problem*

$$u_{tt} - \mu \left(\epsilon; \int_{\mathbf{R}^n} |\nabla u|^2 dx \right) \Delta u = 0,$$

$$u(0,x) = u_0^\epsilon(x), \quad u_t(0,x) = u_1^\epsilon(x),$$

where $\mu(\epsilon, r)$ *is a* C^1 *function on* $[0,\overline{\epsilon}] \times [0,\overline{\nu}]$ *such that*

$$\mu(0,r) \equiv \text{constant}, \qquad \mu(\epsilon, 0) \geq \lambda_0 > 0$$

and $256 c_n (N(\nabla u_0^\epsilon) + \lambda_0^{-1} N(u_1^\epsilon)) \leq \overline{\nu}$, *admits a global solution for* ϵ *small enough. For instance, we can take each of the following functions as* $\mu(\epsilon, r)$:

$$\mu(\epsilon, r) = m(\epsilon r), \quad \mu(\epsilon, r) = 1 + \epsilon \cdot m(r), \quad \mu(\epsilon, r) = (1+r)^\epsilon,$$

with $m(r)$ *as in Corollary* 1.1.

In the examples above, the functions $m_\epsilon(r)$ are uniformly bounded from above and below near $r = 0$. However, Theorem 1.1 can also be applied to some problems which degenerate asymptotically as $\epsilon \to 0$. For instance, if we assume that ∇u_0^ϵ and u_1^ϵ are uniformly bounded in the norm $N(\cdot)$, while $m(r)$ is a C^1 function, then Problem (1.9),(1.10) can be globally solved for small $\epsilon > 0$, in each of the following cases:

$$m_\epsilon(r) = \epsilon^{-1} + m(r),$$
$$m_\epsilon(r) = \epsilon^{\alpha} + \epsilon m(r), \qquad \alpha < \frac{1}{2}, \qquad \operatorname*{Sup}_{r\geq 0} |m'(r)| < \infty,$$
$$m_\epsilon(r) = \epsilon^{\alpha} m(\epsilon r), \qquad \alpha < 1, \qquad \operatorname*{Sup}_{r\geq 0} |m'(r)| < \infty, \qquad m(0) > 0.$$

A crucial assumption in Theorem 1.1 is that the function $m(r)$ is strictly positive and differentiable at $r = 0$. This excludes many degenerate cases (where $m(0) = 0$ or $m'(0) = +\infty$) which are of interest in the theory of Kirchhoff type equations. In particular, the equation

$$u_{tt} - \left(\int |\nabla u|^2 dx\right)^{\lambda} \Delta u = 0 \qquad (\lambda \geq 0) \tag{1.13}$$

has been studied by Y. Yamada[9] and A. Arosio and S. Garavaldi[1] who proved the local existence with data u_0, u_1 in Sobolev spaces, assuming $\lambda \geq 1$, or $0 < \lambda \leq 1$ and $u_0 \not\equiv 0$, respectively. It is to be mentioned that if u_0, u_1 are real analytic functions, then (1.1),(1.2) is always globally solvable as soon as $m(r)$ is a continuous nonnegative function (see [2,5]).

Another interesting equation, related to (1.13), to which Theorem 1.1 does not apply, is

$$u_{tt} - \left[1 + \left(\int |\nabla u|^2 dx\right)^{\lambda}\right] \Delta u = 0 \qquad (0 < \lambda < 1); \tag{1.14}$$

indeed, $m(r) = 1 + r^{\lambda}$ is not a C^1 function when $0 < \lambda < 1$.

In the second part of the paper, we prove some global existence results also for Equations (1.13),(1.14), imposing a suitable relation between the (non analytic) initial data u_0, u_1. More precisely, we require that the pair (u_0, u_1) is close enough to a pair of data generating a traveling wave solution.

In order to illustrate these results, we shall put ourselves, for sake of simplicity, in the one dimensional case ($n = 1$). Then we can easily see that (1.1),(1.2) admits traveling wave solutions, i.e., solutions of the form $u(t, x) = w(x \pm c_0 t)$, if and only if the initial data satisfy $u_1(x) = \pm c_0 \partial_x u_0(x)$ with

$$c_0^2 = m\left(\int |\partial_x u_0|^2 dx\right). \tag{1.15}$$

Thus the closeness of (u_0, u_1) to a couple of data giving rise to a traveling wave can be expressed as follows:

$$u_1(x) = \pm\left[m\left(\int |\partial_x u_0|^2 dx\right)\right]^{1/2} \cdot \partial_x u_0(x) + \epsilon g(x),$$

where $g(x)$ is a given function, and ϵ is small.

Then we prove:

Theorem 1.2.

i) *The Cauchy problem*

$$u_{tt} - \left(\int_{-\infty}^{+\infty} |\partial_x u|^2 dx \right)^{\lambda} \partial_x^2 u = 0 \qquad (\lambda \geq 0), \tag{1.16}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = \pm \delta_0^{\lambda/2} \partial_x u_0(x) + \epsilon g(x) \tag{1.17}$$

with $u_0, g \in C_0^\infty(\mathbf{R})$ *and*

$$\delta_0 = \int_{-\infty}^{+\infty} |\partial_x u_0|^2 dx > 0 \tag{1.18}$$

has a unique global solution for λ, ϵ *small, i.e. provided that*

$$0 \leq \lambda < \overline{\lambda}(u_0, u_1), \quad |\epsilon| < \overline{\epsilon}(u_0, u_1).$$

ii) *The same conclusion as in* (i) *holds for the problem*

$$u_{tt} - \left[1 + \left(\int_{-\infty}^{+\infty} |\partial_x u|^2 dx \right)^{\lambda} \right] \partial_x^2 u = 0 \qquad (\lambda \geq 0), \tag{1.19}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = \pm (1 + \delta_0^{\lambda})^{1/2} \partial_x u_0(x) + \epsilon g(x). \tag{1.20}$$

iii) *Let us now consider again Equation* (1.19), *but with the initial conditions*

$$u(0,x) = \epsilon f(x), \quad u_t(0,x) = \epsilon (1 + \delta_0^{\lambda})^{1/2} \partial_x u_0(x) + \epsilon^2 g(x), \tag{1.21}$$

where $f, g \in C_0^\infty(\mathbf{R})$, *and as above*

$$\delta_0 = \int_{-\infty}^{+\infty} |\partial_x f|^2 dx > 0. \tag{1.22}$$

Then for all (*not necessarily small*) $\lambda \geq 0$ *there exists a unique global solution to* (1.19), (1.21), *provided* $\epsilon \leq \overline{\epsilon}(f, g)$.

Remarks. 1) In part (iii) of Theorem 1.2 we have been forced to assume not only that the initial data are ϵ-small but also that they are ϵ^2-close to a traveling wave. Of course, this additional assumption is not necessary when $\lambda \geq 1$, since in that case $m(r) = 1 + r^{\lambda}$ is a C^1 function at $r = 0$ and we can apply Theorem 1.1.

2) A similar result to part (iii) of Theorem 1.2 holds for Equation (1.1) with

$$m(r) = \frac{1}{|\log r|}, \tag{1.23}$$

(here $u_t(0,x) = \epsilon \sqrt{m(\delta_0)} \partial_x f + \epsilon^2 g$), or more generally for any $m(r)$ such that

$$r \frac{m'(r)}{m(r)} \to 0 \quad \text{as} \quad r \to 0^+. \tag{1.24}$$

3) Theorem 1.2 can be extended to several space dimensions, provided the notion of traveling wave is generalized in the following sense: a solution to Equation (1.1) will be called a traveling wave if

$$\int_{\mathbf{R}^n} |\nabla u(t,x)|^2 dx = \int_{\mathbf{R}^n} |\nabla u_0|^2 dx \qquad \forall t \geq 0. \tag{1.25}$$

By Fourier transform we then find

$$\widehat{u}(t,\xi) = \widehat{u}_0(\xi) e^{\pm i c_0 t |\xi|} \tag{1.26}$$

with

$$c_0^2 = m \left(\int_{\mathbf{R}^n} |\nabla u_0|^2 dx \right), \tag{1.27}$$

so that a pair (u_0, u_1) of initial data generates a traveling wave if and only if

$$\widehat{u}_1(\xi) = \pm ic_0|\xi|\widehat{u}_0(\xi). \tag{1.28}$$

Thus the basic assumption in Theorem 1.2 will be that

$$\widehat{u}_1(\xi) = \pm ic_0|\xi|\widehat{u}_0 + \epsilon\widehat{g}. \tag{1.29}$$

4) We conjecture that the global existence for (1.19) with initial data close to a traveling wave, i.e., satisfying (1.29) with ϵ small enough, holds also for λ large, in particular for the original Kirchhoff equation, corresponding to the choice $m(r) = 1 + r$.

The proof of both Theorems 1.1 and 1.2 (see Section 3 below) is based on a lemma (Lemma 2.1 in Section 2), which consists in an improvement of the existence result of [7] and [4]: in this lemma we find a precise condition on the triple $(m(\cdot), u_0, u_1)$ which is sufficient to ensure the global solvability for Problem (1.1),(1.2).

§2. Two Basic Lemmas

In this section we shall prove two lemmas from which we shall derive Theorems 1.1 and 1.2.

The first lemma consists essentially of an a priori estimate for global solutions to (1.1), (1.2), where $m(r)$ is a C^1 function, $m(r) > 0$. Let $u(t, x) \in C^2(\mathbf{R}^+; C^\infty(\mathbf{R}^n_x))$ be such a global solution; by Fourier transform in x we can write

$$v'' + c^2(t)|\xi|^2 v = 0, \tag{2.1}$$

$$v(0) = v_0, \quad v'(0) = v_1, \tag{2.2}$$

where $v(t) = v(t, \xi) = \widehat{u}$, $v_0 = \widehat{u_0}, v_1 = \widehat{u_1}$, and

$$c(t) = [m(\delta(t))]^{1/2}, \qquad \delta(t) = \int_{\mathbf{R}^n} |\xi|^2 |v(t, \xi)|^2 d\xi. \tag{2.3}$$

Then the (real) quantities

$$\alpha = \frac{|\dot{v}|^2}{c(t)|\xi|} - c(t)|\xi||v|^2,$$
$$\beta = 2\,\mathrm{Re}(\dot{v}\overline{v}),$$
$$\psi = \frac{|\dot{v}|^2}{c(t)|\xi|} + c(t)|\xi||v|^2$$

satisfy the first order system

$$\begin{cases} \alpha' &= -2c(t)|\xi|\beta - \dfrac{c'(t)}{c(t)}\psi, \\ \beta' &= +2c(t)|\xi|\alpha, \\ \psi' &= -\dfrac{c'(t)}{c(t)}\alpha. \end{cases}$$

If we define

$$\gamma(t) = \int_0^t c(s)ds \tag{2.4}$$

and

$$\phi(t, \xi) = e^{2i\gamma(t)|\xi|}(\alpha - i\beta), \tag{2.5}$$

the above system can also be written as

$$\begin{cases} \phi' &= -\dfrac{c'(t)}{c(t)} e^{2i\gamma(t)|\xi|}\psi, \\ \psi' &= -\dfrac{c'(t)}{c(t)} \operatorname{Re}\left(e^{-2i\gamma(t)|\xi|}\phi\right). \end{cases} \tag{2.6}$$

Now we define the following functionals, associated to the quantities ϕ, ψ, for $t \ge 0, \tau \in \mathbf{R}$,

$$J_\phi(t,\tau) = c(t)^{-1} \int_{\mathbf{R}^n} e^{-2i\tau|\xi|}\phi(t,\xi)|\xi|^2 d\xi \tag{2.7}$$

$$J_\psi(t,\tau) = c(t)^{-1} \int_{\mathbf{R}^n} e^{-2i\tau|\xi|}\psi(t,\xi)|\xi|^2 d\xi, \tag{2.8}$$

and we remark that the time derivative $\delta'(t)$ (see (2.3)) can be written in terms of J_ϕ as follows:

$$\delta'(t) = \int |\xi|^2 2\operatorname{Re}(\dot{v}\overline{v})d\xi = \int |\xi|^2 \ \beta \ d\xi = -\operatorname{Im} J_\phi(t,\gamma(t)) \cdot c(t). \tag{2.9}$$

Hence, recalling that $c(t) = (m(\delta(t))^{1/2}$, we get the identity

$$c'(t) = -\frac{1}{2} m'(\delta(t)) \operatorname{Im} J_\phi(t,\gamma(t)). \tag{2.10}$$

Differentiating (2.7),(2.8) with respect to time and using the equations (2.6) and the identity (2.10), we easily obtain

$$J'_\phi(t,\tau) = \operatorname{Im} J_\phi(t,\gamma(t)) \cdot [J_\psi(t,\tau-\gamma(t)) + J_\phi(t,\tau)] \frac{m'(\delta(t))}{2m(\delta(t))} c(t), \tag{2.11}$$

$$J'_\psi(t,\tau) = \operatorname{Im} J_\phi(t,\gamma(t)) \cdot \left[J_\phi(t,\tau-\gamma(t)) + \overline{J_\phi(t,\tau+\gamma(t))} + J_\psi(t,\tau)\right] \frac{m'(\delta(t))}{2m(\delta(t))} c(t). \tag{2.12}$$

Now using (2.11),(2.12) we shall prove that, for all $\tau \in \mathbf{R}$, $t \ge 0$, the inequalities

$$|J_\phi(t,\tau)| \le K_1(1+|\tau|)^{-2}, \quad |J_\psi(t,\tau)| \le K_2(1+|\tau|)^{-2} \tag{2.13}$$

hold for some constants K_1, K_2, provided the function $|m'(r)|/m(r)$ satisfies a suitable "smallness condition". More precisely, (2.13) holds with constants

$$K_1 \equiv 2 \operatorname*{Sup}_{\tau\in\mathbf{R}} (1+|\tau|)^2 |J_\phi(0,\tau)| \tag{2.14}$$

and

$$K_2 \equiv \operatorname{Max}\left\{K_1, 2\operatorname*{Sup}_{\tau\in\mathbf{R}}(1+|\tau|)^2|J_\psi(0,\tau)|\right\} \tag{2.15}$$

as soon as

$$16(K_1+K_2)\frac{|m'(r)|}{m(r)} < 1 \qquad \text{for} \qquad |r-\delta_0| \le K_1, \ r \ge 0 \tag{2.16}$$

where

$$\delta_0 = \delta(0) = \int |\xi|^2 |v_0|^2 d\xi. \tag{2.17}$$

First of all, let us observe that the constants K_1, K_2 given by (2.14),(2.15) are finite. This is an easy consequence of the definition of $\alpha, \beta, \phi, \psi$ and of Lemma A in the Appendix; indeed we have

$$\phi(0,\tau) = \alpha(0,\xi) - i\beta(0,\xi) = \left(\frac{|v_1|^2}{c_0|\xi|} - c_0|\xi||v_0|^2\right) - 2i\operatorname{Re}(v_1\overline{v_0}),$$

$$\psi(0,\tau)=\frac{|v_1|^2}{c_0|\xi|}+c_0|\xi||v_0|^2,$$

where $c_0=c(0)=[m(\delta_0)]^{1/2}$, and hence (see (2.7),(2.8))

$$J_\phi(0,\tau)=\int e^{-2i\tau|\xi|}\left(\frac{|v_1|^2}{m(\delta_0)}-|\xi|^2|v_0|^2\right)|\xi|d\xi-\frac{2i}{\sqrt{m(\delta_0)}}\int e^{-2i\tau|\xi|}\,\mathrm{Re}(v_1\overline{v_0})|\xi|^2d\xi, \quad (2.18)$$

$$J_\psi(0,\tau)=\int e^{-2i\tau|\xi|}\left(\frac{|v_1|^2}{m(\delta_0)}+|\xi|^2|v_0|^2\right)|\xi|d\xi. \quad (2.19)$$

Thus the finiteness of K_1,K_2 follows from Lemma A in the Appendix applied to $\widehat{f_1}=v_1$, $\widehat{f_2}=|\xi|v_0$ (i.e. $f_1=u_1$, $f_2=|\nabla|u_0$), for $\nu=1$ and $k=2$.

Next we prove that (2.16) implies (2.13). To this end, let $[0,T_*[$ be the maximum time interval where (2.13) holds for all $\tau\in\mathbf{R}$ (evidently $T_*>0$ by (2.14),(2.15)), and assume by contradiction that $T_*<\infty$.

Then, integrating (2.11),(2.12) on $[0,T_*]$, and using (2.13), we have, for $t<T_*$,

$$\begin{aligned}|J_\phi(t,\tau)|\le|J_\phi(0,\tau)|+\frac12\operatorname*{Sup}_{[\delta]}\frac{|m'(r)|}{m(r)}\Bigg[&K_1K_2\int_0^{T_*}(1+\gamma(s))^{-2}(1+|\tau-\gamma(s)|)^{-2}c(s)ds\\&+K_1^2\int_0^{T_*}(1+\gamma(s))^{-2}c(s)ds\cdot(1+|\tau|)^{-2}\Bigg],\end{aligned} \quad (2.20)$$

$$\begin{aligned}|J_\psi(t,\tau)|\le|J_\psi(0,\tau)|+\frac12\operatorname*{Sup}_{[\delta]}\frac{|m'(r)|}{m(r)}\Bigg[&K_1^2\int_0^{T_*}(1+\gamma(s))^{-2}(1+|\tau-\gamma(s)|)^{-2}c(s)ds\\&+K_1^2\int_0^{T_*}(1+\gamma(s))^{-2}(1+|\tau+\gamma(s)|)^{-2}c(s)ds\\&+K_1K_2\int_0^{T_*}(1+\gamma(s))^{-2}c(s)ds\cdot(1+|\tau|)^{-2}\Bigg],\end{aligned} \quad (2.21)$$

where the interval

$$[\delta]=\{\delta(t)\ :\ 0\le t<T_*\} \quad (2.22)$$

denotes the range of the values assumed on $[0,T_*[$ by the function $\delta(t)$ defined in (2.3).

Now, by the change of variables $\rho=\gamma(s)$, recalling that $c(s)=\dot\gamma(s)$ and using Lemma B in the Appendix with $\theta_1=\theta_2=2$, we obtain

$$\int_0^{T_*}(1+\gamma(s))^{-2}(1+|\tau-\gamma(s)|)^{-2}c(s)ds\le\int_0^{+\infty}(1+\rho)^{-2}(1+|\tau-\rho|)^{-2}d\rho\le 8(1+|\tau|)^{-2}$$

and an identical inequality holds for the term with $\tau+\gamma(s)$ instead of $\tau-\gamma(s)$, while a simpler computation gives

$$\int_0^{T_*}(1+\gamma(s))^{-2}c(s)ds\le\int_0^{+\infty}(1+\rho)^{-2}d\rho=1.$$

Hence by (2.20),(2.21) and (2.14),(2.15) we obtain, for $t < T_*$,

$$
\begin{aligned}
|J_\phi(t,\tau)| &\le |J_\phi(0,\tau)| + 4(1+|\tau|)^{-2}(K_1K_2+K_1^2)\,\mathop{\mathrm{Sup}}_{[\delta]}\frac{|m'(r)|}{m(r)} \\
&\le \frac{1}{2}K_1(1+|\tau|)^{-2} + 4(1+|\tau|)^{-2}K_1(K_2+K_1)\,\mathop{\mathrm{Sup}}_{[\delta]}\frac{|m'(r)|}{m(r)}, \qquad (2.23)\\
|J_\psi(t,\tau)| &\le |J_\psi(0,\tau)| + 4(1+|\tau|)^{-2}(K_1K_2+2K_1^2)\,\mathop{\mathrm{Sup}}_{[\delta]}\frac{|m'(r)|}{m(r)} \\
&\le \frac{1}{2}K_2(1+|\tau|)^{-2} + 8(1+|\tau|)^{-2}K_1(K_2+K_1)\,\mathop{\mathrm{Sup}}_{[\delta]}\frac{|m'(r)|}{m(r)}. \qquad (2.24)
\end{aligned}
$$

Assume now that $m(r)$ satisfies

$$16(K_1+K_2)\,\mathop{\mathrm{Sup}}_{[\delta]}\frac{|m'(r)|}{m(r)} < 1. \tag{2.25}$$

Then, taking into account that $K_1 \le K_2$ by definition (2.15) we see that (2.23),(2.23) for $t \to T_*$ give

$$|J_\phi(T_*,\tau)| < K_1(1+|\tau|)^{-2}, \quad |J_\psi(T_*,\tau)| < K_2(1+|\tau|)^{-2} \tag{2.26}$$

in contradiction with the definition of T_* (unless $T_* = +\infty$). Thus we have proved that (2.25) implies (2.13) for all $t \ge 0$. On the other hand, condition (2.25) can be made more explicit by estimating the range $[\delta]$ (see (2.22) and (2.3)). Indeed by (2.9) and (2.13) we deduce, for $t < T_*$,

$$|\delta'(t)| \le |\operatorname{Im} J_\phi(t,\gamma(t))| \cdot c(t) \le K_1(1+\gamma(t))^{-2}c(t) \tag{2.27}$$

and hence, integrating on $[0,t]$ and recalling that $c(t) = \dot\gamma(t)$, we see that

$$|\delta(t)-\delta_0| \le K_1\int_0^t (1+\gamma(s))^{-2}\dot\gamma(s)ds \le K_1,$$

or equivalently

$$[\delta] \subseteq [\delta_0 - K_1, \delta_0 + K_1] \cap \mathbf{R}^+ \equiv I_0. \tag{2.28}$$

In conclusion, if (2.16) is fulfilled, then (2.25) holds and hence (2.13) and also (2.27) hold for any $t \ge 0$.

In view of the proof of Theorems 1.1, 1.2, the main consequence of (2.27),(2.28) is the following estimate of the coefficient $c^2(t) = m(\delta(t))$ in Equation (2.1):

$$\left|\frac{d}{dt}c^2(t)\right| \le K_1\,\mathop{\mathrm{Sup}}_{I_0}|m'(r)|\sqrt{m(r)}.$$

To make future reference easier, we collect what we have proved above in the following

Lemma 2.1. *Let $v(t,\xi)$ be a solution to the Cauchy problem*

$$
\begin{aligned}
&v'' + m(\delta(t))|\xi|^2 v = 0,\\
&v(0) = v_0, \quad v'(0) = v_1,
\end{aligned}
$$

where

$$\delta(t) = \int_{\mathbf{R}^n} |\xi|^2 |v(t,\xi)|^2 d\xi,$$

and assume that $m(\delta_0) > 0$*, with*

$$\delta_0 = \delta(0) = \int_{\mathbf{R}^n} |\xi|^2 |v_0|^2 d\xi.$$

Moreover, consider the quantities

$$J_\phi(0,\tau) = \int e^{-2i\tau|\xi|} \left[\left(\frac{|v_1|^2}{m(\delta_0)} - |\xi|^2|v_0|^2 \right) - \frac{2i}{\sqrt{m(\delta_0)}} \mathrm{Re}(v_1\overline{v_0})|\xi| \right] |\xi| d\xi, \tag{2.29}$$

$$J_\psi(0,\tau) = \int e^{-2i\tau|\xi|} \left[\frac{|v_1|^2}{m(\delta_0)} + |\xi|^2|v_0|^2 \right] |\xi| d\xi, \tag{2.30}$$

$$K_1 = 2 \operatorname*{Sup}_{\tau\in\mathbf{R}} (1+|\tau|)^2 |J_\phi(0,\tau)|, \quad K_2 = \mathrm{Max}\{K_1, 2\operatorname*{Sup}_{\tau\in\mathbf{R}}(1+|\tau|)^2|J_\psi(0,\tau)\}, \tag{2.31}$$

and the interval

$$I_0 = [\delta_0 - K_1, \delta_0 + K_1] \cap \mathbf{R}^+. \tag{2.32}$$

Then, if

$$16(K_1+K_2) \operatorname*{Sup}_{I_0} \frac{|m'(r)|}{m(r)} < 1, \tag{2.33}$$

the range of the function $\delta(t)$ *is contained in the interval* I_0*, and for all* $t \geq 0$ *we have*

$$\left| \frac{d}{dt} m(\delta(t)) \right| \leq K_1 \operatorname*{Sup}_{I_0} |m'(r)| \sqrt{m(r)}. \tag{2.34}$$

In the next lemma, we give a simple convergence result for a family of strictly hyperbolic linear equations with coefficients depending on a parameter.

Lemma 2.2. *Cosider the family of Cauchy problems*

$$u_{tt} - a_k(t)\Delta u = 0, \tag{2.35}$$

$$u(0,x) = u_0^k(x), \quad u_t(0,x) = u_1^k(x), \tag{2.36}$$

where the coefficients $a_k(t)$ *are strictly positive* C^1 *functions on the intervals* $[0,\rho_k]$*, with* $\rho_k \to +\infty$ *as* $k \to +\infty$*, and assume that on* $[0,\rho_k]$

$$0 < \lambda_0 \leq a_k(t) \leq \Lambda_0, \qquad |a_k'(t)| \leq \Lambda_0 \tag{2.37}$$

for some constants λ_0, Λ_0 *independent of* k*.*

Then there exists a subsequence $\{a_{k_j}\}$ *which converges in* $L^\infty_{\mathrm{loc}}(\mathbf{R}^+)$ *to some* $a(t) \in C^1(\mathbf{R}^+)$ *satisfying* (2.37) *on* $\mathbf{R}^+$*.*

Moreover, if the initial data u_0^k, u_1^k *converge as* $k \to \infty$ *to some functions* u_0*,* u_1 *in the Sobolev spaces* $H^m(\mathbf{R}^n)$ *for all* m*, then the solutions* $u_{k_j}(t,x)$*, which are in* $C^2([0,\rho_{k_j}]; C^\infty(\mathbf{R}^n))$*, converge for all* $T > 0$ *in the space* $C^2([0,T]; C^\infty(\mathbf{R}^n))$ *to the solution of the limit problem*

$$u_{tt} - a(t)\Delta u = 0, \tag{2.38}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{2.39}$$

which belongs to $C^2(\mathbf{R}^+; C^\infty(\mathbf{R}^n))$*.*

Proof. We will only give a sketch of the proof, since it relies on standard arguments.

The first claim regarding the sequence $\{a_k\}$ is a trivial consequence of the Ascoli-Arzelà theorem.

In order to prove the convergence of the solutions $\{u_{k_j}\}$, which exist by the theory of strictly hyperbolic equations, we introduce the energy of order j of a solution $u_k(t,x)$ to Problem (2.35),(2.36), defined as

$$E_j(u_k,t) = \sum_{|\alpha|=j-1} \int_{\mathbf{R}^n} (|D^\alpha \partial_t u_k|^2 + a_k(t)|\nabla D^\alpha u_k|^2)dx.$$

A standard computation shows that

$$E_j'(u_k,t) \le \frac{|a_k'(t)|}{a_k(t)} \cdot E_j(u_k,t)$$

and hence, by Gronwall's lemma and by (2.37) we have, for any fixed $T > 0$ and k so large that $\rho_k > T$,

$$E_j(u_k,t) \le C(\lambda_0, \Lambda_0, T) \cdot E_j(u_k, 0) \quad \text{on} \quad [0,T].$$

Thus it is clear that the sequence $u_k(t,x)$ is bounded in $C^1([0,T]; H^m(\mathbf{R}^n))$ for all $T > 0$ and $m \ge 0$; hence, using Equation (2.38), we see that it is bounded also in the space $C^2([0,T]; H^m(\mathbf{R}^n))$.

Now assume that the subsequence $\{a_{k_j}\}$ converges as above. Then, using the bounds just proved (and the finite speed of propagation property), it is easy to show that any subsequence of $\{u_{k_j}\}$ admits some convergent subsequence, whose limit must be the unique solution to (2.38),(2.39); thus the whole sequence $\{u_{k_j}\}$ must converge to the same limit, and this concludes the proof of the Lemma.

§3. Proof of the Theorems

A) Proof of Theorem 1.1.

Theorem 1.1 is a direct consequences of Lemmas 2.1, 2.2. We begin by verifying the assumptions of Lemma 2.1. By Lemma A in the Appendix (for $k = 2$, $\mu = \nu = 1$), choosing $\widehat{f}_1, \widehat{f}_2$ equal to one of the functions v_1, $|\xi|v_0$ (where $v_0 = \widehat{u}_0$, $v_1 = \widehat{u}_1$), we obtain that the quantities (2.29),(2.30) can be estimated as follows:

$$|J_\phi(0,\tau)|, |J_\psi(0,\tau)| \le 4c_n(1+|\tau|)^{-2} \cdot [N(\nabla u_0) + N(u_1) \cdot m(\delta_0)^{-1}], \tag{3.1}$$

where $c_n \ge 1$ is the constant defined in (1.3), $N(\nabla u_0) = \sum_{j=1}^{n} N(D_j u_0)$ and

$$N(f) = \sum_{\substack{|\alpha|\le 2\\ |\beta|\le 1}} \int_{\mathbf{R}^n} |x^\alpha \cdot D^\beta f|^2 dx, \qquad \delta_0 = \int_{\mathbf{R}^n} |\nabla u_0|^2 dx. \tag{3.2}$$

Thus the constants K_1, K_2 in (2.31) can be estimated as

$$K_i \le 8c_n[N(\nabla u_0) + N(u_1) \cdot m(\delta_0)^{-1}] \quad (i = 1,2) \tag{3.3}$$

and assumption (2.33) in Lemma 2.1 is implied by

$$128c_n[N(\nabla u_0) + N(u_1) \cdot m(\delta_0)^{-1}] \operatorname*{Sup}_{J_0} \frac{|m'(r)|}{m(r)} < 1, \tag{3.4}$$

where

$$J_0 = [0, \delta_0 + 8c_n(N(\nabla u_0) + N(u_1) \cdot m(\delta_0)^{-1})].$$

We shall now prove that (2.33) is fulfilled as soon as $H < 1/2$, where (see (1.3),(1.4),(1.5))

$$H = \nu_0 \frac{\|m'\|_{L^\infty(0,\nu_0)}}{m(0)}, \qquad \nu_0 = 256c_n \left(N(\nabla u_0) + \frac{N(u_1)}{m(0)} \right). \tag{3.5}$$

We have (see (3.2) and (3.5))

$$\delta_0 \le N(\nabla u_0) \le \frac{\nu_0}{256c_n} \le \nu_0, \tag{3.6}$$

and

$$m(r) \ge m(0) - r \cdot \|m'\|_{L^\infty(0,r)}, \tag{3.7}$$

thus we have easily

$$m(\delta_0) \ge m(0) - \nu_0 \|m'\|_{L^\infty(0,\nu_0)} = m(0)(1-H). \tag{3.8}$$

This implies that

$$N(\nabla u_0) + N(u_1)m(\delta_0)^{-1} \le \frac{1}{1-H}(N(\nabla u_0) + N(u_1)m(0)^{-1}) = \frac{1}{1-H}\frac{\nu_0}{256c_n}. \tag{3.9}$$

Now assume that

$$H < \frac{1}{2}.$$

Then (3.9), together with (3.6), gives

$$\delta_0 + 8c_n(N(\nabla u_0) + N(u_1)m(\delta_0)^{-1}) \le \frac{\nu_0}{256c_n} + \frac{\nu_0}{32(1-H)} \le \nu_0,$$

which allows us to estimate the interval J_0 (see (3.4)) as follows:

$$J_0 \subseteq [0, \nu_0].$$

Hence, using again (3.9), we see that to have (3.4) it is sufficient that

$$\frac{\nu_0}{2(1-H)} \operatorname*{Sup}_{[0,\nu_0]} \frac{|m'(r)|}{m(r)} < 1. \tag{3.10}$$

On the other hand, by (3.7), we know that

$$m(r) \ge m(0) - \nu_0 \|m'\|_{L^\infty(0,\nu_0)} = m(0)(1-H) \qquad \text{for} \qquad r \le \nu_0,$$

thus (3.10) is in turn a consequence of the condition

$$\frac{H}{2(1-H)^2} \equiv \frac{\nu_0}{2(1-H)^2} \frac{\|m'\|_{L^\infty(0,\nu_0)}}{m(0)} < 1,$$

that is to say, $H < 1/2$.

In conclusion, if $H < 1/2$ assumption (2.33) in Lemma 1 is fulfilled, and we obtain that, if $u(t,x)$ is a global solution to Problem (1.1),(1.2), the time derivative of the coefficient in Equation (1.1) satisfies the bound (2.34), and also, by (3.3), the bound

$$\left| \frac{d}{dt} m(\delta(t)) \right| \le 8c_n [N(\nabla u_0) + N(u_1) \cdot m(\delta_0)^{-1}] \cdot \operatorname*{Sup}_{I_0} |m'(r)| \sqrt{m(r)} = C(m, u_0, u_1) \tag{3.11}$$

with a constant C depending only on m and on suitable Sobolev norms of u_0, u_1.

Now we shall apply Lemma 2.2 to conclude the proof of Theorem 1.1. To this end, we approximate in $H^\infty(\mathbf{R}^n) = \cap_m H^m(\mathbf{R}^n)$ the initial data u_0, u_1 with sequences u_0^k, u_1^k of entire real analytic functions, rapidly decreasing at infinity (e.g. by analytic convolution). For such data, Equation (1.1) has global solutions $u_k(t,x)$, as it is proved in [5], which are real analytic in x and belong to $H^\infty(\mathbf{R}^n)$. Moreover, the result of Lemma 2.1 holds for the problems corresponding to the triples (m, u_0^k, u_1^k) with constants independent of k, since $H(m, u_0^k, u_1^k)$ converges to $H(m, u_0, u_1)$ and satisfies eventually $H < 1/2$. Thus, writing

$$a_k(t) = m(\delta_k(t)), \tag{3.12}$$

where

$$\delta_k(t) = \int_{\mathbf{R}^n} |\nabla u_k|^2 dx$$

and applying (3.11), we obtain a common bound for the derivatives $a_k'(t)$. This shows that the first part of assumption (2.37) in Lemma 2.2 is satisfied. In order to verify the second part of (2.37), it is sufficient to observe that, by Lemma 2.1 (see (2.33)), the ranges $[\delta_k]$ of the functions $\delta_k(t)$ are all contained in a common interval on which the function $m(\delta)$ is bounded and strictly positive. Thus we can apply Lemma 2.2, and we obtain that the the sequence $\{u_k\}$ has a converging subsequence $\{u_{k_j}\}$, with a smooth limit $u(t, x)$ which solves Equation (2.38). Moreover, we have

$$a(t) = \lim_j a_{k_j}(t) = \lim_j m\left(\int_{\mathbf{R}^n} |\nabla u_{k_j}|^2 dx\right) = m\left(\int_{\mathbf{R}^n} |\nabla u|^2 dx\right)$$

and this implies that $u(t, x)$ is a solution to (1.1),(1.2).

Uniqueness can be proved by a standard linearization argument. Indeed, consider the solution u obtained by the above argument, and assume there is another solution v, with the same initial data. Then u, v must coincide as long as the corresponding functions $\delta(t)$ stay in the interval where $m(r)$ is Lipschitz continuous; hence they must coincide everywhere.

In particular, from uniqueness it follows that the solution has compact support in space for all times, since we can regard (1.1) as a linear strictly hyperbolic equation, for which the property of finite speed of propagation holds.

The verifications in Corollaries 1.1, 1.2 are fairly obvious.

B) Proof of Theorem 1.2 (Sketch).

The proof follows the same lines as the preceding one. We firstly observe that, in the case $n = 1$, it is possible to prove a version of Lemma 2.1 which is almost identical to the above one, the only difference being that $|\xi|$ is replaced by ξ everywhere. In the following we shall apply this last version.

Consider case (i), that is to say $m(r) = r^\lambda$, assuming e.g. that the initial data are of the form (1.17) with the plus sign. Applying the Fourier transform ($v_0 = \widehat{u}_0, v_1 = \widehat{u}_1$), we have then

$$v_1 = i\delta_0^{\lambda/2}\xi v_0 + \epsilon\widehat{g} \qquad \left(\delta_0 = \int_{-\infty}^{+\infty} |\partial u_0|^2 dx\right),$$

so that

$$|v_1|^2 = \delta_0^\lambda \xi^2 |v_0|^2 + \epsilon^2|\widehat{g}|^2 + 2\epsilon\delta_0^{\lambda/2}\xi \mathrm{Im}(\overline{v_0}\widehat{g}),$$

$$\mathrm{Re}(v_1\overline{v_0}) = -\epsilon\mathrm{Re}(\overline{v_0}\widehat{g})$$

and (2.29),(2.30) become

$$J_\phi(0, \tau) = \int_{-\infty}^{+\infty} e^{-2i\tau\xi}(\epsilon^2\delta_0^{-\lambda}|\widehat{g}|^2 - 2i\epsilon\delta_0^{-\lambda/2}\overline{v_0}\widehat{g})\xi d\xi,$$

$$J_\psi(0,\tau) = \int_{-\infty}^{+\infty} e^{-2i\tau\xi}(2\xi^2|v_0|^2 + \epsilon^2\delta_0^{-\lambda}|\widehat{g}|^2 + 2\epsilon\delta_0^{-\lambda/2}\xi\,\mathrm{Im}(\overline{v_0}\widehat{g}))\xi d\xi.$$

Hence, applying Lemma A in the Appendix, we see that the constants K_1, K_2 defined in (2.31) can be estimated as follows (for $\epsilon \le 1$)

$$K_1 \le C\epsilon(N(\partial_x u_0) + \delta_0^{-\lambda}N(g)), \quad K_2 \le C(N(\partial_x u_0) + \delta_0^{-\lambda}N(g))$$

for some universal constant C. Let us now choose ϵ so small with respect to u_0, g that

$$K_1 \le \delta_0/2,$$

then the interval I_0 in (2.32) can be estimated as follows

$$I_0 \subseteq \left[\frac{\delta_0}{2}, 2\delta_0\right] \tag{3.13}$$

and assumption (2.33) in Lemma 2.1 is fulfilled as soon as

$$32C(N(\nabla u_0) + \delta_0^{-\lambda}N(g)) \operatorname*{Sup}_{[\delta_0/2,2\delta_0]} \frac{|m'(r)|}{m(r)} < 1. \tag{3.14}$$

But we have $\delta_0 > 0$,

$$\operatorname*{Sup}_{[\delta_0/2,2\delta_0]} \frac{|m'(r)|}{m(r)} \le \frac{2}{\delta_0} \operatorname*{Sup}_{[\delta_0/2,2\delta_0]} \frac{|m'(r)|r}{m(r)},$$

and for $m(r) = r^\lambda$ we have also

$$\frac{|m'(r)|r}{m(r)} \equiv \lambda,$$

thus (3.14) is fulfilled provided λ is small enough with respect to u_0, g.

Thus we have proved that the assumptions of Lemma 2.1 are fulfilled for small ϵ and λ. Note in particular that by (3.13) the range $[\delta]$ of $\delta(t)$ is bounded away from 0. The proof now proceeds exactly as for Theorem 1.1.

The proof of cases (ii),(iii) is completely analogous.

Appendix

In the following, $\|\cdot\|_s$ for real s will denote the Sobolev norm on $\mathbf{R}^n$

$$\|f\|_s = \left(\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2(1+|\xi|)^{2s}d\xi\right)^{1/2}.$$

Lemma A. *Let $f_1(x), f_2(x)$ be two smooth functions with compact support on $\mathbf{R}^n$, let $\mu > 0$, $\nu > -n$ be real numbers, and define*

$$F(\tau) = \int e^{i\tau|\xi|^\mu}\widehat{f_1}(\xi)\widehat{f_2}(\xi)|\xi|^\nu d\xi \tag{1}$$

for $\tau \in \mathbf{R}$. Then:

i) *For every nonnegative integer $k \le (\nu+n)/\mu$ we have*

$$|F(\tau)| \le C(n,\mu,\nu)(1+|\tau|)^{-k}N_k(f_1)^{1/2}N_k(f_2)^{1/2}, \tag{2}$$

where

$$N_k(f) = \sum_{|\alpha|\le k}\left(\|x^\alpha f\|^2_{(\nu+k(1-\mu))/2} + \delta\cdot\|x^\alpha f\|^2_{L_1}\right),$$
$$\delta = 0 \quad \text{if} \quad \nu + k(1-\mu) \ge 0, \qquad \delta = 1 \quad \text{otherwise}. \tag{3}$$

ii) *If $(\nu+n)/\mu$ is not integer, we have in addition*

$$|F(\tau)| \le C(n,\mu,\nu)(1+|\tau|)^{-(\nu+n)/\mu} N_*(f_1)^{1/2} N_*(f_2)^{1/2}, \tag{4}$$

where

$$N_*(f) = \sum_{|\alpha|\le k_*+1} \left(\|x^\alpha f\|^2_{(k_*+1-n)/2} + \delta_* \cdot \|x^\alpha f\|^2_{L_1} \right), \qquad k_* = \left[\frac{\nu+n}{\mu}\right], \tag{5}$$
$$\delta_* = 0 \quad if \quad k_*+1-n \ge 0, \qquad \delta_* = 1 \quad otherwise.$$

In particular, for $k=2$ and $\mu=\nu=1$, we have

$$\left| \int e^{i\tau|\xi|} \widehat{f}_1(\xi)\widehat{f}_2(\xi)|\xi| d\xi \right| \le c(n)(1+|\tau|)^{-2} N(f_1)^{1/2} N(f_2)^{1/2}$$

with

$$N(f) = \sum_{\substack{|\alpha|\le 2 \\ |\beta|\le 1}} \int_{\mathbf{R}^n} |x^\alpha D^\beta f|^2 dx.$$

Proof.

i) Setting $|\xi| = \rho$, $\xi/|\xi| = \eta$ and

$$\widehat{f}_1(\xi) \cdot \widehat{f}_2(\xi) = w(\xi) = w(\rho,\eta),$$

we can write (1) in the form

$$F(\tau) = \int_{|\eta|=1} G(\tau,\eta) d\eta,$$

with

$$G(\tau,\eta) = \int_0^\infty e^{i\tau\rho^\mu} w(\rho,\eta)\rho^\nu \rho^{n-1} d\rho \qquad (|\eta|=1). \tag{6}$$

We shall prove the following estimate:

$$|G(\tau,\eta)| \le C(\mu,\nu)|\tau|^{-k} \int_0^\infty |\partial_\rho^k w| \cdot \rho^{\nu+k(1-\mu)} \rho^{n-1} d\rho, \tag{7}$$

for all real $\tau \ne 0$, and all integer $k \le (n+\nu)/\mu$ ($k \ge 0$).

Before proving (7), we show how (2) follows from (7). Observing that

$$|\partial_\rho^k w| \le c(k) \sum_{h_1+h_2=k} |\partial_\rho^{h_1} \widehat{f}_1| \cdot |\partial_\rho^{h_2} \widehat{f}_2|,$$
$$|\partial_\rho^h \widehat{f}| \le c(n,h) \sum_{|\alpha|\le h} |\partial_\xi^\alpha \widehat{f}|, \quad |\partial_\xi^\alpha \widehat{f}| = |\widehat{x^\alpha f}|,$$

we easily see that (7) implies

$$|G(\tau,\eta)| \le C(n,\mu,\nu)|\tau|^{-k} \widetilde{N}_k(f_1)^{1/2} \widetilde{N}_k(f_2)^{1/2}, \tag{8}$$

where

$$\widetilde{N}_k(f) = \sum_{|\alpha|\le k} \int_{\mathbf{R}^n} |\widehat{x^\alpha f}|^2 |\xi|^{\nu+k(1-\mu)} d\xi. \tag{9}$$

On the other hand we have, for any $s > -n$,

$$\int_{\mathbf{R}^n} |\widehat{g}|^2 |\xi|^{2s} d\xi \le c_1(n) \operatorname*{Sup}_{|\xi|\le 1} |\widehat{g}(\xi)|^2 + \int_{|\xi|\ge 1} |\widehat{g}(\xi)|^2 (1+|\xi|)^{2s} d\xi$$
$$\le c_2(n) \left(\|g\|^2_{L_1} + \|g\|^2_s \right),$$

hence for $g = x^\alpha f$ and $s = (\nu + k(1-\mu))/2$ (note that $\nu + k(1-\mu) > -n$ since $\mu > 0$, $\nu > -n$ and $0 \le k \le (\nu+n)/\mu$) we get

$$\widetilde{N}_k(f) \le c(n) N_k(f), \qquad 0 \le k \le \frac{\nu+n}{\mu}.$$

Thus we have proved that (8), and hence (7), implies (2) for $|\tau| \ge 1$. As to the values $|\tau| \le 1$, (2) can be proved directly observing that

$$\operatorname*{Sup}_{|\tau|\le 1} |F(\tau)| \le \int_{\mathbf{R}^n} |w(\xi)| \cdot |\xi|^\nu d\xi \le C(n,\nu) \widetilde{N}_0(f_1)^{1/2} \widetilde{N}_0(f_2)^{1/2}$$

by the Schwartz inequality, and that (see (9) and (3))

$$\widetilde{N}_0(f) \le N_k(f) \qquad \text{if } \ k \ge 0.$$

The same argument shows that (2) holds for $k = 0$.

In conlusion, it remains to prove (7) for integer $k \in [1, (\nu+n)/\mu]$, under the assumption that $(\nu+n)/\mu \ge 1$. To this end, we perform the change of variables

$$\rho \mapsto \rho^\mu \equiv r, \qquad w(\rho,\mu) \mapsto w(r^{1/\mu}, \eta) \equiv \psi(r)$$

for a fixed η such that $|\eta| = 1$, so that (writing for brevity $G(\tau)$ instead of $G(\tau,\eta)$) (6) becomes

$$G(\tau) = \mu^{-1} \int_0^\infty e^{i\tau r} \psi(r) r^{\lambda-1} dr, \qquad \lambda \equiv \frac{\nu+n}{\mu} \ge 1. \tag{10}$$

We claim that

$$|G(\tau)| \le \mu^{-1} C(\lambda) |\tau|^{-k} \sum_{j=0}^{k} \int_0^\infty |(r\partial_r)^j \psi| \cdot r^{\lambda-k-1} dr \qquad \text{if } k < \lambda \tag{11}$$

while

$$|G(\tau)| \le \mu^{-1} C(\lambda) |\tau|^{-k} \sum_{j=1}^{k} \int_0^\infty |(r\partial_r)^j \psi| \cdot r^{-1} dr \qquad \text{if } k = \lambda. \tag{12}$$

Indeed, integrating by parts in (10) we have

$$G(\tau) = \mu^{-1}(-i\tau)^{-1} \left\{ \int_0^\infty e^{i\tau r} [(r\partial_r)\psi + (\lambda-1)\psi] r^{\lambda-2} dr + \delta(\lambda-1) \cdot \psi(0) \right\}, \tag{13}$$

where $\delta(s)$ is defined by $\delta(0) = 1$, $\delta(s) \equiv 0$ for $s \ne 0$, and if we apply (13) k times we find the identity

$$G(\tau) = \mu^{-1}(-i\tau)^{-k} \left\{ \int_0^\infty e^{i\tau r} \left(\sum_{j=0}^{k} A_j(\lambda)(r\partial_r)^j \psi \right) + B(\lambda)\delta(\lambda-k) \cdot \psi(0) \right\} \tag{14}$$

for some constants $A_0, \cdots, A_k, B$ depending only on λ, with

$$A_0(\lambda) = (\lambda-1)(\lambda-2)\cdots(\lambda-k).$$

Now, (11) and (12) follow directly from (14), taking into account that

$$\|\psi\|_{L_\infty} \le \int_0^\infty |\psi'| dr = \int_0^\infty |(r\partial_r)\psi| \cdot r^{-1} dr. \tag{15}$$

Finally, we have to prove that (11),(12) give (7). Going back to the original variable $\rho = r^{1/\mu}$, observing that

$$r\partial_r = \mu^{-1} \rho \partial_\rho, \quad r^{-1} dr = \mu \rho^{-1} d\rho,$$

$$(\rho\partial_\rho)^j = \sum_{h=1}^{j} C(h,\mu)\rho^h\partial_\rho^h \qquad \text{for } \ j \geq 1, \tag{16}$$

and recalling that $\lambda = (\nu+n)/\mu$, we easily see that (11) and (12) imply respectively that

$$|G(\tau)| \leq C(\mu,\nu)|\tau|^{-k}\sum_{h=0}^{k}\int_0^\infty |\partial_\rho^h w|\rho^{h+\nu-k\mu+(n-1)}d\rho \qquad \text{if } k < \frac{\nu+n}{\mu}, \tag{17}$$

$$|G(\tau)| \leq C(\mu,\nu)|\tau|^{-k}\sum_{h=1}^{k}\int_0^\infty |\partial_\rho^h w|\rho^{h-1}d\rho, \quad \text{if } k = \frac{\nu+n}{\mu}. \tag{18}$$

But using the inequality(*)

$$\int_0^\infty |\phi(\rho)|\rho^{\alpha-1}d\rho \leq \frac{1}{\alpha}\int_0^\infty |\phi'(\rho)|\rho^\alpha d\rho, \qquad \alpha > 0 \tag{19}$$

with $\phi = \partial_\rho^h w$ and $\alpha = h+\nu-k\mu+n$ (in (17)) or $\alpha = h-1$ (in (18)) and observing that

$$k-1 = k+\nu-k\mu+(n-1) \ \text{ if } \ k = (\nu+n)/\mu,$$

we see that (16),(17) imply

$$|G(\tau)| \leq C_1(\mu,\nu)|\tau|^{-k}\int_0^\infty |\partial_\rho^k w|\rho^{\nu+k(1-\mu)}\rho^{n-1}d\rho,$$

as claimed above. This concludes the proof of part (i).

ii) Assume now that $\lambda = (\nu+n)/\mu$ is not an integer, say $\lambda = k_* + \alpha$ with $0 < \alpha < 1$. From (14) with $k = k_*$ we know that

$$G(\tau) = \mu^{-1}(-i\tau)^{-k}H(\tau) \tag{20}$$

with

$$H(\tau) = \int_0^\infty e^{i\tau r}\phi(r)r^{\alpha-1}dr, \qquad \phi(r) = \sum_{j=0}^{k_*} A_j(\lambda)(r\partial_r)^j\psi.$$

Therefore, splitting the interval of integration as $[0,\delta]\cup[\delta,+\infty[$, we can write

$$H(\tau) = H_1(\tau) + H_2(\tau)$$

and we have

$$|H_1(\tau)| = \left|\int_0^\delta e^{i\tau r}\phi(r)r^{\alpha-1}dr\right| \leq \|\phi\|_{L_\infty}\frac{\delta^\alpha}{\alpha}$$

and

$$\begin{aligned}|H_2(\tau)| &= \left|\int_\delta^\infty e^{i\tau r}\phi(r)r^{\alpha-1}dr\right| \\ &= \left|(-i\tau)^{-1}\left\{\int_\delta^\infty e^{i\tau r}(\phi' r^{\alpha-1} + (\alpha-1)\phi r^{\alpha-2})dr + e^{i\tau\delta}\phi(\delta)\delta^{\alpha-1}\right\}\right| \\ &\leq |\tau|^{-1}(\|\phi'\|_{L_1}\delta^{\alpha-1} + \|\phi\|_{L_\infty}\delta^{\alpha-1} + |\phi(\delta)|\delta^{\alpha-1}).\end{aligned}$$

(*) Actually (19) holds for any smooth function $\phi(\rho)$ with compact support in $[0,+\infty[$, and can be proved by applying the identity

$$\int_0^\infty \phi(\rho)\rho^{\alpha-1}d\rho = -\frac{1}{\alpha}\int_0^\infty \phi'\rho^\alpha d\rho, \quad \alpha > 0$$

to the functions $\phi_\epsilon = (\epsilon+|\phi|^2)^{1/2}$ and then letting $\epsilon \to 0$; note that $|\phi| \leq \phi_\epsilon$, $|\phi'_\epsilon| \leq |\phi'|$.

Thus, if we choose $\delta = |\tau|^{-1}$ and recall (15), we obtain $|H(\tau)| \leq C(\alpha)|\tau|^{-\alpha}\|\phi'\|_{L_1}$. Hence, by (20),(16) and (19), we get

$$\begin{aligned}|G(\tau)| &\leq \mu^{-1}C(\lambda)|\tau|^{-k_*-\alpha}\sum_{j=0}^{k_*}\int_0^\infty |(r\partial_r)^{j+1}\psi|\cdot r^{-1}dr \\ &= c(n,\mu,\nu)|\tau|^{-(\nu+n)/\mu}\sum_{j=0}^{k_*}\int_0^\infty |(\rho\partial_\rho)^{j+1}w|\rho^{-1}d\rho \\ &\leq c_1(n,\mu,\nu)|\tau|^{-(\nu+n)/\mu}\int_0^\infty |\partial_\rho^{k_*+1}w|\rho^{k_*}d\rho.\end{aligned}$$

Then (4) follows by a similar argument as in the beginning of the proof.

Lemma B. *If* $\operatorname{Max}(\theta_1,\theta_2) > 1$, *then*

$$\int_0^{+\infty}(1+|t-s|)^{-\theta_1}(1+|s|)^{-\theta_2}ds \leq c(\theta_1,\theta_2)(1+|t|)^{-\min(\theta_1,\theta_2)}$$

with $c(\theta_1,\theta_2) = 2^{\min(\theta_1,\theta_2)+1}/(\operatorname{Max}(\theta_1,\theta_2)-1)$.

Proof. Write for brevity $\varphi = (1+|t-s|)^{-\theta_1}(1+|s|)^{-\theta_2}$. We have immediately

$$\int_t^\infty \varphi\, ds \leq \frac{1}{\operatorname{Max}(\theta_1,\theta_2)-1}(1+t)^{-\min(\theta_1,\theta_2)}.$$

Moreover, for $s \in [0,t]$ the expression φ is symmetric in θ_1, θ_2, thus we can assume that $\theta_1 \geq \theta_2$, $\theta_1 > 1$. We have then

$$\int_{t/2}^t \varphi\, ds \leq \left(1+\frac{t}{2}\right)^{-\theta_2}\int_{t/2}^t (1+t-s)^{-\theta_1}ds \leq \frac{2^{\theta_2}}{\theta_1-1}(1+t)^{-\theta_2}$$

and

$$\begin{aligned}\int_0^{t/2}\varphi\, ds &= \int_0^{t/2}(1+t-s)^{-\theta_2}(1+t-s)^{\theta_2-\theta_1}(1+s)^{-\theta_2}ds \\ &\leq \left(1+\frac{t}{2}\right)^{-\theta_2}\int_0^{t/2}(1+s)^{-\theta_1}ds \leq \frac{2^{\theta_2}}{\theta_1-1}(1+t)^{-\theta_2},\end{aligned}$$

and this concludes the proof of the Lemma.

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