Chin. Ann. of Math.
16B: 4(1995),459-466.

A NECESSARY AND SUFFICIENT CONDITION OF EXISTENCE OF GLOBAL SOLUTIONS FOR SOME NONLINEAR HYPERBOLIC EQUATIONS

ZHANG QUANDE*

Abstract

The author considers the Klien–Gordon equations $u_{tt} - \Delta u + \mu u = f(u)$ $(\mu > 0,\ |f(u)| \le c|u|^{\alpha+1})$. The necessary and sufficient condition of existence of global solutions is obtained for $E(0) = \frac{1}{2}(\|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \mu\|u_0\|_{L^2}^2) - \int_{R^n}\int_0^{u_0} f(s)dsdx < d$ (d is the given constant).

Keywords Global solution, Blow up of the solution, Existence, Nonexistence.
1991 MR Subject Classification 35L05, 35L15, 35L70.

In this paper, we consider the following Cauchy problem of nonlinear hyperbolic equation

$$u_{tt} - \Delta u + \mu u = | u |^{\alpha} u, \quad \mu > 0,\ \alpha > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in R^n, \tag{2}$$

where $u_0(x) \in H^2(R^n), u_1(x) \in H^1(R^n)$.

Over the last 20 years there has been considerable interest in existence and nonexistence of the global solutions for nonlinear hyperbolic equations. As far as I know, there are few results which give the necessary and sufficient conditions of existence of global solutions. We mention a remarkable work of F. John[1]. In [1] he showed an "almost" necessary and sufficient condition of global existence of Cauchy problem for $u_{tt} - \Delta u =| u |^p$ in three-dimensional space, that is,

1) if $1 < p < 1 + \sqrt{2}$, then the global solution vanishes identically for the initial data satisfying $u_0 = 0, u_1 \ge 0$;

2) if $p > 1+\sqrt{2}$, then there exists a unique C^2-solution for small initial data with compact support.

Maybe it is the best result at present. F. Asakura[4] generalized to the case of initial data without compact support. In general n dimensional space, the papers [2,3] gave the sufficient condition of blow up in finite time for generalized solutions in $L^1(R^n)$. The equation of the form (1) occurs in the classical modelling of certain phenomena in field theory[9]. It is also called Klein-Gordon equation. Berger[6,7] discussed the stationary states of (1) and (2). H. A. Levine[10] discussed the blow up of solutions for more abstract equations $pu_{tt}-Au = \mathcal{F}(u)$. The object of this paper is to give the necessary and sufficient condition of global existence or nonexistence in $C^0(R^+, H^1(R^n))$ for (1) and (2).

Manuscript received August 9, 1993.
*Department of Mathematics, Shaanxi Normal University, Xi'an 710062, China.

Our main result is the following

Theorem 1. *Let* $\mu > 0, 0 < \alpha < \frac{2}{n-2}, n > 2$ $(n \le 2, 0 < \alpha < +\infty)$.

$$E(0) = \frac{1}{2}(\|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \mu\|u_0\|_{L^2}^2) - \frac{1}{\alpha+2}\|u_0\|_{L^{\alpha+2}}^{\alpha+2} < d, \tag{3}$$

where

$$d = \inf_{u\in H^1, u\neq 0} \alpha(\|\nabla u\|_{L^2}^2 + \mu\|u\|_{L^2}^2)^{\frac{\alpha+2}{\alpha}} / 2(\alpha+2)(\|u\|_{L^{\alpha+2}}^{\alpha+2})^{\frac{2}{\alpha}}.$$

Then the global solution of (1) *and* (2) *in* $C^0(R^+, H^1)$ *exists if and only if either*

$$(\|\nabla u_0\|_{L^2}^2 + \mu\|u_0\|_{L^2}^2) > \|u_0\|_{L^{\alpha+2}}^{\alpha+2}, \tag{4}$$

or

$$(\|\nabla u_0\|_{L^2}^2 + \mu\|u_0\|_{L^2}^2) = 0 \quad (i.e., \ \|u\|_{H^1} = 0). \tag{4'}$$

Lemma 1. *Let*

$$\begin{aligned} F(u) &= \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \mu\|u\|_{L^2}^2) - \frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2} \\ &\triangleq \frac{1}{2}a(u) - \frac{1}{\alpha+2}b(u). \end{aligned}$$

If $a(u) = b(u) \neq 0$, *then* $F(u) \ge d > 0$.

Proof. Since $F(\lambda u) = \frac{\lambda^2}{2}a(u) - \frac{\lambda^{\alpha+2}}{\alpha+2}b(u)$,

$$\begin{aligned} &\sup_{\lambda\ge 0} F(\lambda u) = F\Big(\Big(\frac{a(u)}{b(u)}\Big)^{\frac{1}{\alpha}} u\Big) = \frac{\alpha}{2(\alpha+2)} \frac{(a(u))^{\frac{\alpha+2}{\alpha}}}{(b(u))^{\frac{2}{\alpha}}}, \\ &\qquad b(u) \neq 0. \end{aligned}$$

Therefore, when $a(u) = b(u) \neq 0$,

$$\sup_{\lambda\ge 0} F(\lambda u) = F(u) \ge d.$$

Furthermore, from Sobolev embedding theorem,

$$\|u\|_{L^{\alpha+2}} \le C_0\|u\|_{H^1}, \text{ if } n > 2, \quad 2 < \alpha+2 \le \frac{2n}{n-2} \ (n \le 2, 0 < \alpha < +\infty).$$

Therefore

$$d \ge \begin{cases} \dfrac{\alpha}{\alpha+2}(\dfrac{1}{2C_1})^{\frac{\alpha+2}{\alpha}}, & \mu \ge 1, \\ \dfrac{\alpha}{\alpha+2}(\dfrac{\mu}{2C_1})^{\frac{\alpha+2}{\alpha}}, & 0 < \mu < 1. \end{cases} \tag{5}$$

Lemma 2. *Let* $1 < \alpha + 1 \le \frac{n}{n-2}$, *for* $n > 2$ $(0 < \alpha < \infty$, *for* $n \le 2)$. *Then the local solution of* (1), (2) *exists in* $C^0([0, T_0], H^1(R^n))$ *for some* $T_0 > 0$.

For a detailed proof of Lemma 2 see [5]. In fact, put $u = e^{\sigma t}v$, then equation (1) is transformed to

$$v_{tt} - \Delta v + \lambda v_t + \mu' v = e^{\sigma t} \mid v \mid^\alpha v, \ \lambda > 0, \ \mu' > 0.$$

The local existence of $\|u\|_{H^2}^2$ and $\|u_t\|_{H^1}^2$ can be derived via energy methods and fixed point theorems. Furthermore, from

$$\begin{aligned} \mid \|u(t_1)\|_{H^1} - \|u(t_2)\|_{H^1} \mid &\le \|u(t_1) - u(t_2)\|_{H^1} \\ &\le \sup_{0\le t\le T} \|u_t(t)\|_{H^1} \mid t_1 - t_2 \mid, \end{aligned}$$

we see that $\|u\|_{H^1}(t)$ is continuous.

Proof of Theorem 1.

I) Sufficiency. From Lemma 2, the solution of (1) and (2) is continuous with respect to t (within the interval of existence). We will show that the solution is global.

Multiplying the equation (1) by u_t and integrating we have

$$\frac{1}{2}\|u_t\|_{L^2}^2 + F(u) = E(0) < d. \tag{6}$$

We assert that for all $t \geq 0$, either

$$a(u) - b(u) > 0,\ a(u) \neq 0, \tag{7}$$

or

$$a(u) = b(u) = 0. \tag{7'}$$

Suppose that neither (7) nor (7') holds and let t_1 be the smallest time for which

$$a(u) \leq b(u) \text{ for } t > t_1.$$

We consider two cases $a(u(t_1)) = 0$ and $a(u(t_1)) \neq 0$ respectively.

a) The case $a(u(t_1)) = 0$. Noting that $b(u) \leq C_0\|u\|_{H^1}^{\alpha+2}$, we have $b(u(t_1)) = 0$. Since t_i is the smallest time such that neither (7) nor (7′) holds, we have $0 < a(u) \leq b(u)$ for some $\varepsilon > 0$ and $t_1 < t < t_1 + \varepsilon$, i.e.,

$$\frac{a(u)}{b(u)} \leq 1 \text{ for } t_1 < t < t_1 + \varepsilon.$$

On the other hand, we have

$$\frac{a(u)}{b(u)} \geq \frac{a(u)}{C_0\|u\|_{H^1}^{\alpha+2}} \geq \frac{1}{C_1}[a(u)]^{-\frac{2}{\alpha}} \text{ for } t_1 < t < t_1 + \varepsilon.$$

Therefore, we have

$$\frac{1}{C_1}[a(u)]^{-\frac{2}{\alpha}} \leq 1 \text{ for } t_1 < t < t_1 + \varepsilon.$$

Then in virtue of Lemma 2, $a(u(t))$ is continuous in t and $\lim\limits_{t\to t_1+0} a(u(t)) = 0$. But from the above we see that

$$\lim_{t\to t_1+0} a(u)^{-\frac{2}{\alpha}} \leq C_1 < +\infty.$$

This implies a contradiction.

b) The case $a(u(t_1)) \neq 0$. Since

$$\begin{aligned} |\ \|u(t)\|_{L^{\alpha+2}} - \|u(s)\|_{L^{\alpha+2}}\ | &\leq \|u(t) - u(s)\|_{L^{\alpha+2}} \\ &\leq C\|u(t) - u(s)\|_{H^1} \\ &\leq \sup_{t\geq 0}\|u_t\|_{H^1}\ |\ t - s\ |, \end{aligned}$$

we see that $b(u(t))$ is continuous in t (within the interval of existence).

From (7) and (8) we have $a(u(t_1)) = b(u(t_1)) > 0$.

It follows from Lemma 1 that $F(u(t_1)) \geq d$. This contradicts (6).

Therefore, from a) and b) we see that (7) or (7') holds.

From (6) and (7) or (7') we have

$$\|u_t\|_{L^2}^2 \leq d, \quad \frac{\alpha}{2(\alpha+2)}(\|\nabla u\|_{L^2}^2 + \mu\|u\|_{L^2}^2) \leq F(u) \leq E(0). \tag{9}$$

Hence $\|u\|_{H^1} \leq$ const.

We now prove that $\|u\|_{H^1}(t)$ is continuous with respect to t in R^+.

Let $u(t)$ be a generalized solution of (1) and (2) satisfying (3) and (4) or (4'). From (1) we have

$$u_{x_i t} + \Delta u_{x_i} + \mu u_{x_i} = (\alpha+1) \mid u \mid^\alpha u_{x_i}, \ i = 1, 2, \cdots, n, \tag{10}$$

where u_{x_i} is a weak derivative. Therefore, we have

$$\begin{aligned}(\|u_{x_i t}\|_{L^2}^2 + \|\nabla u_{x_i}\|_{L^2}^2 + \mu\|u_{x_i}\|_{L^2}^2)' &= (\alpha+1)(\mid u \mid^\alpha u_{x_i}, u_{x_i t}) \\ &\leq C\|u\|_{L^{n\alpha}}^\alpha \|u_{x_i}\|_{L^{\frac{2n}{n-2}}} \|u_{x_i t}\|_{L^2}.\end{aligned}$$

Since $0 < \alpha < \frac{2}{n-2}$, from Sobolev's embedding theorem and (9) we have

$$\|u\|_{L^{\alpha n}} \leq C\|u\|_{H^1} \leq \text{ const.}$$

and

$$\|u_{x_i}\|_{L^{\frac{2n}{n-2}}} \leq C\|u_{x_i}\|_{H^1} \leq \text{ const. } (\|\nabla u_{x_i}\|_{L^2}^2 + \mu\|u_{x_i}\|_{L^2}^2)^{\frac{1}{2}}.$$

Therefore we have

$$\begin{aligned}&(\|u_{x_i t}\|_{L^2}^2 + \|\nabla u_{x_i}\|_{L^2}^2 + \mu\|u_{x_i}\|_{L^2}^2)' \\ &\leq C(\|u_{x_i t}\|_{L^2}^2 + \|\nabla u_{x_i}\|_{L^2}^2 + \mu\|u_{x_i}\|_{L^2}^2), \quad i = 1, 2, \cdots, n.\end{aligned} \tag{11}$$

In virtue of Gronwall's inequality we have

$$\|u_{x_i t}\|_{L^2}^2 + \|\nabla u_{x_i}\|_{L^2}^2 + \|u_{x_i}\|_{L^2}^2 \leq C(T) \text{ for } 0 \leq t < \infty. \tag{12}$$

$C(T)$ is constant depending on T. Hence we see that for any $T > 0$, $t \in [0, T]$, $u_t(t) \in H^1(R^n), u(t) \in H^2(R^n)$. Therefore, we have $u \in C^0([0, T], H^1(R^n))$ for any $T > 0$.

Therefore, the global solution of (1) and (2) exists in $C^0([0, +\infty], H'(R^n))$.

II) Necessity. If (4) or (4') dose not hold, from (3) and Lemma 1 we have

$$\|\nabla u_0\|_{L^2}^2 + \mu\|u_0\|_{L^2}^2 < \|u_0\|_{L^{\alpha+2}}^{\alpha+2}. \tag{13}$$

From Lemma 2 we know that the local solution of (1) and (2) exists in $C^0([0, T_0], H^1(R^n))$ for some $T_0 > 0$. It is similar to the proof of sufficiency that we can assert that (within the interval of existence)

$$a(u) < b(u) \text{ for all } t \geq 0. \tag{14}$$

If (14) does not hold, there is $t_1 \geq 0$ such that (14) holds for $0 \leq t < t_1$ and $a(u(t)) \geq b(u(t))$ for $t \geq t_1$.

By the continuity of $a(u(t))$ and $b(u(t))$,

$$a(u(t_1)) = b(u(t_1)).$$

Similarly, we consider two cases $a(u(t_1)) \neq 0$ and $a(u(t_1)) = 0$ respectively.

a) If $a(u(t_1)) \neq 0$, from Lemma 1 we have $F(u(t_1)) \geq d$. This contradicts (6).

b) Let $a(u(t_1)) = 0$. Since $b(u) > a(u) \geq 0$ for $0 \leq t < t_1$, we have

$$1 > \frac{a(u)}{b(u)} \geq \frac{a(u)}{C\|u\|_{H^1}^{\alpha+2}} \geq c(\mu)(a(u))^{-\frac{\alpha}{2}} \ for \ t < t_1.$$

This contradicts $\lim\limits_{t \to t_1 - 0} a(u(t)) = 0$. Hence from a) and b) our assertion (14) holds for all $t \geq 0$.

On the other hand, multiplying equation (1) by u we have

$$(\|u\|_{L^2}^2)'' = 2\|u_t\|_{L^2}^2 + 2(b(u(t)) - a(u(t))) > 0 \text{ for all } t \geq 0. \tag{15}$$

From (6) we have

$$(\|u\|_{L^2}^2)'' = (\alpha+4)\|u_t\|_{L^2}^2 + \alpha(\|\nabla u\|_{L^2}^2 + \mu\|u_t\|_{L^2}^2) - (\alpha+2)E(0). \tag{16}$$

By (15), $(\|u\|_{L^2}^2)'' > 0$. $\|u\|_{L^2}^2$ is a convex function in t. Hence we say that there exist t_1 and t_2 $(t_1 \leq t_2)$ such that

$$(\|u\|_{L^2}^2)'(t_1) \geq 0, \text{ and } \alpha\mu(\|u\|_{L^2}^2)(t_2) \geq (\alpha+2)E(0).$$

(This assertion will be proved later). Hence

$$(\|u\|_{L^2}^2)'' \geq (\alpha+4)\|u_t\|_{L^2}^2 \text{ for } t \geq t_2, \tag{17}$$

$$(\|u\|_{L^2}^2)(\|u\|_{L^2}^2)'' - \frac{\alpha+4}{4}[(\|u\|_{L^2}^2)']^2 \geq (\alpha+4)[\|u\|_{L^2}^2\|u_t\|_{L^2}^2 - (u_t,u)^2] \geq 0, \tag{18}$$

$$(\|u\|_{L^2}^{-\frac{\alpha}{2}})'' = -\frac{\alpha}{4}\|u\|_{L^2}^{-(\frac{\alpha}{2}+4)}\{\|u\|_{L^2}^2(\|u\|_{L^2}^2)'' - \frac{\alpha+4}{4}[(\|u\|_{L^2}^2)']^2\} \leq 0. \tag{19}$$

Therefore, $\|u\|_{L^2}^{-\frac{\alpha}{2}}$ is concave for $t \geq t_2$ and

$$(\|u\|_{L^2}^{-\frac{\alpha}{2}})' = -\frac{\alpha}{4}\|u\|_{L^2}^{-\frac{\alpha}{2}}(\|u\|_{L^2}^2)' < 0$$

for $t \geq t_2 \geq t_1$. Then there exists a finite time T for which $\|u\|_{L^2}^{-\frac{\alpha}{2}} \to 0$ as $t \to T-0$. In other words

$$\lim_{t\to T-0}\|u\|_{L^2} = +\infty. \tag{20}$$

We now prove that there exist t_1 and t_2 $(t_1 \leq t_2)$, such that $(\|u\|_{L^2}^2)' \geq 0$ for $t \geq t_1$ and $\alpha\mu\|u\|_{L^2}^2 \geq E(0)$ for all $t \geq t_2$.

If $(\|u\|_{L^2}^2)' \leq 0$ for all $t \leq 0$, then from $(\|u\|_{L^2}^2)'' > 0$ we have

$$\lim_{t\to+\infty}(\|u\|_{L^2}^2)' = B \leq 0, \quad \lim_{t\to+\infty}\|u\|_{L^2}^2 = A \geq 0.$$

It is clear that $B = 0$. Therefore, there is a sequence $\{t_n\}$ such that, as $t_n \to \infty$, $(\|u\|_{L^2}^2)''(t_n) \to 0$. From (14) and (15) we have

$$\lim_{t_n\to\infty}\|u_t\|_{L^2}^2 = 0 \text{ and } \lim_{t_n\to\infty}(b(u) - a(u)) = 0.$$

From (6) we have

$$\lim_{t_n\to\infty}F(u) = E(0). \tag{21}$$

On the other hand, from the definition of $F(u)$ we have

$$\begin{aligned}\lim_{t_n\to\infty}b(u) &= \lim_{t_n\to\infty}\frac{2(\alpha+2)}{\alpha}[F(u) + \frac{1}{2}(b(u) - a(u))]\\ &= \frac{2(\alpha+2)}{\alpha}E(0) = \lim_{t_n\to\infty}a(u).\end{aligned}$$

If $E(0) > 0$, $\lim_{t_n\to\infty}\dfrac{a(u)}{b(u)} = 1$. From Lemma 1 and (21) we have

$$E(0) = \lim_{t_n\to\infty}F(u) = \lim_{t_n\to\infty}F\Big(\Big[\frac{a(u(t_n))}{b(u(t_n))}\Big]^{\frac{1}{\alpha}}, u(t_n)\Big) \geq d.$$

This contradicts condition (3).

If $E(0) < 0$, $\lim\limits_{t_n\to\infty} b(u) = \lim\limits_{t_n\to\infty} a(u) < 0$. It contradicts the fact that $a(u) \geq 0$ and $b(u) \geq 0$.

If $E(0) = 0$, $\lim\limits_{t_n\to\infty} b(u) = \lim\limits_{t_n\to\infty} a(u) = 0$. But, from (14) and $b(u) \leq c[a(u)]^{\frac{\alpha+2}{2}}$ we have

$$1 > \frac{a(u)}{b(u)} \geq \frac{1}{c}(a(u))^{-\frac{\alpha}{2}}.$$

This contradiction is clear.

It follows therefore that $\|u\|_{L^2}^2 \to \infty$ in a finite time, and the proof of Theorem 1 is completed.

We also consider the initial boundary-value problem.

Corollary 1. *Let* $u\mid_{\partial\Omega}= 0, \mu \geq 0$. *If initial data satisfy conditions* (3) *and* (4), *then the results of Theorem* 1 *hold.*

The results are applicable to more general nonlinearity $f(u)$ satisfying $|f(u)| \leq c|u|^{\alpha+1}$. In this case, we define (cf. Lemma 1)

$$a(u(t)) = \|\nabla u\|_{L^2}^2 + \mu\|u\|_{L^2}^2, \quad b(u(t)) = \int_{R^n}\int_0^u f(s)dsdx,$$

$$F(\lambda,\ u) = \frac{\lambda^2}{2}a(u(t)) - \lambda^{\alpha+2}b(u(t)).$$

$$\sup_{\lambda\geq 0} F(\lambda,\ u) = \begin{cases} +\infty, & b(u) \leq 0, \\ \frac{\alpha}{2(\alpha+2)}\frac{a^{\frac{\alpha+2}{\alpha}}(u)}{((\alpha+2)b(u))^{\frac{2}{\alpha}}}, & b(u) > 0, \end{cases}$$

$$d' = \inf_{u\in H^1,\ b(u)>0} \frac{\alpha}{2(\alpha+2)}\frac{a^{\frac{\alpha+2}{\alpha}}(u)}{((\alpha+2)b(u))^{\frac{2}{\alpha}}}.$$

It is similar to Lemma 1 that if $a(u) = (\alpha+2)b(u) \neq 0$, then $\sup\limits_{\lambda\geq 0} F(\lambda,\ u) = F(1,\ u) = \frac{1}{2}a(u) - b(u) \geq d'$.

Corollary 2. *Let* $f(u)$ *satisfy* $|f(u)| \leq c|u|^{\alpha+1}$ *and* $\alpha,\ \mu$ *be as in Theorem* 1 *and*

$$E(0) = \frac{1}{2}(\|u_1\|_{L^2}^2 + a(u_0)) - b(u_0) < d'. \tag{22}$$

Then

$$\begin{cases} u_{tt} - \Delta u + \mu u = f(u), \\ u(0,x) = u_0, \quad u_t(0,x) = u_1 \end{cases} \tag{23}$$

has a global solution in $C^0(\mathbf{R}^1,\ H^1)$ *if and only if*

$$a(u_0) > (\alpha+2)b(u_0) \ \textit{or}\ a(u_0) = 0. \tag{24}$$

As in [11], to consider the 'averaged' version of (1)

$$u_{tt} - \nabla u + \mu u = \|u\|_{L^2}^{\alpha} u, \quad \alpha > 0, \quad \mu > 0, \tag{25}$$

we can improve the conditions of blow up of the solutions in Theorem 1 of paper [10].

Corollary 3. *If* $\mu \geq 0$, $(u_1, u_0) = \int_{R^n} u_0u_1dx \geq 0$,

$$\|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \mu\|u_0\|_{L^2}^2 < \|u_0\|_{L^2}^{\alpha+2}, \tag{26}$$

then the solution of (25) *blows up infinite time.*

Proof. It is similar to the proof of Theorem 1 that we have

$$\frac{1}{2}(\|u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\mu\|u\|_{L^2}^2)-\frac{1}{\alpha+2}\|u\|_{L^2}^{\alpha+2}=E_1(0), \tag{27}$$

where $E_1(0)=\frac{1}{2}(\|u_1\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2+\mu\|u_0\|_{L^2}^2)-\frac{1}{\alpha+2}\|u_0\|_{L^2}^{\alpha+2}$,

$$\begin{aligned}(\|u\|_{L^2}^2)''&=2[\|u_t\|_{L^2}^2+\|u\|_{L^2}^{2+\alpha}-(\|\nabla u\|_{L^2}^2+\mu\|u\|_{L^2}^2)]\\&\geq 2[2\|u_t\|_{L^2}^2+\|u\|_{L^2}^{2+\alpha}-(\|u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\mu\|u\|_{L^2}^2)].\end{aligned} \tag{27}$$

From (26) we have

$$(\|u\|_{L^2}^2)''\mid_{t=0}>0, (\|u\|_{L^2}^2)'\mid_{t=0}\geq 0.$$

Therefore there exists $\varepsilon>0$ such that $(\|u\|_{L^2}^2)'(t)>0$ for $0<t\leq\varepsilon$. Hence $\|u\|_{L2}^2(t)$ is increasing for $0\leq t\leq\varepsilon$. Let

$$\|u\|_{L^2}^{\alpha+2}(t)=\|u_0\|_{L^2}^{\alpha+2}+\delta(t),$$
$$(\|u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\mu\|u\|_{L^2}^2)(t)=\|u_1\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2+\mu\|u_0\|_{L^2}^2+\sigma(t)$$

for $0\leq t\leq\varepsilon$. From (27) we have $\delta(t)=\frac{\alpha+2}{2}\sigma(t)$. Therefore we have

$$(\|u\|_{L^2}^2)''>0, \quad (\|u\|_{L^2}^2)'>0 \text{ for } 0\leq t\leq\varepsilon.$$

Hence we can assert

$$(\|u\|_{L^2}^2)''>0, \text{ and } (\|u\|_{L^2}^2)'>0 \text{ for all } t>0.$$

From (26) and (27) we have

$$(\|u\|_{L^2}^2)''\geq(\alpha+4)\|u_t\|_{L^2}^2+\alpha\|u\|_{L^2}^2-2(\alpha+2)E_1(0). \tag{29}$$

Since $\|u\|_{L^2}^2$ is increasing, there is a t_1 such that $\alpha\|u\|_{L^2}^2\geq 2$ for $t\geq t_1$, and we have

$$(\|u\|_{L^2}^2)''\geq(\alpha+4)\|u_t\|_{L^2}^2 \text{ for } t\geq t_1. \tag{30}$$

From the proof of Theorem 1 there is a constant T, such that

$$\lim_{t\to T-0}\|u\|_{L^2}^2=+\infty.$$

The proof of Theorem 2 is completed.

If $\mu<0$ in (1) and (2) (or (23)), we have the following result.

$$u_{tt}-\Delta u-\lambda^2 u=\phi(u), \tag{31}$$
$$u(x,0)=\varepsilon f(x), \quad u_t(x,0)=\varepsilon g(x), \quad \varepsilon>0, \tag{32}$$

where $\phi(u)=\mid u\mid^\alpha u$ (or $\|u\|_{L^2}^\alpha u$).

Corollary 4. *If* $f(x)\in H^1,\ g(x)\in L^2,\ \int_{R^n}fgdx\geq 0$,

$$\lambda^2\|f\|_{L^2}^2-\|\nabla f\|_{L^2}^2+\|g\|_{L^2}^2\geq 0, \tag{33}$$

then the global solution of (30) *and* (31) *vanishes identically.*

References

[1] John, F., Blow up of solutions of nonlinear wave equations in three dimensions, *Manuscripta Math.,* **28** (1979), 235-269.
[2] Kato, T., Blow up of some nonlinear hyperbolic equations, *Comm. Pure Appl. Math.,* **33** (1980), 501-503.
[3] Glassey, R. T., Finite-time blow up for solutions of nonlinear wave equations, *Math. Z.,* **177** (1981), 323-340.

[4] Asakura, F., Existence of global solution to semilinear wave equation with slowly decreasing initial data in three space dimensions, *Comm. P. D. E.,* **11:**13 (1986), 1459-1487.

[5] Zhang Quande, Existence and uniqueness of global solution to Cauchy problem of nonlinear wave equation, *J. of shaanxi normal university,* **20:**4(1992), 81-82.

[6] Berger. M. S., On morse theory and stationary states for nonlinear wave equations, *Bull. Amer. Math. Soc.,* **76** (1970), 827-829.

[7] Berger. M. S., On the existence and structure of stationary states for a nonlinear Klein-Gordon equation, *J. functional Analysis,* **9** (1972), 249-261.

[8] Sideris, T. C., A decay estimate for the three-dimensional inhomogeneous Klein-Gordon equation and global existence for nonlinear equation, Microcal Analysis Nonlinear wave, Springer-Verlay, 1991.

[9] Vuillermot, P. A., Small divisors and the construction of stable manifolds for nonlinear Klein-Gordon equations on $R_0^+ \times R$, Nonlinear hyperbolic equation and field theory, Great Britain by Biddle Ltd., 1992.

[10] Levine, H. A., Instability and nonexistence of global solutions to nonlinear wave equations of the $Pu_{tt} = -Au = \mathcal{F}(u)$, *Trans. Amer. Math. Soc.,* **192** (1972), 1-21.

[11] Ball, J. M., Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Quant. J. Math.,* **28** (1977), 473-486.