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IRRATIONAL ROTATION C^*-ALGEBRA FOR GROUPOID C^*-ALGEBRA**

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Abstract

This paper characterizes the irrational rotaion C^*-algebra associated with the Toeplitz C^*-algebra over the L-shaped domain in $\mathbb{C}^2$ in the sense of the maximal radical series, which is an isomorphism invariant.

Keywords L-shaped domain, Toeplitz C^*-algebra, Irrational rotation, Invariant subset, Maximal ideal.
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§1. Preliminaries

Irrational rotation C^*-algebra on the unit circle was first studied by M. Rieffel in [13]. Since 1980's many people have paid great attention to this subject (see [15], [16], [17], [18] and [19]). It has played an important part in the analysis of C^*-algebras, K-theory and index theory. In recent years the study of rotation C^*-algebrs on group C^*-algebras and Toeplitz C^*-algebras has been developed. For Example, in [19] Handelman and Yin obtained a complete invariant for rotation C^*-algebra of Toeplitz C^*-algebra on the polydisk. In this paper from the view of groupoid we establish the structure of rotation C^*-algebras of Toeplitz C^*-algebras on L-shaped domain in $\mathbb{C}^2$. This idea will get further developing in our other papers.

Suppose that Y is a locally compact Hausdoff and second countable space, and X is a both open and compact subset of Y. $\mathbb{Z}^n$ acts on Y on the right continuously so that $(Y\mathbb{Z}^n)$ is a transfomation group. For the n-tuple θ in $\mathbb{T}^n$, define the homomorphism $C_\theta : Y \times \mathbb{Z}^n \to \mathbb{T}$ by

$$C_\theta(y,p) = \theta^p.$$

Denote the reduction of $Y \times \mathbb{Z}^n$ on X by G. Then the reduction of the skew product $(Y \times \mathbb{Z}^n)(C_\theta)$ on $X \times \mathbb{T}$ is the skew product $G(C_\theta)$.

Proposition 1.1. *The groupoid G and $G(C_\theta)$ are r-discrete and amenable.*

The skew product $G(C_\theta) = G \times_{C_\theta} \mathbb{T}$ is a locally compact groupoid with composable pairs

$$G(C_\theta)^{(2)} = \{((x,p,a),(y,q,b)) | ((x,p),(y,q)) \in G^{(2)} \text{and } b = a\theta^p\}.$$

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The product is

$$(x,p,a)(y,q,b)=(x,p+q,a),$$

and the inverse is

$$(x,p,a)^{-1}=(x+p,-p,a\theta^p).$$

The domain map is

$$d(x,p,a)=(x+p,0,\theta^p),$$

and the range map is

$$r(x,p,a)=(x,0,a).$$

So the unit space may be identified with $X\times \mathbb{T}$.

Remark 1.1. Indeed, $G(C_\theta)$ is the reduction of the skew product of $(Y\times \mathbb{Z}^p)(C_\theta)$ on $X\times\mathbb{T}$.

Proposition 1.2. $C^*(G(C_\theta))\cong C^*(G)\times_{\alpha_\theta}\mathbb{Z}$.

Proposition 1.3. *The groupoid $(Y\times\mathbb{Z}^p)(C_\theta)$ is principal if there is no solution of nonzero integers to the equation $\theta^p=1$.*

Proof. We have to prove that the isotropy group $(Y\times\mathbb{Z}^n)(C_\theta)|_u$ at every point u in the unit space is trivial. Suppose that (x,m,a) is in the isotropy group of u. Then we have

$$(x+p,0,\theta^p a)=(x,0,a)=u.$$

It follows that $\theta^p=1$. Therefore $p=0$ and $(x,m,a)=u$.

Remark 1.2. $G(C_\theta)$ is principal if $(Y\times\mathbb{Z}^n)(C_\theta)$ is.

It is easy to prove the following lemma.

Lemma 1.1. *Suppose that G is a groupoid and A a group. Let $c:G\to A$ be a homomorphism. $G(c)$ is the skew product $G\times_c A$. Suppose that E is a subset of G^0. Then $E\times A$ is invariant in $G(c)$ iff E is invariant in G.*

Let $\Omega=\{(z_1,z_2)\in\mathbb{C}^2\,||z_1|<\delta_1,|z_2|<1 \text{ or } |z_1|<1,|z_2|<\delta_2\}$, where $\delta_1,\delta_2<1$. Then Ω, named L-shaped domain, is a Reinhardt domain in $\mathbb{C}^2$. P. E. Curto and P. S. Muhly have represented the Toeplitz C^*-algebra $C^*(\Omega)$ faithfully by a groupoid C^*-algebra $C^*(G)$[7]. Let us repeat the procedure briefly here with some new notations introduced. Let $T(p):A^2(\Omega)\to A^2(\Omega)$ be the Toeplitz operator of the symbol z^p. Then $\{T(p)|p\in\mathbb{Z}_+^2\}$ is a contractable representation of $\mathbb{Z}_+^2$ by a weighted function

$$w_+(p,q)=\frac{\|z^{p+q}\|}{\|z^q\|},$$

for $p,q\in\mathbb{Z}_+^2$.

A direct calculation shows

$$w_+(\epsilon_1,p)=\sqrt{\frac{(p_1+1)(\delta_1^{2p_1+4}+\delta_2^{2p_2+2}-\delta_1^{2p_1+4}\delta_2^{2p_2+2})}{(p_1+2)(\delta_1^{2p_1+2}+\delta_2^{2p_2+2}-\delta_1^{2p_1+2}\delta_2^{2p_2+2})}},$$

and

$$w_+(\epsilon_2,p)=\sqrt{\frac{(p_2+1)(\delta_1^{2p_1+2}+\delta_2^{2p_2+4}-\delta_1^{2p_1+2}\delta_2^{2p_2+4})}{(p_2+2)(\delta_1^{2p_1+2}+\delta_2^{2p_2+2}-\delta_1^{2p_1+2}\delta_2^{2p_2+2})}}.$$

Extend each $w_+(p,\cdot)$ to $\mathbb{Z}^2$ by taking zero on $\mathbb{Z}^2\backslash\mathbb{Z}^2_+$. Let A denote the translation-invariant C^*-subalgebra of $l^\infty(\mathbb{Z}^2)$ generated by the family $\{w(p,\cdot)|p\in\mathbb{Z}_+\}$ not including the identity. The maximal ideals space of A, denoted by Y, is locally compact and second countable. The natural action, $\tau:\mathbb{Z}^2\to \mathrm{Aut}(A)$, defined by translation induces an action of $\mathbb{Z}^2$ on Y according to this prescription: $(y+p)(a)=y(\tau_p(a))$. Since the evaluation at p gives a multiplicative linear functional, say $\alpha(p)$, we get an injection $\alpha:\mathbb{Z}^2\to Y$ both open and continuous. The subset $\overline{\alpha(\mathbb{Z}^2_+)}$, denoted by X, is open and compact. G is the reduction of $Y\times\mathbb{Z}^2$ by X as defined above. Then $C^*(\Omega)$ is faithfully represented by $C^*(G)$ (see [7]).

According to [7], Y consists of four parts, i.e.,

$$Y=\alpha(\mathbb{Z}^2)\cup\alpha(\mathbb{Z}\times\{\infty\})\cup\alpha(\{\infty\}\times\mathbb{Z})\cup\beta([-\infty,+\infty]),$$

where

$$\alpha(p_1,\infty)=\lim_{p_2\to+\infty}\alpha(p_1,p_2)$$

and

$$\alpha(\infty,p_2)=\lim_{p_1\to+\infty}\alpha(p_1,p_2)$$

in Y; and $\beta:[-\infty,+\infty]\to\infty_G$ is the realization of the subset, ∞_G, of Y consisting of all the possible limits $\lim\limits_{k_1,k_2\to+\infty}\alpha(k)$ in Y. Indeed, $\beta(t)$ is uniquely determined by

$$(\beta(t)(w(\epsilon_1,\cdot)),\beta(t)(w(\epsilon_2,\cdot))=\begin{cases}(\delta_1,1) & \text{for } \ t=-\infty,\\ \left(\sqrt{\dfrac{\delta_1^2+\exp(t)}{1+\exp(t)}},\sqrt{\dfrac{1+\delta_2^2\exp(t)}{1+\exp(t)}}\right) & \text{for } \ t\in\mathbb{R},\\ (1,\delta_2) & \text{for } \ t=+\infty.\end{cases}$$

Thus

$$X=\alpha(\mathbb{Z}^2_+)\cup\alpha(\mathbb{Z}_+\times\{\infty\})\cup\alpha(\{\infty\}\times\mathbb{Z}_+)\cup\beta([-\infty,+\infty]).$$

Given a pair of numbers $\theta=(\theta_1,\theta_2)\in\mathbb{T}^2$, satisfying the condition that there is no nonzero integer n such that $\theta_1^n=1$ or $\theta_2^n=1$, which is weaker than that in Proposition 1.3, there is an automorphism $\varphi_\theta:\Omega\to\Omega$ defined via

$$\varphi_\theta(z_1,z_2)=(\theta_1z_1,\theta_2z_2),\ \text{for } (z_1,z_2)\in\Omega.$$

Thus there is an induced C^*-dynamical system $(C^*(\Omega),\mathbb{Z},\tilde\varphi_\theta)$, where $\tilde\varphi_\theta$ is the induced automorphism of $C^*(\Omega)$ such that $\tilde\varphi_\theta(T_f)=T_{f\bullet\varphi_\theta^{-1}}$ for f in $C(\overline{\Omega})$.

Proposition 1.4. *$C^*(G)\times_{\alpha_\theta}\mathbb{Z}\cong C^*(\Omega)\times_{\tilde\varphi_\theta}\mathbb{Z}$.*

Remark 1.3. $\lim\limits_{p_1\to+\infty}\alpha(p_1,+\infty)=\beta(-\infty)$ and $\lim\limits_{p_2\to+\infty}\alpha(+\infty,p_2)=\beta(+\infty)$.

§2. Invariant Maximal Radical Series of $C^*(G(C_\theta))$

The maximal radical series of a C^*-algebra is invariant under the isomorphism. It plays an important part in the classification of some C^*-algebras. By the definition[20], the maximal radical of a C^*-algebra A is the intersection of all closed two-sided maximal ideals of A, and is denoted by $m(A)$, the composition series

$$\cdots\triangleleft m(m(A))\triangleleft m(A)\triangleleft A.$$

is called the maximal radical series. In this section we will determine the maximal radical series of the rotational C^*-algebra $C^*(G(C_\theta))$.

By [1], there is an order-preserving homomorphism from the family of the invariant open subsets to the family of the closed ideals in the reduced groupoid C^*-algebra. And now, we will first determine the minimal invariant closed subsets in the groupoid $G(C_\theta)$.

Lemma 2.1. *There are only two minimal invariant closed subsets in the unit space of the groupoid $G(C_\theta)$, i.e., $\{\beta(+\infty)\} \times \mathbb{T}$ and $\{\beta(-\infty)\} \times \mathbb{T}$, denoted by F_1 and F_2 respectively. Their complements are denoted by B_1 and B_2 respectively. Any invariant closed subset contains at least one of the F_i's.*

Proof. The F_i's are obviously minimal invariant and closed. Given an invariant closed subset F, take any u in F.

1) If u is in either F_1 or F_2, then $F_1 \subseteq F$ or $F_2 \subseteq F$.

2) If u is in $\beta(\mathbb{R}) \times \mathbb{T}$, then

$$\lim_{m\to+\infty}(u + (0, m)) = (\beta(+\infty), t)$$

for some t in $\mathbb{T}$. Hence $F \cap F_1 \neq \emptyset$, and by 1) $F_1 \subset F$.

3) If u is in $\alpha(\mathbb{Z}_+ \times \{\infty\}) \times \mathbb{T}$, then

$$\lim_{m\to+\infty}(u + (m, 0)) = (\beta(-\infty), t)$$

for some t in $\mathbb{T}$. Hence $F \cap F_2 \neq \emptyset$, and by 1) $F_2 \subset F$.

4) If u is in $\alpha(\{\infty\} \times \mathbb{Z}_+) \times \mathbb{T}$, then by the same reason as above, $F_1 \subset F$.

5) If u is in $\alpha(\mathbb{Z}_+^2) \times \mathbb{T}$, then

$$\lim_{m\to+\infty}(u + (m, 0)) = (\alpha(\infty, n), t)$$

for some t in $\mathbb{T}$. Hence by the same reason as in 4), $F_1 \subset F$.

The lemma follows now.

Remark 2.1. We have used the fact that $\{\theta^p | p \in \mathbb{Z}_+^2\}$ is dense in $\mathbb{T}$ if there is no integer n of nonzero such that $\theta_1^n = 1$ or $\theta_2^n = 1$.

Lemma 2.2. *If the ratio $\dfrac{\ln \delta_1}{\ln \delta_2}$ is irrational, the isotropy group $G(C_\theta)|_u$ is trivial for $u \notin F_1 \cup F_2$, i.e., $u \in B$.*

Proof. For any (x, p, t) in $G(C_\theta)|_u$, we have two equalities

$$x + p = x, \tag{I}$$

$$\theta^p = 1. \tag{II}$$

(1) If x is in $\alpha(\mathbb{Z}_+^2)$, say $x = \alpha(q)$, then equality (I) becomes $\alpha(q+p) = \alpha(q)$. Consequently, $p = 0$ since α is injective.

(2) If x is in $\alpha(\mathbb{Z}_+ \times \{\infty\})$, say $x = \alpha(m, +\infty)$, then equality (I) becomes $\alpha(m+p_1, \infty) = \alpha(m, \infty)$. Thus $p_1 = 0$. It follows that $p_2 = 0$ from equality (II).

(3) If x is in $\alpha(\{\infty\} \times \mathbb{Z}_+)$, then $p = 0$ by the same reason as in case (2).

(4) If x is in $\beta(-\infty, +\infty)$, say $x = \beta(s)$, then equation (I) becomes $\beta(s + 2p_2 \ln \delta_2 - 2p_1 \ln \delta_1) = \beta(s)$. It follows that $p_2 \ln \delta_2 = p_1 \ln \delta_1$. Therefore $p = 0$.

Finally we get $(x, p, t) = u$. The lemma follows.

Theorem 2.1. *The maximal radical of the groupoid C^*-algebra $C^*(G(C_\theta))$ is $\mathbf{I}(B)$.*

Proof. If $1, \dfrac{\arg\theta_1}{2\pi}$ and $\dfrac{\arg\theta_2}{2\pi}$ are linearly independent over the field $\mathbb{Q}$ of the rational numbers, the groupoid is principal, the maximal closed ideals are $\mathbf{I}(B_1)$ and $\mathbf{I}(B_2)$. Therefore the intersection of the maximal closed ideals is $\mathbf{I}(B)$.

If $1, \dfrac{\arg\theta_1}{2\pi}$ and $\dfrac{\arg\theta_2}{2\pi}$ are linearly dependent over the field $\mathbb{Q}$, we will prove the following claims

(1) There is indeed a maximal closed ideal in the groupoid C^*-algebra and the intersection of the maximal closed ideals is contained in $\mathbf{I}(B)$.

(2) Each maximal closed ideal I in the groupoid C^*-algebra contains $\mathbf{I}(B)$.

And now

$$\begin{aligned} C^*(G(C_\theta))/\mathbf{I}(B_1) &\cong C^*(G(C_\theta)|_{F_1}) \\ &= C^*(\beta\{+\infty\} \times \mathbb{T} \times \mathbb{Z}^2) \\ &\cong C^*(\mathbb{T} \times \mathbb{Z}^2) \\ &\cong C(\mathbb{T}^2) \times_{\alpha_\theta} \mathbb{Z} \end{aligned}$$

where $\alpha_\theta(f)(\lambda_1,\lambda_2) = f(\theta_1\lambda_1, \theta_2\lambda_2) = f \cdot \varphi_\theta(\lambda_1,\lambda_2)$. Since the homeomophism φ_θ is not minimal, the crossed product $C(\mathbb{T}^2)\times_{\alpha_\theta}\mathbb{Z}$ is not simple by [1]. However, the nontrivial closed ideal must be contained in some maximal ones since the crossed product is unital. Suppose that I is the maximal closed ideal in the crossed product. Then the quotient $C(\mathbb{T}^2)\times_{\alpha_\theta}\mathbb{Z}/I$ is simple. So there is a surjective homomorphism from $C^*(G(C_\theta))$ onto $C(\mathbb{T}^2)\times_{\alpha_\theta}\mathbb{Z}/I$, whose kernel is a maximal closed ideal in $C^*(G(C_\theta))$.

Since the closed orbit $\overline{\{\varphi_\theta^n(\lambda) | n \in \mathbb{Z}\}}$ is minimal for every $\lambda \in \mathbb{T}^2$, by [1] the intersection of the maximal closed ideals in the crossed product is $\{0\}$. Hence the maximal radical is contained in $\mathbf{I}(B_1)$. A similar argument shows that the maximal radical is contained in $\mathbf{I}(B_2)$. The claim (1) follows.

For each maximal closed ideal I, there is an integrated representation, $\pi = (\mu, L, \mathcal{H})$, of $C^*(G(C_\theta))$ with kernel I. It follows that the representation π is weakly contained in the induced left regular representation living on μ. Therefore $\mathbf{I}(F) \subseteq \ker(\pi)$, where F denotes the support of μ. F is an invariant closed subset. By the proof of Lemma 2.1, F contains either F_1 or F_2.

If $F = F_1$ or $F = F_2$, then $\mathbf{I}(B_1) \subseteq I$ or $\mathbf{I}(B_2) \subseteq I$; thus $\mathbf{I}(B) \subseteq I$.

If $F \neq F_1$ and $F \neq F_2$, then

1) F only contains F_1. Then $\overline{F \setminus F_1}$ is a nontrivial invariant closed subset, say $\widetilde{F}_2$, contained in F. Therefore it contains F_1, i.e., $\widetilde{F}_2 = F$. By the proof of Proposition 4.4 in [1],

$$\sup_{u \in F} |f(u)| \le \|\pi(f)\|,$$

it follows that $I \subseteq \mathbf{I}(B_1)$. Therefore $I = \mathbf{I}(B_1)$. Thus $\mathbf{I}(B) \subseteq I$.

2) F only contains F_2. Then $\mathbf{I}(B) \subseteq I$ by the same reason as in case 1.

3) F contains both F_1 and F_2. Set

$$F' = F \setminus F_1 \cup F_2,\ \mu'(E) = \mu(E \cap F'),\ \mu_1(E) = \mu(E \cap (F_1 \cup F_2)).$$

Then $\pi_1 = (\mu_1, L, \mathcal{H})$ and $\pi' = (\mu', L, \mathcal{H})$ are the integrated representations of the groupoid

C^*-algebra $C^*(G(C_\theta))$. Moreover we have

$$\pi_1(f)(\xi) = \pi(f)(\chi_{F_1\cup F_2}\xi) = \chi_{F_1\cup F_2}\pi(f)(\xi),$$
$$\pi'(f)(\xi) = \pi(f)(\chi_{F'}\xi) = \chi_{F'}\pi(f)(\xi),$$
$$\ker(\pi) = \ker(\pi_1)\cap\ker(\pi').$$

Since $\ker(\pi)$ is a maximal closed ideal, it coincides with either $\ker(\pi_1)$ or $\ker(\pi')$.

(1) If $I = \ker(\pi_1)$, it follows immediately that $\mathbf{I}(B)\subseteq I$.

(2) If $I = \ker(\pi')$, one of the following cases occurs.

(i) $\mathrm{supp}(\mu') = \overline{F'}$ contains $F_1\cup F_2$. By the proof of Proposition 4.4 in [1] we get

$$\sup_{u\in F}|f(u)| \le \|\pi(f)\|,$$

and it follows immediately that $I = \mathbf{I}(G(C_\theta)^0\setminus\overline{F'})$. Since B_1 and B_2 are the maximal invariant open subsets and

$$\mathbf{I}(G(C_\theta)^0\setminus\overline{F'})\subseteq\mathbf{I}(B_1)\cap\mathbf{I}(B_2),$$

this case can not occur.

(ii) $\mathrm{supp}(\mu')$ contains only one of the F_i's. Then by the above discussion we have $\mathbf{I}(B)\subseteq I$.

The claim (2) follows now. The theorem follows from the above claims.

Lemma 2.3. *The groupoid $G(C_\theta)|_B$, denoted by $G(C_\theta)'$, is r-discrete, principal and amenable.*

Lemma 2.4. *The maximal radical of $C^*(G(C_\theta)')$ is $\mathbf{I}(\alpha(\mathbb{Z}_+^2)\times\mathbb{T})$, denoted by $C^*(G(C_\theta)'')$.*

Proof. By Lemma 2.2 and [1], there is an order-preserving isomorphism between the family of the maximal closed ideals in $C^*(G(C_\theta))$ and the family $\mathfrak{B}$ of the maximal invariant open subsets in $G(C_\theta)$. Let $\bigcap\limits_{B\in\mathfrak{B}}\mathbf{I}(B) = I$. Then there is an invariant open subset $\widetilde{B}$ such that $I = \mathbf{I}(\widetilde{B})$. We find that $\widetilde{B} = \mathrm{int}\bigcap\limits_{B\in\mathfrak{B}} B$. Let us determine the minimal invariant closed subsets in the unit space of the groupoid $G(C_\theta)'$. Note first that any minimal invariant closed subset of the unit space must be a closed orbit $\overline{[t]}$ for some t in the unit space.

The unit space of $G(C_\theta)'$ consists of four disjoint parts,

$$\alpha(\mathbb{Z}_+^2)\times\mathbb{T},\quad \alpha(\mathbb{Z}_+\times\{\infty\})\times\mathbb{T},\quad \alpha(\{\infty\}\times\mathbb{Z}_+)\times\mathbb{T}$$

and $\beta(\mathbb{R})\times\mathbb{T}$. The first part is an invariant open subset, while the last ones are invariant closed subsets.

Given u in the unit space, we proceed in the following four cases.

(1) $u\in\beta(\mathbb{R})\times\mathbb{T}$, say $u = (\beta(s),t)$. Define the distance function d on $\beta(\mathbb{R})\times\mathbb{T}$ by

$$d((\beta(s),t),(\beta(s'),t')) = |s-s'| + |t-t|'.$$

Then the distance is an invariance under the action of $\mathbb{Z}^2$ on $\beta(\mathbb{R})\times\mathbb{T}$. For each $v\in\overline{[u]}$ there is a sequence $\{p_m\}_{m=1}^\infty$ in $\mathbb{Z}^2$ such that $v = \lim\limits_{m\to\infty}(u+p_m)$. However

$$\lim_{m\to\infty} d(u, v-p_m) = \lim d(u+p_m, v) = 0.$$

It follows that the closed orbits in $\beta(\mathbb{R})\times\mathbb{T}$ are either disjoint or identical. So the closed orbits in $\beta(\mathbb{R})\times\mathbb{T}$ are the minimal invariant closed subsets in the unit space.

(2) $u\in\alpha(\mathbb{Z}_+\times\{\infty\})\times\mathbb{T}$, say $u = (\alpha(n,\infty),t)$. Now the subset

$$S := \{u+(0,m) = (\alpha(n,\infty),t\theta_2^m)|m\in\mathbb{Z}\}$$

is contained in the orbit $[u]$. It follows that $\{\alpha(n,\infty)\}\times \mathbb{T}$ is contained in the closed orbit $\overline{[u]}$. For any $k\in \mathbb{Z}_+$,

$$u+(k-n,0)=(\alpha(k,\infty),t\theta_1^{k-n}).$$

It follows that

$$\alpha(\mathbb{Z}_+\times\{\infty\})\times \mathbb{T}=\overline{[u]}.$$

Therefore $\alpha(\mathbb{Z}_+\times\{\infty\})\times \mathbb{T}$ is a minimal invariant closed subset in the unit space.

(3) $u\in\alpha(\{\infty\}\times\mathbb{Z}_+)\times\mathbb{T}$. By the same reason as that in case (2), $\alpha(\{\infty\}\times\mathbb{Z}_+)\times\mathbb{T}$ is a minimal invariant closed subset in the unit space.

(4) $u\in\alpha(\mathbb{Z}_+^2)\times\mathbb{T}$, say $u=(\alpha(p),t)$. Now the closed orbit $\overline{[u]}$ contains at least one point in $\alpha(\mathbb{Z}_+\times\{\infty\})\times\mathbb{T}$ and therefore contains $\alpha(\mathbb{Z}_+\times\{\infty\})\times\mathbb{T}$, so the closed orbit $\overline{[u]}$ is not minimal.

So the family of the minimal invariant closed subsets in the unit space is

$$\{\alpha(\mathbb{Z}_+\times\{\infty\})\times\mathbb{T},\alpha(\{\infty\}\times\mathbb{Z}_+)\times\mathbb{T},\overline{[u]}|u\in\beta(\mathbb{R})\times\mathbb{T}\},$$

whose union is

$$\alpha(\mathbb{Z}_+\times\{\infty\})\times\mathbb{T}\cup\alpha(\{\infty\}\times\mathbb{Z}_+)\times\mathbb{T}\cup\beta(\mathbb{R})\times\mathbb{T}.$$

Therefore the intersection of the maximal invariant open subsets in the unit space is $\alpha(\mathbb{Z}_+^2)\times\mathbb{T}$. The lemma follows now.

Lemma 2.5. *The intersection of the maximal invariant open subsets in* $G(C_\theta)|_{\alpha(\mathbb{Z}_+^2)\times\mathbb{T}}$ *is empty. Consequently the maximal radical of* $C^*(G(C_\theta)'')$ *is zero.*

Proof. Given u in $\alpha(\mathbb{Z}_+^2)\times\mathbb{T}$ the closed orbit created by u is exactly the orbit created by u. So every orbit in the unit space $\alpha(\mathbb{Z}_+^2)\times\mathbb{T}$ is a minimal invariant closed subset. The lemma follows now.

In summary, we obtain the maximal radical series,

$$\{0\}\lhd C^*(G(C_\theta)'')\lhd C^*(G(C_\theta)')\lhd C^*(G(C_\theta)),$$

for the groupoid C^*-algebra $C^*(G(C_\theta))$ in the case that both $\dfrac{\arg\theta_1}{2\pi}$ and $\dfrac{\arg\theta_2}{2\pi}$ are irrational. It is invariant under the isomorphism.

Remark 2.2. The classification and the K-theory of the rotational C^*-algebras will be given in our following paper.

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