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ANTI-SADDLES OF A POLYNOMIAL SYSTEM**

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Abstract

By using the generalized Poincaré index theorem it is proved that if the n^2 critical points of an n-polynomial system form a configuration of type $(2n-1)-(2n-3)+(2n-5)-\cdots+(-1)^{n-1}$, and the $2n-1$ outmost anti-saddles form the vertices of a convex $(2n-1)$-polygon, then among these $2n-1$ anti-saddles at least one must be a node.

Keywords Polynomial system, Anti-saddle, Poincaré index theorem, Equator.
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We have conjectured in [1] that when a cubic polynomial differential system has 9 finite critical points among which 3 are saddles and 6 are anti-saddles, then in the latter there must be at least one node. We have proved in [2] that at this time the configuration of the 9 critical points may have the type $5-3+1$, $6-3$ or $4-3+2$. In this paper we will prove that our conjecture is true when configuration of type $5-3+1$ appears and the outmost pentagon is convex by using the generalized Poincaré index theorem and triangulation method introduced in [3] and [4]. Moreover, we can generalize this result to the n-polynomial system which shows that conjecture I in [5] is true, if only the outmost $(2n-1)$-polygon is convex.

Since under the triangulation of the Poincaré hemisphere the equator is divided into segments and each segment is a trajectory or a part of a trajectory, on which there may situate saddles, nodes or saddle-nodes, we must know when an equator-segment is a side of a triangle, quadrilateral, pentagon or an m-polygon, how to determine the value of ν and σ (for the meaning see [2]) of this segment in order to apply the index theorem.

There are 8 cases to be considered:

1) If the equator-segment $\overline{AB}$ contains no critical point, then $\nu=1,\ \sigma=0$, i.e., $\overline{AB}$ is equivalent to an outer contact point.

In order to prove this, we draw a triangle ΔABC containing no critical point as in Fig. 1(a). Contracting $\overline{AB}$ to a point D we get Fig. 1(b). Evidently we have $\nu=1,\sigma=0$ both at C and D, and $\text{ind}\Sigma' = \text{ind}\Sigma = 1+\frac{1}{2}(0-2)=0$, and hence the result.

2) If $\overline{AB}$ contains a saddle point, then $\nu=\sigma=0$.

This is shown by Figs.2(a) and (b), where E, F, and E', F' each with $\nu=1,\sigma=0$ are outer contact points.

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3) If $\overline{AB}$ contains two saddle points, then $\nu = 0, \sigma = 1$.

This is shown by Figs.3(a) and 3(b), where I', H', E' are outer contact points, so D is an inner contact point with $\nu = 0, \sigma = 1$.

4) If $\overline{AB}$ contains a node, then $\nu = 2, \sigma = 0$ (see Figs. 4(a),(b)).

5) If $\overline{AB}$ contains a node and a saddle, or a sddle-node, then $\nu = 1, \sigma = 0$ (see Figs. 5(a),(b)), i.e., it is equivalent to Case 1).

6) If $\overline{AB}$ contains two saddles and a node, then $\nu = \sigma = 0$ (see Figs. 6(a),(b), (c), (d)), i.e., it is equivalent to Case 2).

7) If $\overline{AB}$ contains 3 saddles, then $\nu = 0, \sigma = 2$ (see Figs. 7(a),(b)).

8) If $\overline{AB}$ contains 3 saddles and 1 node, then $\nu = 0, \sigma = 1$ (equivalent to Case 3)).

Let us denote the whole equator by E. We can summarize these 8 cases for the equivalent values V_i of the equator-segments $E_i \neq E$ on the boundary of a plane region G in a table as follows, where σ =number of inner contact points, ν =number of outer contact points.

$s =$ no. of saddles	0	1	2	0	1	2	3	3	4	n	2	4	6	8	$2n$
$n =$ no. of nodes	0	0	0	1	1	1	0	1	1	1	2	6	2	2	2
$V = \sigma$ or $-\nu$	-1	0	1	-2	-1	0	2	1	2	$n-2$	-1	-3	3	5	$2n-3$

Table 1

From this table we can easily get the following rule:

Rule. Assume (s_i, n_i) and $V_i (i = 1, 2)$ denote any two cases in the Table 1, and let

$$(s_3, n_3) = (s_1, n_1) + (s_2, n_2) = (s_1 + s_2, n_1 + n_2) \text{ with } E_1 \cap E_2 = \emptyset,\ E_1 \cup E_2 = E_3 \neq E.$$

Then

$$V_3 = V_1 + V_2 + 1. \tag{1}$$

The reason is, e.g., when Fig.2 and Fig.5 are put together to make Fig.6, the number of outer contact points not lying on the equator decreases in number by 1. By this rule we can get other results not contained in Figs.1-8, such as columns 9-15 in Table 1.

The following Table 2 refers to cases when the whole equator E is a boundary of G. It is easily seen that in these cases s and n must be both even, and columns in this table cannot be added as in Table 1.

$s =$ no. of saddles	2	4	6	8	$2n$
$n =$ no. of nodes	2	6	2	2	2
$V = \sigma$ or $-\nu$	0	-2	4	6	$2n-2$

Table 2

In order to get Table 2 from Table 1, we should use the formula

$$V = V_i + 1, \tag{2}$$

if the equator E contains the same number of saddles and nodes as E_i. This is because if $E = E_i \cup E_i'$, then E_i' contains no critical point, since E_i and E_i' are connected on both of their ends, so we have $V = V_i - 1 + 2 = V_i + 1$. Thus from columns 11-15 we can get the whole Table 2 by Formula (2).

On the other hand, we have

$$V = V_1 + V_2 + 2, \tag{3}$$

if $E_1 \cup E_2 = E,\ E_1 \cap E_2 = \emptyset$, but $E_i \neq E$ for $i = 1, 2$.

Now, assume that we have a configuration of type $5-3+1$ for a certain cubic system (Fig.9), where A, B, C, D, E are assumed to be all foci on the convex pentagon Γ. Then there must be at least one outer contact point on the sides of Γ, since the sum of indices of critical points within Γ is -2. Assume this outer contact point P lies on the side $\overline{CD}$ of Γ. Then the straight line S determined by $\overline{CD}$ divides the upper hemisphere into two simply-connected regions G_l and G_r, such that the interior of Γ lies in G_l. Since each of G_l and G_r has a half equator as its boundary, which contains 3 saddles and one node or 2 saddles (hence $V_l = V_r = 1$), and S as a part of the boundary of $G_l(G_r)$ contains 3 outer contact points C, D, P (2 outer contact points C, D and one inner contact point P), we can calculate the sum of indices of critical points within G_l and G_r by Poincaré index formula:

$$\Sigma_l = \sum_i \text{ind}O_i \text{ in } G_l = 1 + \frac{1-3}{2} = 0,$$
$$\Sigma_r = \sum_i \text{ind}O_i \text{ in } G_r = 1 + \frac{2-2}{2} = 1.$$

But actually we have $\Sigma_l = 1, \Sigma_r = 0$, as is easily seen, so Fig.9 is impossible.

So we get

Theorem 1. *If the* 9 *critical points of a cubic system make a configuration of type* $5-3+1$*, and the* 5 *outmost anti-saddles form the vertices of a convex pentagon, then they cannot be all foci or centers.*

This method of proof can be easily generalized to polynomial systems of degree greater than 3. For example, assume the 16 critical points of a quartic system has a distribution of type $7-5+3-1$, and the 7 outmost critical points A, B, C, D, E, F, G forming the vertices of a convex polygon Γ' are all foci (Fig.10). Since the sum of indices of critical points within Γ' is -3, there must be an outer contact point on the sides of Γ'. Assume P is an outer contact point on the segment $\overline{AB}$ of Γ'. There must be a fourth contact point Q on the straight line l determined by $\overline{AB}$. As before, let G_L and G_R denote the 2 simple-connected regions when the upper hemisphere is divided by l. From Column 9 of Table 1, the totality of critical points at infinity on the boundary of G_L or G_R is equivalent to 2 inner contact points. So, when Q is an outer contact point with respect to G_L, we have

$$\Sigma_L = 1 + \frac{2-4}{2} = 0, \quad \Sigma_R = 1 + \frac{4-2}{2} = 2. \tag{4}$$

When Q is an inner contact point with respect to G_L (dotted line in Fig.10), we have

$$\Sigma_L = 1 + \frac{3-3}{2} = 1, \quad \Sigma_R = 1 + \frac{3-3}{2} = 1. \tag{5}$$

But actually we should have

$$\Sigma_l = 2, \quad \Sigma_R = 0,$$

so Fig.10 is impossible.

In the general case, i.e., for the n-polynomial differential system, if the n^2 critical points form a configuration of type

$$(2n-1)-(2n-3)+(2n-5)+\cdots+(-1)^{n-1}, \tag{6}$$

and the outmost $2n-1$ anti-saddles are all foci and form the vertices of a convex $(2n-1)$-polygon, then from Column 10 of Table 1 the totality of critical points at infinity on the boundary of G_L or G_R is equivalent to $n-2$ inner contact points, so instead of (4) and (5) we will have

$$\Sigma_L = 0, 1, 2, \cdots, \text{ or } n-3, \quad \Sigma_R = n-2, n-3, n-4 \cdots, \text{ or } 1,$$

according as G_L has $n, n-1, \cdots$, or 3 outer contact points on its boundary line l. But actually we should have $\Sigma_L = n-2$, $\Sigma_R = 0$. So the assumption that these vertices are all foci is absurd.

So we get

Theorem 2. *When the n^2 critical points of an n-polynomial system make a configuration of type* (6) *and the outmost* $(2n-1)$*-polygon is convex, at least one of its vertices must be a node.*

Notice that Fig.11 shows: in case of a cubic system with 9 critical points forming a configuration of type $5-3+1$, if the outmost pentagon $\Gamma = ABCDEA$ is not convex, there is a possibility for the 5 vertices to be all foci or centers.

On the other hand, if in Fig.9 instead of a focus we put on the vertex A of Γ a node with $\nu = 2, \sigma = 0$ as shown in Fig.12, then we can arrange the 5 contact points $P_1, \cdots, P_5$ each on a line determined by a side of Γ in such a way that there are no finite critical point outside Γ, and there are 3 saddles and one node on the equator. We believe that Fig.12 can be realized by a certain cubic system.

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Fig.2

Fig.3

Fig.4

Fig.5

Fig.6

Fig.7

Fig.8

Fig.9

Fig.10

Fig.11

Fig.12