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LARGE-TIME BEHAVIOR OF SOLUTIONS FOR THE SYSTEM OF COMPRESSIBLE ADIABATIC FLOW THROUGH POROUS MEDIA***

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Abstract

Consider the system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \quad \alpha > 0, \\ s_t = 0, \end{cases} \tag{1}$$

which can be used to model the adiabatic gas flow through porous media. Here v is specific volume, u denotes velocity, s stands for entropy, p denotes pressure with $p_v < 0$ for $v > 0$.

It is proved that the solutions of (1) tend to those of the following nonlinear parabolic equation time-asymptotically:

$$\begin{cases} v_t = -\frac{1}{\alpha} p(v,s)_{xx}, \\ s_t = 0, \\ u = -\frac{1}{\alpha} p(v,s)_x. \end{cases}$$

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§1. Introduction

Consider the system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \quad \alpha > 0, \\ s_t = 0, \end{cases} \tag{1.1}$$

which can be used to model the adiabatic gas flow through porous media. Here v is specific volume, u denotes velocity, s stands for entropy, p denotes pressure with $p_v < 0$ for $v > 0$. This system is strictly hyperbolic with eigenvalues $\lambda_1 = -\sqrt{-p_v}, \lambda_2 \equiv 0, \lambda_3 = \sqrt{-p_v}$.

The intent of this paper is to contribute to the program of elucidating the role of damping mechanism, in particular, the influence to the asymptotic behavior of the processes under consideration.

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For the case of isentropic flow, namely, $s(x,t) \equiv$ constant, it has been proved[1] that the solution of the Cauchy problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad \alpha > 0,\ p'(v) < 0 \text{ for } v > 0, \\ v(x,0) = v_0(x), u(x,0) = u_0(x), \text{ with } \lim\limits_{x\to\mp\infty}(v_0(x), u_0(x)) = (v^{\mp}, u^{\mp}) \end{cases} \tag{1.2}$$

can be described by the solution of the following problem

$$\begin{cases} v_t = -\frac{1}{\alpha}p(v)_{xx}, \\ u = -\frac{1}{\alpha}p(v)_x, \\ v(x,0) = \widetilde{v}_0(x) \text{ with } \lim\limits_{x\to\mp\infty}\widetilde{v}_0(x) = v^{\mp} \end{cases} \tag{1.3}$$

time-asymptotically. The system in (1.3) is obtained from (1.2) by approximating the momentum equation in $(1.2)_2$ with Darcy's law. Moreover, the L_2-norm and L_∞-norm of the difference between these two solutions tend to zero with a rate $t^{-\frac{1}{2}}$ as time tends to infinity[1]. This shows that certain nonlinear diffusive phenomena occur for the solution of (1.2) which is caused by the damping mechanism.

For the system (1.1), the corresponding simplified system takes the form

$$\begin{cases} v_t = -\frac{1}{\alpha}p(v,s)_{xx}, \\ u = -\frac{1}{\alpha}p(v,s)_x, \\ s_t = 0. \end{cases} \tag{1.4}$$

We will compare the solution of (1.1) with initial data $(u(x,0), v(x,0), s(x,0))$ to the solution of (1.4) with initial data $(v(x,0), s(x,0))$ and prove that the difference between these two solutions tends to zero as time tends to infinity. This shows that the large time behavior of solutions for nonlinear hyperbolic system (1.1) can be well approximated by corresponding simplified nonlinear diffusion equations and certain nonlinear diffusive phenomena may occur also, as in the isentropic case, for the solutions of (1.1), caused by the damping mechanism.

Denote the solution of (1.1) with

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad s(x,0) = s_0(x) \tag{1.5}$$

by (u_1, v_1, s_1); the solution of (1.4) with

$$v(x,0) = v_0(x), \quad s(x,0) = s_0(x)$$

by (u_2, v_2, s_2). It is obvious that

$$s_1(x,t) = s_2(x,t) = s(x) = s_0(x). \tag{1.6}$$

For simplicity, we assume that

$$\lim_{x\to\mp\infty} u_0(x) = 0, \quad \lim_{x\to\mp\infty} v_0(x) = \overline{v} > 0$$

and

$$\lim_{x\to\mp\infty} s_0(x) = \overline{s}, \tag{1.7}$$

where $\overline{v}$ and $\overline{s}$ are constants.

Remark 1.1. The same result as in the present paper can be obtained in the case when the initial data for (1.4) is $v(x,0) = \widetilde{v}_0(x) \not\equiv v_0(x)$ but $\lim\limits_{x\to\mp\infty}\widetilde{v}_0(x) = \overline{v}$.

Let $w = v_1 - v_2, z = u_1 - u_2$. It follows by (1.1) and (1.4) that

$$\begin{cases} w_t - z_x = 0, \\ z_t + [p(w + v_2, s) - p(v_2, s)]_x + \alpha z - \frac{1}{\alpha} p(v_2, s)_{xt} = 0, \end{cases} \tag{1.8}$$

where $s = s(x)$ is known.

By introducing $y(x,t) \equiv \int_{-\infty}^{x} w(\xi, t) d\xi$, the system (1.8) can be reduced into a single equation for y since $y_x = w$ and $y_t = z$, due to (1.7). Namely,

$$y_{tt} + [p(y_x + v_2, s) - p(v_2, s)]_x + \alpha y_t - \frac{1}{\alpha} p(v_2, s)_{xt} = 0, \tag{1.9}$$

where v_2 satisfies $(1.4)_1$ and $(1.5)_2$, $s = s(x) = s_0(x)$.

Ignore the subscript with v and assume $\alpha = 1$ for convenience. We study the following initial-value problem

$$\begin{cases} y_{tt} + [p(y_x + v, s) - p(v, s)]_x + y_t - p(v, s)_{xt} = 0, & (1.10) \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), & (1.11) \end{cases}$$

where

$$\begin{aligned} y_0(x) &= 0, \\ y_1(x) &= u_0(x) + p(v_0(x), s_0(x))_x, \end{aligned}$$

while $v(x,t)$ satisfies

$$\begin{cases} v_t = -p(v, s)_{xx}, & (1.12) \\ v(x, 0) = v_0(x). & (1.13) \end{cases}$$

For any given initial data $(u_0(x), v_0(x), s_0(x))$ with $u_0(x) \in H^2(R), [v_0(x) - \overline{v}] \in H^5(R)$ and $[s_0(x) - \overline{s}] \in H^4(R)$, we show that the Cauchy problem (1.10) – (1.13) has a unique smooth solution in the large in time provided the initial data are small. (We will give the precise description for the smallness in the next section). Furthermore, the solution y and it's derivatives y_t and y_x decay to zero in the L_∞-norm as $t \to \infty$, which implies that the system (1.1) is accurately approximated by (1.4) time-asymptotically.

Remark 1.2. A similar result for the case of isentropic flow was obtained in [2].

Remark 1.3. A different approach for the global existence of solutions of (1.1) can be found in [4] where

$$u(x, 0) = u_0(x), \quad v(x, 0) = \overline{v} + v_0(x) \text{ and } s(x, 0) = \overline{s} + s_0(x)$$

such that $(u_0(x), v_0(x), s_0(x))$ is smooth with a compact support.

§2. Premininary Remarks

For any given $\overline{v}, \overline{s}$, we choose r such that $0 < r < \overline{v}$ and define

$$\Omega = \{(v, s) : 0 < \overline{v} - r \le v \le \overline{v} + r, \overline{s} - r \le s \le \overline{s} + r\}.$$

Hypothesis 2.1. The function $p(v,s)$ is smooth in Ω such that $0 < a_0 \le -p_v(v,s) \le a_1$ for $(v,s) \in \Omega$ and the derivatives $\frac{\partial^{(i)} p(v,s)}{\partial v^{(l)} \partial s^{(i-l)}}, 0 \le l \le i, 1 \le i \le 4$, are bounded in Ω. Without loss of generality, we assume that $a_1 \ge 1, 0 < a_0 \le 1$.

We seek a smooth solution $(y(x,t),v(x,t))\in C^2(t\geq 0,x\in R)$ and

$$\begin{aligned}\|y(t),v(t)\|_2\equiv{}&|y(\cdot,t)|_{C^2}+|y_t(\cdot,t)|_{C^1}+|y_{tt}(\cdot,t)|_{C^0}\\&+|v(\cdot,t)-\overline{v}|_{C^2}+|v_t(\cdot,t)|_{C^1}+|v_{tt}(\cdot,t)|_{C^0}<\infty\quad\text{for } t\geq 0,\end{aligned}\tag{2.1}$$

where $|f(\cdot)|_{C^l}\equiv\sum\limits_{0\leq j\leq l}\sup\limits_{R}|d^jf(x)/dx^j|$.

By using Sobolev's lemma, it is known that

$$|f(\cdot)|_{C^l}\leq K\|f(\cdot)\|_{H^{l+1}}.\tag{2.2}$$

Thus, using the L_2-energy method as in [2] we will solve the Cauchy problem (1.10)–(1.13) in the space X_3 defined by

$$X_m=\left\{\begin{array}{l}y(t)\in L_\infty(t;H^m),\ y_t(t)\in L_\infty(t;H^{m-1}),\ y_{tt}\in L_\infty(t;H^{m-2}),\\(v(t)-\overline{v})\in L_\infty(t;H^m),\ v_t(t)\in L_\infty(t;H^{m-1}),\ v_{tt}(t)\in L_\infty(t;H^{m-2}),\\0\leq t\leq T,\ \text{for any } T>0.\end{array}\right\}\tag{2.3}$$

Hypothesis 2.2.

$$\begin{aligned}&u_0(x)\in H^2(R),\quad v_0(x)-\overline{v}\in H^5(R),\\&s_0(x)-\overline{s}\in H^4(R)\text{ and }(v_0(x),s_0(x))\in\Omega^*,\end{aligned}\tag{2.4}$$

where

$$\Omega^*=\{(v,s):0<\overline{v}-r^*\leq v\leq\overline{v}+r^*,\overline{s}-r^*\leq s\leq\overline{s}+r^*,0<r^*<r\}.$$

It can be proved that the classical local existence theorem gives the solution for the Cauchy problem (1.10)–(1.13) in the space X_3 locally in time. For the global existence in $t>0$ we only need the a priori estimates in the norm (2.1) for which the a priori estimates in the norm of X_3 is sufficient by (2.2), i.e.,

$$\begin{aligned}|||y(t),v(t)|||_3^2\equiv{}&\|y(t)\|_{H^3}^2+\|y_t(t)\|_{H^2}^2+\|y_{tt}(t)\|_{H^1}^2\\&+\|v(t)-\overline{v}\|_{H^3}^2+\|v_t(t)\|_{H^2}^2+\|v_{tt}(t)\|_{H^1}^2<\infty\text{ for } t\geq 0.\end{aligned}\tag{2.5}$$

In order to obtain the a priori estimates in the norm (2.5), we introduce

$$E(t)=\sum_{j=1}^{3}E_j(t)\tag{2.6}$$

for the solution (y,v) with $(v,s)\in\Omega$ and $(y_x+v,s)\in\Omega$ in each (t,x), where

$$\begin{aligned}E_1(t)={}&\frac{1}{2}\int_{-\infty}^{\infty}\Big\{\frac{1}{2}\Big(y\cdot y_t+\frac{y^2}{2}\Big)+y_t^2+[-p_v(\sigma y_x+v,s)]y_x^2\\&+M_1\Big[Q+\frac{q_x^2}{2}+\frac{\theta_t^2}{2}\Big]\Big\}(x,t)dx,\\E_2(t)={}&\frac{1}{2}\int_{-\infty}^{\infty}\{y_{tt}^2+[1-p_v(y_x+v,s)]y_{tx}^2+[-p_v(y_x+v,s)]y_{xx}^2\\&+M_2[\theta_{tt}^2+(1-p_v(v,s))\theta_{tx}^2]\}(x,t)dx,\\E_3(t)={}&\frac{1}{2}\int_{-\infty}^{\infty}\{y_{ttx}^2+[1-p_v(y_x+v,s)]y_{xxt}^2\\&+M_3[\theta_{xxt}^2+\theta_{ttx}^2]\}(x,t)dx,\end{aligned}$$

$M_i(i=1,2,3)$ is a positive constant, given later, $0<\sigma<1$, $s=s(x)$, θ is defined by

$$\theta(x,t)=v(x,t)-\widehat{v}(x), \tag{2.7}$$

and $\widehat{v}(x)$ is determined by

$$p(\widehat{v}(x),s(x))=p(\overline{v},\overline{s}), \tag{2.8}$$

$$q(\theta,x)\equiv p(\widehat{v}(x),s(x))-p(\theta+\widehat{v}(x),s(x)), \tag{2.9}$$

$Q(\theta,x)$ is defined by

$$Q(\theta,x)=\int_0^\theta q(\eta,x)d\eta. \tag{2.10}$$

It is claimed that $\int_{-\infty}^{\infty}\theta_x^2dx$ can be expressed in terms of $\int_{-\infty}^{\infty}q_x^2dx$ and $\int_{-\infty}^{\infty}\theta^2dx$ by differentiating (2.8) and (2.9) with respect to x respectively and combining them; $\int_{-\infty}^{\infty}\theta_{xx}^2dx$ can be expressed in terms of $\int_{-\infty}^{\infty}q_{xx}^2dx, \int_{-\infty}^{\infty}q_x^2dx$ and $\int_{-\infty}^{\infty}\theta^2dx$ by differentiating (2.8) and (2.9) with respect to x twice respectively and combining the resulting equalities; $\int_{-\infty}^{\infty}\theta_{xxx}^2dx$ can be expressed in terms of $\int_{-\infty}^{\infty}\theta_{tx}^2dx, \int_{-\infty}^{\infty}q_{xx}^2dx$, $\int_{-\infty}^{\infty}q_x^2dx$ and $\int_{-\infty}^{\infty}\theta^2dx$ with the help of $(2.8)_{xxx}$, $(2.9)_{xxx}$, (1.12) and (2.7). Moreover, $\int_{-\infty}^{\infty}\theta^2dx$ can be expressed in terms of $\int_{-\infty}^{\infty}Qdx$, (by 2.10). This, together with (2.1), (2.2), (2.5)–(2.10), implies

$$\begin{aligned}\|y(t),v(t)\|_2^2&\le\|y(t),\theta(t)\|_2^2+K_1|s-\overline{s}|_{C^2}\\&\le K|||y(t),\theta(t)|||_3^2+K_1|s-\overline{s}|_{C^2}\\&\le K^2E(t)+K_1|s-\overline{s}|_{C^2}\end{aligned} \tag{2.11}$$

for $(y,v)(x,t)$ with $(v,s)\in\Omega,(\widehat{v}(x),s)\in\Omega$ and $(y_x+v,s)\in\Omega$, provided

$$|s-\overline{s}|_{C^3}=\delta_0\le 1, \tag{2.12}$$

where the constants K and K_1 depend only on Ω and p.

Lemma 2.1. *Under the Hypotheses* 2.1 *and* 2.2, *there exists an* $\varepsilon=\varepsilon(\Omega,p)$ *with* $0<\varepsilon\le 1$, *such that if the solution* $(y(t),v(t))\in X_3$ *with* $(y_x+v,s)\in\Omega,(v,s)\in\Omega$ *and* $(\widehat{v},s)\in\Omega$ *to the Cauchy problem* (1.10)–(1.13) *is small as*

$$\|y(t),v(t)\|_2<\varepsilon \quad in\ 0\le t\le T \tag{2.13}$$

and

$$|s-\overline{s}|_{C^3}<\varepsilon \tag{2.14}$$

then one has the a priori estimate

$$E(t)\le K_2E(0) \quad in\ 0\le t\le T \tag{2.15}$$

where $K_2\ge 1$ *depends only on* p *and* Ω.

For proving, we first assume that the solution $(y(t),v(t))$ belongs to the space X_4 with $y_1(x)\in H^3(R), v_0(x)-\overline{v}\in H^6(R)$. This a priori estimate (2.15) is also valid for the solution $(y(t),v(t))$ in X_3 by use of the Friedrich's mollifier under the same assumptions (2.13) and (2.14), which we omit.

To obtain the a priori estimate of $E(t)$, we establish certain L_2-estimates on $\theta(x,t)$ first.

By the definition (2.7) of θ, the equation (1.12) can be written as

$$\theta_t=q(\theta,x)_{xx}, \tag{2.16}$$

where $q(\theta,x)$ is defined in (2.9).

Multiplying (2.16) by $q(\theta,x)$, using the definition of (2.10) on $Q(\theta,x)$ and (2.9) on $q(\theta,x)$, integrating then over $[t_0,t]\times(-\infty,\infty)$, after the integration by parts we get

$$\int_{-\infty}^{\infty} Q(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty} q_x^2(x,\tau)dxd\tau=\int_{-\infty}^{\infty} Q(x,t_0)dx, \tag{2.17}$$

where q_x denotes $\frac{\partial}{\partial x}[q(\theta,x)]$.

Define $\overline{V}=\frac{\partial}{\partial x}[q(\theta,x)]$. Then

$$\overline{V}_t=q_{xt}.$$

Multiplying the above equation by $\overline{V}$ and integrating over $[t_0,t]\times(-\infty,\infty)$ with the help of integration by parts, we arrive at

$$\int_{-\infty}^{\infty}\frac{1}{2}q_x^2(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}(-p_v)\theta_t^2(x,\tau)dxd\tau=\int_{-\infty}^{\infty}\frac{1}{2}q_x^2(x,t_0)dx. \tag{2.18}$$

Differentiate (2.16) with respect to t and multiply the equation by θ_t. Then one obtains the following equation by integration:

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\theta_t^2}{2}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}[-p_v(v,s)]\theta_{tx}^2(x,\tau)dxd\tau\\&=\int_{-\infty}^{\infty}\frac{\theta_t^2}{2}(x,t_0)dx-\int_{t_0}^{t}\int_{-\infty}^{\infty}\theta_{tx}\cdot\theta_t\{p_{vv}(\theta_x+\widehat{v}')+p_{vs}s'\}(x,\tau)dxd\tau.\end{aligned} \tag{2.19}$$

(2.17)–(2.19) yield, by using Cauchy inequality,

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big\{Q+\frac{1}{2}q_x^2+\frac{1}{2}\theta_t^2\Big\}(x,t)dx+(a_0-\delta M)\int_{t_0}^{t}\int_{-\infty}^{\infty}[\theta_t^2+\theta_{tx}^2](x,\tau)dxd\tau\\&\le\int_{-\infty}^{\infty}\Big\{Q+\frac{1}{2}q_x^2+\frac{1}{2}\theta_t^2\Big\}(x,t_0)dx.\end{aligned} \tag{2.20}$$

Hereafter, M denotes the constant which only depends on the bound of $\frac{\partial^i p}{\partial v^l\partial s^{(i-l)}}$ $(1\le i\le 4,\ 0\le l\le i)$ in Ω, and

$$\delta=\max\{\|\theta(t)\|_2,\delta_0\}.$$

Differentiate (2.16) with respect to t and multiply the resulting equation by θ_{tt}, integrate it over $[t_0,t]\times(-\infty,\infty)$ then, we get

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big[-p_v(v,s)\cdot\frac{\theta_{xt}^2}{2}\Big](x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}\theta_{tt}^2(x,\tau)dxd\tau\\&=\int_{-\infty}^{\infty}\Big[-p_v(v,s)\cdot\frac{\theta_{xt}^2}{2}\Big](x,t_0)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}(-p_v)_t\cdot\frac{\theta_{xt}^2}{2}dxd\tau\\&\quad-\int_{t_0}^{t}\int_{-\infty}^{\infty}\theta_{tt}\{[(p_{vv}(\theta_x+\widehat{v}')+p_{vs}s')\theta_t]_x\}(x,\tau)dxd\tau.\end{aligned} \tag{2.21}$$

Differentiating (2.16) with respect to t and x successively and multiplying the resulting equation by θ_{xt}, integrating it over $[t_0,t]\times(-\infty,\infty)$, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\theta_{xt}^2}{2}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}[-p_v(v,s)\cdot\theta_{xxt}^2](x,\tau)dxd\tau\\&=\int_{-\infty}^{\infty}\frac{\theta_{xt}^2}{2}(x,t_0)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}\theta_{xxt}\{2\theta_{xt}[p_{vv}(\theta_x+\widehat{v}')\\&\quad+p_{vs}s']+\theta_t[p_{vv}(\theta_x+\widehat{v}')+p_{vs}s']_x\}(x,\tau)dxd\tau.\end{aligned} \tag{2.22}$$

Differentiating (2.16) with respect to t twice and multiplying the equation by θ_{tt}, integrating it over $[t_0,t]\times(-\infty,\infty)$, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\theta_{tt}^2}{2}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}[-p_v(v,s)\theta_{xtt}^2](x,\tau)dxd\tau\\&=\int_{-\infty}^{\infty}\frac{\theta_{tt}^2}{2}(x,t_0)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}\theta_{ttx}\{\theta_{tt}[p_{vv}(\theta_x+\widehat{v}')+p_{vs}s']\\&\quad+\theta_t[p_{vv}(\theta_x+\widehat{v}')+p_{vs}s']_t+\theta_{xt}(p_v)_t\}(x,\tau)dxd\tau.\end{aligned}\tag{2.23}$$

It follows from (2.21)–(2.23) that

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac{\theta_{tt}^2}{2}+\frac{[1-p_v(v,s)]}{2}\theta_{xt}^2\right\}(x,t)dx+(1-\delta M)\int_{t_0}^{t}\int_{-\infty}^{\infty}\theta_{tt}^2dxd\tau\\&+(a_0-\delta M)\int_{t_0}^{t}\int_{-\infty}^{\infty}[\theta_{xxt}^2+\theta_{ttx}^2](x,\tau)dxd\tau\\&\le\int_{-\infty}^{\infty}\left\{\frac{\theta_{tt}^2}{2}+\frac{[1-p_v(v,s)]}{2}\theta_{xt}^2\right\}(x,t_0)dx+\delta M\int_{t_0}^{t}\int_{-\infty}^{\infty}(\theta_t^2+\theta_{tx}^2)dxd\tau.\end{aligned}\tag{2.24}$$

Differentiating (2.16) with respect to t twice and x once successively, multiplying the equation by θ_{ttx} and integrating it over $[t_0,t]\times(-\infty,\infty)$, we arrive at

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\theta_{ttx}^2}{2}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}[-p_v(v,s)\cdot\theta_{ttxx}^2](x,\tau)dxd\tau\\&=\int_{-\infty}^{\infty}\frac{\theta_{ttx}^2}{2}(x,t_0)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}\theta_{ttxx}\{2(p_v)_t\cdot\theta_{txx}+2(p_v)_x\cdot\theta_{ttx}+(p_v)_{xx}\cdot\theta_{tt}\\&\quad+2(p_v)_{tx}\theta_{tx}+2(p_{vv})_x\theta_t\cdot\theta_{tx}+(p_{vv})_{xx}\theta_t^2\}(x,\tau)dxd\tau.\end{aligned}\tag{2.25}$$

Differentiating (2.16) with respect to t once and x twice successively, multiplying it by θ_{xxt} and integrating over $[t_0,t]\times(-\infty,\infty)$, it turns out

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\theta_{xxt}^2}{2}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}[-p_v(v,s)\cdot\theta_{xxxt}^2](x,\tau)dxd\tau\\&=\int_{-\infty}^{\infty}\frac{\theta_{xxt}^2}{2}(x,t_0)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}\theta_{xxxt}\{\theta_{xxt}\cdot3(p_v)_x+(p_v)_t\theta_{xxx}\\&\quad+3(p_v)_{xx}\cdot\theta_{tx}+\theta_t[2(p_{vv})_x\cdot\theta_{xx}+(p_{vv})_{xx}\cdot\theta_x+(p_{vv}\widehat{v}'+p_{vs}s')_{xx}]\}(x,\tau)dxd\tau,\end{aligned}\tag{2.26}$$

where θ_{xxx} can be expressed in terms of $\theta_{tx},\theta_t,\theta_x,\theta$ by differentiating (2.16) with respect to x.

(2.25) and (2.26) yield

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\frac{\theta_{xxt}^2}{2}+\frac{\theta_{ttx}^2}{2}\right)(x,t)dx+(a_0-\delta M)\int_{t_0}^{t}\int_{-\infty}^{\infty}(\theta_{xxxt}^2+\theta_{xxtt}^2)(x,\tau)dxd\tau\\&\le\int_{-\infty}^{\infty}\left(\frac{\theta_{xxt}^2}{2}+\frac{\theta_{ttx}^2}{2}\right)(x,t_0)dx+\delta M\int_{t_0}^{t}\int_{-\infty}^{\infty}[\theta_{xxt}^2+\theta_{xtt}^2+\theta_{tt}^2+\theta_{xt}^2+\theta_t^2]dxd\tau.\end{aligned}\tag{2.27}$$

We turn to the L_2-estimates on $y(t,x)$ next

$$y_{tt}+[p(y_x+v,s)-p(v,s)]_x+y_t-p(v,s)_{xt}=0.\tag{2.28}$$

Multiply the equation (2.28) by y and y_t respectively and integrate then over $[t_0,t]\times$

$(-\infty,\infty)$. After the integration by parts, we arrive at

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big(y\cdot y_t+\frac{y^2}{2}\Big)(x,t)dx+\int_{t_0}^{t}\int_{\infty}^{\infty}[-p_v(\sigma y_x+v,s)\cdot y_x^2](x,\tau)dxd\tau\\ =&\int_{-\infty}^{\infty}\Big(y\cdot y_t+\frac{y^2}{2}\Big)(x,t_0)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_t^2(x,\tau)dxd\tau\\ &+\int_{t_0}^{t}\int_{-\infty}^{\infty}[-p_v(v,s)v_t\cdot y_x](x,\tau)dxd\tau,\end{aligned}\tag{2.29}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac{y_t^2}{2}+\frac{[-p_v(\sigma y_x+v,s)]}{2}\cdot y_x^2\right\}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_t^2dxd\tau\\ =&\int_{-\infty}^{\infty}\left\{\frac{y_t^2}{2}+\frac{[-p_v(\sigma y_x+v,s)]}{2}\cdot y_x^2\right\}(x,t_0)dx\\ &+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_t\cdot[p_{vv}(v,s)\cdot v_x\cdot v_t+p_{vs}(v,s)v_t\cdot s'+p_v(v,s)\cdot v_{tx}]dxd\tau\\ &-\int_{t_0}^{t}\int_{-\infty}^{\infty}\frac{y_x^2}{2}[p_v(\sigma y_x+v,s)]_t(x,\tau)dxd\tau,\end{aligned}\tag{2.30}$$

where $0<\sigma<1, (y_x+v,s)\in\Omega$ and $(v,s)\in\Omega$.

By using Cauchy inequality with (2.29) and (2.30), it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac14\Big[y\cdot y_t+\frac{y^2}{2}\Big]+\frac{y_t^2}{2}+\frac{[-p_v(\sigma y_x+v,s)]}{2}\cdot y_x^2\right\}(x,t)dx\\ &+\Big(\frac{3a_0}{16}-\delta_1\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}y_x^2dxd\tau+\Big(\frac12-\delta_1\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}y_t^2dxd\tau\\ \le&\int_{-\infty}^{\infty}\left\{\frac14\Big[y\cdot y_t+\frac{y^2}{2}\Big]+\frac{y_t^2}{2}+\frac{[-p_v(\sigma y_x+v,s)]}{2}\cdot y_x^2\right\}(x,t_0)dx\\ &+a_1^2\int_{t_0}^{t}\int_{-\infty}^{\infty}v_{tx}^2dxd\tau+\Big(\frac{a_1^2}{4a_0}+\delta_1\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}v_t^2dxd\tau,\end{aligned}\tag{2.31}$$

where $\delta_1=\delta_1(\|y(t),\theta(t)\|_2,\delta_0)$.

It is clear that there exists an $\varepsilon>0$ such that if (2.13) and (2.14) are true, then

$$\delta_1\le\min\left\{\frac{a_0}{16},\frac14\right\}=\frac{a_0}{16}\text{ and }\delta M\le\frac{a_0}{2}.\tag{2.32}$$

Thus, it follows from (2.31), (2.32) and (2.20) that

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac14\Big[y\cdot y_t+\frac{y^2}{2}\Big]+\frac{y_t^2}{2}+\frac{[-p_v(\sigma y_x+v,s)]}{2}\cdot y_x^2\right\}(x,t)dx\\ &+\frac{a_0}{8}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_x^2dxd\tau+\frac14\int_{t_0}^{t}\int_{-\infty}^{\infty}y_t^2dxd\tau\\ \le&\int_{-\infty}^{\infty}\left\{\frac14\Big[y\cdot y_t+\frac{y^2}{2}\Big]+\frac{y_t^2}{2}+\frac{[-p_v(\sigma y_x+v,s)]}{2}\cdot y_x^2\right\}(x,t_0)dx\\ &+a_1^2\int_{t_0}^{t}\int_{-\infty}^{\infty}v_{tx}^2dxd\tau+\Big(\frac{a_1^2}{4a_0}+\frac{a_0}{16}\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}v_t^2dxd\tau\end{aligned}$$

and

$$\int_{-\infty}^{\infty}\Big\{Q+\frac{1}{2}q_x^2+\frac{1}{2}\theta_x^2\Big\}(x,t)dx+\frac{a_0}{2}\int_{t_0}^{t}\int_{-\infty}^{\infty}(\theta_t^2+\theta_{tx}^2)dxd\tau$$
$$\le\int_{-\infty}^{\infty}\Big\{Q+\frac{1}{2}q_x^2+\frac{1}{2}\theta_t^2\Big\}(x,t_0)dx.$$

Combining these two inequalities, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac{1}{4}\Big[y\cdot y_t+\frac{y^2}{2}\Big]+\frac{y_t^2}{2}+\frac{[-p_v(v+\sigma y_x,s)]}{2}\cdot y_x^2+M_1\Big[Q+\frac{1}{2}q_x^2+\frac{1}{2}\theta_t^2\Big]\right\}(x,t)dx\\&+\frac{a_0}{8}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_x^2dxd\tau+\frac{1}{4}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_t^2dxd\tau+M_1\cdot\frac{a_0}{4}\int_{t_0}^{t}\int_{-\infty}^{\infty}(\theta_t^2+\theta_{tx}^2)dxd\tau\\&\le\int_{-\infty}^{\infty}\left\{\frac{1}{4}\Big[y\cdot y_t+\frac{y^2}{2}\Big]+\frac{y_t^2}{2}+\frac{[-p_v(v+\sigma y_x,s)]}{2}\cdot y_x^2+M_1\Big[Q+\frac{1}{2}q_x^2+\frac{1}{2}\theta_t^2\Big]\right\}(x,t_0)dx,\end{aligned}\tag{2.33}$$

where

$$M_1=\max\Big\{\frac{4a_1^2}{a_0},\Big(\frac{1}{4}+\frac{a_1^2}{a_0^2}\Big)\Big\}.\tag{2.34}$$

This gives

$$\begin{aligned}&E_1(t)+\frac{a_0}{8}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_x^2dxd\tau+\frac{1}{4}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_t^2dxd\tau+a_1^2\int_{t_0}^{t}\int_{-\infty}^{\infty}(\theta_t^2+\theta_{tx}^2)dxd\tau\\&\le E_1(t_0),\end{aligned}\tag{2.35}$$

where $E_1(t)$ is defined as in (2.6).

Differentiate (2.28) with respect to t and multiply by y_{tt} then. We obtain the following equation by integration.

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{tx}^2\right\}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tt}^2dxd\tau\\&=\int_{-\infty}^{\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{tx}^2\right\}(x,t_0)dx-\int_{t_0}^{t}\int_{-\infty}^{\infty}[p_v(y_x+v,s)]_t\cdot\frac{y_{tx}^2}{2}dxd\tau\\&\quad-\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tt}\{[p_v(y_x+v,s)-p_v(v,s)]v_t\}_xdxd\tau+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tt}\{p_v(v,s)v_t\}_{xt}dxd\tau.\end{aligned}\tag{2.36}$$

Differentiate (2.28) with respect to x and multiply by y_{xt} then. We obtain the equation below by integration

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac{y_{tx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xx}^2\right\}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tx}^2dxd\tau\\&=\int_{-\infty}^{\infty}\left\{\frac{y_{tx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}y_{xx}^2\right\}(x,t_0)dx-\int_{t_0}^{t}\int_{-\infty}^{\infty}[p_v(y_x+v,s)]_t\cdot\frac{y_{xx}^2}{2}dxd\tau\\&\quad-\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tx}\{[p_v(y_x+v,s)-p_v(v,s)]v_x+[p_s(y_x+v,s)-p_s(v,s)]s'\}_xdxd\tau\\&\quad+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tx}\{p_v(v,s)v_t\}_{xx}dxd\tau.\end{aligned}\tag{2.37}$$

By using Cauchy inequality to (2.36) and (2.37), it reads

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[1-p_v(y_x+v,s)]}{2}y_{tx}^2+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xx}^2\right\}(x,t)dx\\&+(\frac{1}{2}-\delta_1)\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_{tt}^2+y_{tx}^2)dxd\tau\\\le&\int_{-\infty}^{\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[1-p_v(y_x+v,s)]}{2}\cdot y_{tx}^2+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xx}^2\right\}(x,t_0)dx\\&+\frac{a_1^2}{2}\int_{t_0}^{t}\int_{-\infty}^{\infty}(v_{ttx}^2+v_{txx}^2)dxd\tau+\delta_1\int_{t_0}^{t}\int_{-\infty}^{\infty}[y_t^2+y_x^2+v_{tt}^2+v_{tx}^2+v_t^2]dxd\tau.\end{aligned}\tag{2.38}$$

Due to (2.32) and (2.24), it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[1-p_v(y_x+v,s)]}{2}\cdot y_{tx}^2+\frac{[-p_v(y_x+v,s)]}{2}y_{xx}^2\right.\\&\left.+M_2\left[\frac{\theta_{tt}^2}{2}+\frac{[1-p_v(v,s)]}{2}\cdot\theta_{xt}^2\right]\right\}(x,t)dx\\&+\frac{7}{16}\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_{tt}^2+y_{tx}^2)dxd\tau+\frac{a_0}{4}M_2\int_{t_0}^{t}\int_{-\infty}^{\infty}(\theta_{ttx}^2+\theta_{xxt}^2+\theta_{tt}^2)dxd\tau\\\le&\int_{-\infty}^{\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[1-p_v(y_x+v,s)]}{2}\cdot y_{tx}^2+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xx}^2\right.\\&\left.+M_2\left[\frac{\theta_{tt}^2}{2}+\frac{[1-p_v(v,s)]}{2}\theta_{xt}^2\right]\right\}(x,t_0)dx\\&+\frac{a_0}{16}\int_{t_0}^{t}\int_{-\infty}^{\infty}[y_t^2+y_x^2]dxd\tau+\left(\frac{a_0}{2}M_2+\frac{a_0}{16}\right)\int_{t_0}^{t}\int_{-\infty}^{\infty}(v_t^2+v_{tx}^2)dxd\tau,\end{aligned}\tag{2.39}$$

where

$$M_2=\frac{2a_1^2}{a_0}.\tag{2.40}$$

This yields

$$\begin{aligned}&E_2(t)+\frac{7}{16}\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_{tt}^2+y_{tx}^2)dxd\tau+\frac{a_1^2}{2}\int_{t_0}^{t}\int_{-\infty}^{\infty}(\theta_{ttx}^2+\theta_{xxt}^2+\theta_{tt}^2)dxd\tau\\\le&E_2(t_0)+\frac{a_0}{16}\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_t^2+y_x^2)dxd\tau+\left(a_1^2+\frac{a_0}{16}\right)\int_{t_0}^{t}\int_{-\infty}^{\infty}(v_t^2+v_{tx}^2)dxd\tau\\\le&E_2(t_0)+K_0E_1(t_0),\quad\text{due to (2.35)},\end{aligned}\tag{2.41}$$

where $E_2(t)$ is defined as in (2.6), $K_0>0$ depends only on Ω and p.

Differentiate (2.28) with respect to x and t successively and multiply the resulting equation by y_{ttx}, integrate it then over $[t_0,t]\times(-\infty,\infty)$. One obtains the following inequality with the help of (2.28) and differentiating (2.28) with respect to x,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xxt}^2\right\}(x,t)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{ttx}^2dxd\tau\\
&\le\int_{-\infty}^{\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xxt}^2\right\}(x,t_0)dx\\
&\quad+\int_{t_0}^{t}\int_{-\infty}^{\infty}|(-p_v(v,s))\cdot y_{ttx}\cdot v_{ttxx}|dxd\tau\\
&\quad+\delta_1\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_{ttx}^2+y_{xxt}^2+y_{tt}^2+y_{tx}^2+y_t^2+y_x^2)dxd\tau\\
&\quad+\delta_1\int_{t_0}^{t}\int_{-\infty}^{\infty}(v_{ttx}^2+v_{txx}^2+v_{tt}^2+v_{tx}^2+v_t^2)dxd\tau.
\end{aligned}
\tag{2.42}
$$

To estimate the term of $\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{xxt}^2$, we differentiate (2.28) with respect to t and multiply it by y_{xxt}. Integrate over $[t_0,t]\times(-\infty,\infty)$ then, we arrive at

$$
\begin{aligned}
&\int_{t_0}^{t}\int_{-\infty}^{\infty}[-p_v(y_x+v,s)]\cdot y_{xxt}^2dxd\tau\\
&\le\frac{1}{2}\int_{-\infty}^{\infty}y_{tt}^2(x,t)dx+\frac{1}{2}\int_{-\infty}^{\infty}y_{xxt}^2(x,t)dx\\
&\quad-\int_{-\infty}^{\infty}(y_{tt}\cdot y_{xxt})(x,t_0)dx+\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{ttx}^2dxd\tau\\
&\quad+\int_{t_0}^{t}\int_{-\infty}^{\infty}|y_{xxt}\{y_{tt}-p_v(v,s)\cdot v_{xtt}\}|dxd\tau\\
&\quad+\delta_1\int_{t_0}^{t}\int_{-\infty}^{\infty}[y_{xxt}^2+y_{xt}^2+y_x^2+v_{tt}^2+v_t^2]dxd\tau.
\end{aligned}
\tag{2.43}
$$

Due to (2.41) and (2.42), (2.43) implies that

$$
\begin{aligned}
&\frac{5a_0}{8}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{xxt}^2dxd\tau\\
&\le 2E_2(t_0)+K_0E_1(t_0)+\frac{2}{a_0}\int_{-\infty}^{\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xxt}^2\right\}(x,t_0)dx\\
&\quad+\Big(\frac{5}{4}+\frac{a_0}{16}\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{ttx}^2dxd\tau+\frac{a_0}{8}\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_{tx}^2+y_t^2+y_x^2)dxd\tau\\
&\quad+\Big(\frac{2}{a_0}+\frac{a_0}{16}\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tt}^2dxd\tau+\frac{a_1^2}{a_0^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}v_{ttxx}^2dxd\tau\\
&\quad+\Big(\frac{2a_1^2}{a_0}+\frac{a_0}{16}\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}v_{ttx}^2dxd\tau+\frac{a_0}{8}\int_{t_0}^{t}\int_{-\infty}^{\infty}(v_{xxt}^2+v_{tt}^2+v_{tx}^2+v_t^2)dxd\tau
\end{aligned}
\tag{2.44}
$$

provided

$$
\delta_1\le\frac{a_0^2}{16}. \tag{2.45}
$$

Substituting (2.44) into (2.42), we end up with

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xxt}^2\right\}(x,t)dx+\frac{1}{4}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{ttx}^2dxd\tau\\
&\le(1+\frac{1}{5})\int_{-\infty}^{\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xxt}^2\right\}(x,t_0)dx\\
&\quad+\frac{a_0}{10}[2E_2(t_0)+K_0E_1(t_0)]+\Big(\frac{1}{5}+\frac{11}{160}a_0^2\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tt}^2dxd\tau\\
&\quad+\frac{3a_0^2}{40}\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_{tx}^2+y_t^2+y_x^2)dxd\tau+\frac{a_1^2}{10a_0}\int_{t_0}^{t}\int_{-\infty}^{\infty}v_{ttxx}^2dxd\tau\\
&\quad+\Big(\frac{a_1^2}{5}+\frac{11}{160}a_0^2\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}v_{ttx}^2dxd\tau\\
&\quad+\frac{3a_0^2}{40}\int_{t_0}^{t}\int_{-\infty}^{\infty}(v_{xxt}^2+v_{tt}^2+v_{tx}^2+v_t^2)dxd\tau
\end{aligned}\tag{2.46}$$

provided (2.45) holds.

By combining (2.46) with (2.27), it reads

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xxt}^2+M_3[\frac{\theta_{xxt}^2}{2}+\frac{\theta_{ttx}^2}{2}]\right\}(x,t)dx\\
&\quad+\frac{1}{4}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{ttx}^2dxd\tau+\frac{a_0}{4}M_3\int_{t_0}^{t}\int_{-\infty}^{\infty}[\theta_{xxxt}^2+\theta_{xxtt}^2]dxd\tau\\
&\le(1+\frac{1}{5})\int_{-\infty}^{\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\cdot y_{xxt}^2+M_3[\frac{\theta_{xxt}^2}{2}+\frac{\theta_{ttx}^2}{2}]\right\}(x,t_0)dx\\
&\quad+\Big(\frac{1}{5}+\frac{11}{160}a_0^2\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tt}^2dxd\tau+\frac{a_0}{10}[2E_2(t_0)+K_0E_1(t_0)]\\
&\quad+\frac{3a_0^2}{40}\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_{tx}^2+y_t^2+y_x^2)dxd\tau+\Big(\frac{a_1^2}{5}+\frac{11a_0^2}{160}+\frac{a_0}{2}M_3\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}v_{ttx}^2dxd\tau\\
&\quad+\Big(\frac{3a_0^2}{40}+\frac{a_0}{2}M_3\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}(v_{xxt}^2+v_{tt}^2+v_{tx}^2+v_t^2)dxd\tau,
\end{aligned}$$

where

$$M_3=\frac{2a_1^2}{5a_0^2}.\tag{2.47}$$

Namely,

$$\begin{aligned}
&E_3(t)+\frac{1}{4}\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{ttx}^2dxd\tau+\frac{a_1^2}{10a_0}\int_{t_0}^{t}\int_{-\infty}^{\infty}(\theta_{xxxt}^2+\theta_{xxtt}^2)dxd\tau\\
&\le(1+\frac{1}{5})E_3(t_0)+\frac{a_0}{10}[2E_2(t_0)+K_0E_1(t_0)]+\frac{3a_0^2}{40}\int_{t_0}^{t}\int_{-\infty}^{\infty}(y_{tx}^2+y_t^2+y_x^2)dxd\tau\\
&\quad+\Big(\frac{1}{5}+\frac{11}{160}a_0^2\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}y_{tt}^2dxd\tau+\Big(\frac{a_1^2}{5}+\frac{11a_0^2}{160}+\frac{a_1^2}{5a_0}\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}v_{ttx}^2dxd\tau\\
&\quad+\Big(\frac{3a_0^2}{40}+\frac{a_1^2}{5a_0}\Big)\int_{t_0}^{t}\int_{-\infty}^{\infty}(v_{xxt}^2+v_{tt}^2+v_{tx}^2+v_t^2)dxd\tau\\
&\le(1+\frac{2}{a_0})E_3(t_0)+\widehat{K}[E_2(t_0)+E_1(t_0)],\quad\text{due to (2.35) and (2.41),}
\end{aligned}\tag{2.48}$$

where $E_3(t)$ is defined as in (2.6).

By (2.35), (2.41) and (2.48), we arrive at the a priori estimate (2.15) under the assumption (2.13) and (2.14), where ε is chosen so that (2.32) and (2.45) are satisfied.

This a priori estimate is also valid for the solution $(y(t), v(t))$ belonging to the space X_3 under the same assumption (2.13) and (2.14) by use of the Friedrich's mollifier. Lemma 2.1 follows then.

§3. The Main Theorem

Theorem 3.1. *Under the Hypotheses* 2.1 *and* 2.2 *there exists a constant* $\varepsilon_0 > 0$ *such that if the initial data are small as* $E(0) < \varepsilon_0$ *and* $|s_0(x) - \overline{s}|_{C^3} < \varepsilon_0$*, then the Cauchy problem* (1.10)–(1.13) *has a unique smooth solution in the large in time. The solution* $(y(t), v(t) - \widehat{v})$ *decays to zero in the* L_∞ *norm as* $t \to \infty$ *and so do their first derivatives.*

Proof. We choose the initial data so small that

$$E(0) < \frac{\varepsilon^2}{4K^2K_2^2} \tag{3.1}$$

and

$$|s_0(x) - \overline{s}|_{C^3} < \frac{\varepsilon^2}{2K_1}, \tag{3.2}$$

where ε and K_2 are the same as in Lemma 2.1, K and K_1 are the same as in (2.11).

By the local existence theorem there exists $t_1 > 0$ such that the solution $(y(t), v(t))$ exists in $0 \le t \le t_1$ and satisfies

$$E(t) \le 2K_2E(0) \text{ and } (y_x + v, s) \in \Omega, (v, s) \in \Omega, \quad \text{in } 0 \le t \le t_1.$$

It follows by (2.11) then that

$$\begin{aligned} \|y(t), v(t)\|_2^2 &\le K^2E(t) + K_1|s - \overline{s}|_{C^2} \\ &\le 2K^2K_2E(0) + K_1|s_0(x) - \overline{s}|_{C^2} \\ &< \varepsilon^2 \quad \text{in } 0 \le t \le t_1. \end{aligned} \tag{3.3}$$

Thus, Lemma 2.1 implies that

$$E(t) \le K_2E(0) \quad \text{in } 0 \le t \le t_1. \tag{3.4}$$

Therefore, $(y_x + v, s) \in \widehat{\Omega}, (v, s) \in \widehat{\Omega}$, in $0 \le t \le t_1$, where

$$\widehat{\Omega} = \{(v, s) : 0 < \overline{v} - \widehat{r} \le v \le \overline{v} + \widehat{r}, \overline{s} - \widehat{r} \le s \le \overline{s} + \widehat{r}, 0 < r^* < \widehat{r} < r\}.$$

Next, by the local existence theorem for $t \ge t_1$, there exists $\widetilde{t} > 0$ such that the solution $(y(t), v(t))$ exists in $0 \le t \le t_1 + \widetilde{t}$ and satisfies

$$E(t) \le 2K_2E(t_1) \text{ and } (y_x + v, s) \in \Omega, \quad (v, s) \in \Omega, \quad \text{in } t_1 \le t \le t_1 + \widetilde{t}. \tag{3.5}$$

In view of (2.11), (3.1), (3.2), (3.4) and (3.5), it reads

$$\begin{aligned} \|y(t), v(t)\|_2^2 &\le K^2E(t) + K_1|s - \overline{s}|_{C^2} \\ &\le 2K^2K_2^2E(0) + K_1|s_0(x) - \overline{s}|_{C^2} \\ &< \varepsilon^2 \quad \text{in } t_1 \le t \le t_1 + \widetilde{t}. \end{aligned} \tag{3.6}$$

Therefore, (3.3), (3.6) and Lemma 2.1 imply

$$E(t) \le K_2E(0) \text{ in } 0 \le t \le t_1 + \widetilde{t}. \tag{3.7}$$

Also

$$(y_x+v,s)\in\widehat{\Omega},\quad (v,s)\in\widehat{\Omega},\ \text{in } 0\le t\le t_1+\widetilde{t}.$$

Repeating the same procedure with the same time interval $\widetilde{t}>0$, we complete the proof of the global existence of the solution.

We prove the decay of solution now.

By using Cauchy inequality, it follows from (2.35) with taking $t_0=0$ that

$$\int_{-\infty}^{\infty}\Big(\frac{1}{4}y_t^2+\frac{a_0}{2}y_x^2\Big)(x,t)dx+\frac{a_0}{8}\int_0^t\int_{-\infty}^{\infty}y_x^2dxd\tau+\frac{1}{4}\int_0^t\int_{-\infty}^{\infty}y_t^2dxd\tau\le E_1(0),$$

which implies that

$$\int_{-\infty}^{\infty}\Big(y_t^2+y_x^2\Big)(x,t)dx\to 0 \text{ as } t\to\infty. \tag{3.8}$$

On the other hand, it can be shown, due to (2.35), that

$$y^2(x,t)\le\widetilde{k}\Big(\int_{-\infty}^{\infty}y_x^2dx\Big)^{1/2}, \tag{3.9}$$

where $\widetilde{k}$ is a positive constant independent of t.

(3.8) and (3.9) show that

$$y^2(x,t)\to 0 \text{ as } t\to\infty,\ \text{uniformly for } x\in(-\infty,\infty). \tag{3.10}$$

Next, it reads from (2.41) that

$$\int_{-\infty}^{\infty}\Big[\frac{y_{tt}^2}{2}+\frac{(1+a_0)}{2}y_{tx}^2\Big](x,t)dx+\frac{7}{16}\int_0^t\int_{-\infty}^{\infty}(y_{tt}^2+y_{tx}^2)dxd\tau\le E_2(0)+K_0E_1(0),$$

which yields

$$\int_{-\infty}^{\infty}(y_{tt}^2+y_{tx}^2)(x,t)dx\to 0 \text{ as } t\to\infty. \tag{3.11}$$

In view of (2.35), it is also true that

$$y_t^2(x,t)\le\widetilde{k}\left[\int_{-\infty}^{\infty}y_{tx}^2dx\right]^{1/2}. \tag{3.12}$$

This, combined with (3.11), gives

$$y_t^2(x,t)\to 0 \text{ as } t\to\infty \text{ uniformly for } x\in(-\infty,\infty). \tag{3.13}$$

Similarly, it can be shown that

$$y_x^2(x,t)\to 0 \text{ as } t\to\infty \text{ uniformly for } x\in(-\infty,\infty). \tag{3.14}$$

The decay of $(v(x,t)-\widehat{v})$ can be discussed in the same way, as in [3].

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