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# DYNAMICS ON WEAKLY PSEUDOCONVEX DOMAINS***

ZHANG WENJUN* REN FYUAO**

**Abstract**

This paper studies the iterations of holomorphic self-maps which have nonwandering points over general pseudoconvex domains in $\mathbf{C}^2$. The authors give especially a Denjoy-Wolff-type theorem on pseudoconvex domains with real-analytic boundaries, or even more general, on domains of finite type.

**Keywords** Iteration, Horosphere, Domains of finite type, Nonwandering point.
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## §1. Introduction

In a recent paper (see [17]), we discussed the dynamics of holomorphic self-maps of strongly pseudoconvex domains. The strongly pseudoconvex domain is in a sense the weakest domain that we can handle completely. The complex analytic properties of the weakly pseudoconvex domains in $\mathbf{C}^N$, even if with smooth boundaries, can differ very much from those of the strongly pseudoconvex domains (see [7]); the dynamical property is by no means excepted. In a recent paper (see [10]), Hriljac studied the dynamics for some kind of two-dimensional weakly pseudoconvex domains. He got the following theorem in quite a great space.

**Theorem 1.1.** *Let $X$ be a two-dimensional compact complex-analytic manifold, with a smoothly varying* (1,1) *form which induces a metric $d$ on $X$. Let $\Omega \subset X$ be an open connected submanifold satisfying the following condition* (C):

$$\begin{aligned}&\text{for any } \xi \in \partial\Omega \text{ there exists a neighborhood } U_\xi \text{ of } \xi \text{ in } X\\ &\text{and a continuous plurisubharmonic function } h_\xi : U_\xi \to \mathbf{R}\\ &\text{such that } \Omega \cap U_\xi = \{x \in U_\xi \mid h_\xi(x) < 0\}.\end{aligned} \tag{C}$$

*If $f \in H(\Omega, \Omega)$, $\{f^n\}$ is normal, and $f$ has a nonwandering point $p \in \Omega$. Then one of the following holds:*

(I) *$p$ is an attracting fixed point of $f$ ;*

(II) *There is a submanifold $S \subset \Omega$ of dimension* 1, *such that $p \in S, p \in f(S)$, and $f|_S$ is an isomorphism. Furthermore, there exists a subsequence $f^{n_j}$ such that $f^{n_j}|_S \to id_S$, and*

$$\lim_{n\to\infty} d(f^n(z), S) = 0, \tag{1.1}$$

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*Department of Mathematics, Henan University, Kaifeng 475001, China.
**Institute of Mathematics, Fudan University, Shanghai 200433, China.
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*where $d$ is the distance given on $\Omega$.*

(III) $f \in \mathrm{Aut}(\Omega)$, *and* $\exists f^{n_j}, f^{n_j} \to id_{\Omega.}$

When $X = \overline{\mathbf{C}^2}$, a domain that satisfies Condition (C) must be a weakly pseudoconvex domain. So Hriljac asked on page 728 of [10]: can the requirement that $\Omega$ satisfy Condition (C) be replaced by the milder requirement that $\Omega$ be pseudoconvex in order that Theorem 1.1 still holds? We shall give an affirmative answer to this question in §2. Obviously, our method is more direct and more simple. Furthermore, our proof can be applied to give similar results for some weakly pseudoconvex domains of $\mathbf{C}^N$ for general $N > 2$.

We got in [17] a complete description of the iterations of holomorphic self-maps over strongly pseudoconvex domains. Many examples show that this is false for general weakly pseudoconvex domains. As mentioned in the first paragraph, the complex analytic properties of the weakly pseudoconvex domains can differ very much from those of strongly pseudoconvex domains. However, for bounded weakly pseudoconvex domains with real-analytic boundaries, it seems to be in many respects more similar to the strongly pseudoconvex case. In §3 we shall give the Denjoy-Wolff-type theorem for such domains in $\mathbf{C}^2$ with exactly the same form as that of the strongly pseudoconvex domains. As an application, we obtain the Denjoy-Wolff-type theorem for contractible weakly pseudoconvex domains with real-analytic boundaries in exactly the same form as the classical Denjoy-Wolff-type theorem; an analogous result for contractible strongly pseudoconvex domains with $C^\infty$boundaries was got by Daowei Ma in 1991 (see [13]).

## §2. General Case

**Theorem 2.1.** *Let $\Omega \subset\subset \mathbf{C}^2$be a weakly pseudoconvex domain $f \in H(\Omega, \Omega)$. If $\{f^n\}$ is normal, and $f$ has a nonwandering point $p \in \Omega$, then one of the following holds:*

(I) *$p$ is an attracting fixed point of $f$ ;*

(II) *There is a submanifold $S \subset \Omega$ of dimension 1, such that $p \in S, p \in f(S)$, and $f|_S$ is an isomorphism. Furthermore, there exists a subsequence $f^{n_j}$such that $f^{n_j}|_S \to id_{S,}$ and*

$$\lim_{n\to\infty} d(f^n(z), S) = 0, \tag{2.1}$$

*where $d$ is any distance on $\Omega$.*

(III) $f \in \mathrm{Aut}(\Omega)$, *and* $\exists f^{n_j}, f^{n_j} \to id_{\Omega.}$

**Proof.** Denote by $\Gamma$ the closure of $\{f^n\}$ under the compact open topology, and by $\Gamma'$ the set of all limiting maps of $\{f^n\}$. Then the proof of the theorem follows immediately from the following assertions.

**Assertion 2.1.** For any $g \in \Gamma'$, we have $g(\Omega) \subset \partial\Omega$ or $g \in H(\Omega, \Omega)$.

In fact, by the definition of the normality, any convergent subsequence $f^{n_j}, \{f^{n_j}\}$, is either compactly divergent to $g, g(\Omega) \subset \partial\Omega$, or converges uniformly on compact subsets of $\Omega$ to $g \in H(\Omega, \Omega)$.

**Assertion 2.2.** There exists a $g \in \Gamma'$, such that $g \in H(\Omega, \Omega)$.

Since $f$ has a nonwandering point $p$, for any $k$ let

$$U_k = \{z \in \Omega |\ |z - p| < \frac{1}{k}\},$$

then $\exists n_k$ such that

$$f^{n_k}(U_k) \cap U_k \neq \emptyset. \tag{2.2}$$

If $\{n_k\}$ is bounded, then there is an $n_{k_0}$ such that

$$f^{n_{k_0}}(U_k) \cap U_k \neq \emptyset \tag{2.3}$$

holds for infinitely many $k'$s, this implies $f^{n_{k_0}}(p) = p$. So by Assertion 2.1 any convergent subsequence of $\{f^{kn_{k_0}}\}$ converges to a $g \in H(\Omega, \Omega)$.

If $\{n_k\}$ is unbounded, then again by Assertion 2.1 and (2.2) we know that the convergent subsequence of $\{f^{n_k}\}$ converges to a $g \in H(\Omega, \Omega)$.

**Assertion 2.3.** There is a subsequence of $\{f^n\}$ converging to a retraction $R$, $S = R(\Omega)$ is a submanifold of $\Omega$, $R|_S = id|_S$.

To prove this, as in the proof of Theorem 3.1 of [17], we use the skills of H. Cartan (see [5], and also [3], [15]).

By Assertion 2.2, $\exists f^{m_k} \to g \in H(\Omega, \Omega)$. Taking a subsequence if necessary, we can assume that

$$k_j = m_{j+1} - m_j \to \infty,$$

and

$$l_j = k_j - m_j = m_{j+1} - 2m_j \to \infty.$$

Again taking the subsequences if necessary, we may assume that both subsequences $\{f^{k_j}\}$ and $\{f^{l_j}\}$ are convergent. Let

$$f^{k_j} \to R \in \Gamma'(f) \subset H(\Omega, \bar{\Omega}),$$

$$f^{l_j} \to h \in \Gamma'(f) \subset H(\Omega, \bar{\Omega}).$$

By the relation

$$f^{m_{j+1}} = f^{k_j} \circ f^{m_j},$$

passing to the limit as $j \to \infty$ we get $g = R \circ g$. So $R \in H(\Omega, \Omega)$ by Assertion 2.1, and this again implies

$$g = g \circ R = R \circ g. \tag{2.4}$$

Similarly we have $h \in H(\Omega, \Omega)$, and

$$R = g \circ h = h \circ g. \tag{2.5}$$

Now from (2.4) and (2.5) we know

$$R^2 = h \circ g \circ h \circ g = h \circ R \circ g = h \circ g = R,$$

so $R \in \Gamma(f)$ is a retraction and $f^{k_j} \to R$.

By [6] (or [15], [3]), $S$ is a submanifold of $\Omega, R|_S = id|_S$.

Now we go to the proof of Theorem 2.1.

If $\dim S = 2$, then (III) holds since in this case $S = \Omega, f^{k_j} \to R = id_\Omega$, and so $f \in \mathrm{Aut}(\Omega)$.

If $\dim S \leq 1$, we first prove that $\Gamma(f) \subset H(\Omega, \Omega)$.

In fact, if $\dim S = 0$, then $S = \{a\}$ for some $a \in \Omega$ and $f^{k_j}(z) \to R(z) \equiv a$, so $f(a) = a$. By Assertion 2.1, for any $g \in \Gamma'(f)$, we have $g \in H(\Omega, \Omega)$, so $\Gamma(f) \subset H(\Omega, \Omega)$.

If dim$S = 1$, we use the reduction to absurdity. Suppose not, then $\exists f^{p_j}$,

$$f^{p_j} \to \rho \in H(\Omega, \bar{\Omega}), \quad \rho(\Omega) \subset \partial\Omega.$$

On the other hand, it is easily seen that $R \circ f = f \circ R$. So we have

$$f(S) = f \circ R(\Omega) = R \circ f(\Omega) \subset R(\Omega) = S. \tag{2.6}$$

Consequently $\rho(S) \subset \overline{f(S)} \subset \bar{S}$, this implies $\rho(S) \subset \bar{S} \setminus S$. By the proof of Proposition 12 of [10], we know that this is impossible. So $\Gamma(f) \subset H(\Omega, \Omega)$.

Now by $\Gamma(f) \subset H(\Omega, \Omega)$, we know that $\Gamma(f)$ is a compact Abel semigroup. So the theory of semigroup (see [16]) provides a unique retraction in $\Gamma(f)$, that is exactly the $R$ we get in Assertion 2.3. Repeating the proof of Assertion 2.3, we conclude that for any convergent subsequence $f^{p_j}$ of $\{f^n\}$, if $f^{p_j} \to F$, then $F = R \circ F$.

So if dim$S = 0$, then $F(z) \equiv a$. This implies $f^n(z) \to a, \forall z \in \Omega$. It is clear that $a = p$, this yields Conclusion (I).

If dim$S = 1$, then $F(\Omega) = R \circ F(\Omega) \subset R(\Omega) = S$, so $\mathrm{d}(F(z), S) = 0$, that is,

$$\lim_{n \to \infty} d(f^n(z), S) = 0.$$

Furthermore $f^{k_j}|_S \to R|_S = id|_S$, it is clear that $p \in f(S), p \in S$ (since $f^n(\Omega) \to S$), this means that (II) holds.

**Remarks.**

1. When $\Omega$ is taut, $H(\Omega, \Omega)$ is normal, so Theorem 2.1 holds without the additional assumption that $\{f^n\}$ is normal.

2. When $\Omega$ is a weakly pseudoconvex domain with $C^1$boundary, $\Omega$ is taut, so Theorem 2.1 holds without the additional assumption that $\{f^n\}$ is normal.

3. For general weakly pseudoconvex domain with $C^1$boundary, general $f \in H(\Omega, \Omega)$, we have four possibilities for $\{f^n\}$, besides (I)–(III), we may have that $g(\Omega) \subset \partial\Omega$ holds for all $g \in \Gamma$.

4. For general $\Omega \subset \mathbf{C}^N$, whether Theorem 2.1 is true or not is still open. But if $\Omega$ is complete under the Kobayashi metric, then we can show that an analogue of Theorem 2.1 holds without the additional assumption that $\{f^n\}$ is normal.

## §3. Domains With Real-Analytic Boundaries

The weakly pseudoconvex domains with real-analytic boundaries, besides the strongly pseudoconvex domains, are in a sense the most completely understood domains. In this section, we shall set up the Denjoy-Wolff-type theorem on such domains. We discuss first the properties of the horospheres beginning with a necessary definition and a sequence of lemmas and propositions.

**Definition 3.1.** *Let $\Omega \subset\subset \mathbf{C}^N$ be a domain, and choose $a \in \Omega, x \in \partial\Omega$ and $R > 0$. Then the small horosphere $E_a(x, R)$ and the big horosphere $F_a(x, R)$ with center $x$, pole $a$ and radius $R$ are defined by*

$$E_a(x, R) = \{z \in \Omega | \limsup_{w \to x}[K_\Omega(z, w) - K_\Omega(a, w)] < \frac{1}{2}\log R\},$$

$$F_a(x, R) = \{z \in \Omega | \liminf_{w \to x}[K_\Omega(z, w) - K_\Omega(a, w)] < \frac{1}{2}\log R\},$$

*where $K_\Omega(\cdot,\cdot)$ is the Kobayashi distance on $\Omega$.*

**Lemma 3.1** ([8], Theorem 2.3). *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous function satisfying the Dini condition $\int_0^1 \frac{\phi(x)}{x}dx < \infty$. Let $\Omega \subset\subset \mathbf{C}^N$ be a domain and $\Omega_1 \subset \Omega$ be an open subset of $\Omega$. Assume that for every point $A \in \partial\Omega_1 \cap \partial\Omega$ there exists a function $P_A \in C(\bar{\Omega}_1) \cap H(\Omega_1), |P_A| < 1$ on $\bar{\Omega}_1 \setminus \{A\}$, which peaks at $A$ and satisfies*

$$C|1 - P_A(z)| \le |z - A| \le \phi(1 - |P_A(z)|, \quad z \in \Omega_1. \tag{3.1}$$

*Then for every point $p$ in the relative interior of $\partial\Omega_1 \cap \partial\Omega$ there is a neighborhood $U$ of $p$ and a constant $K$ such that*

$$K_\Omega(z, w) \ge \frac{1}{2}\log\frac{1}{d(z)} - K, \quad \forall z \in U \cap \Omega_1, w \in \Omega \setminus \Omega_1.$$

An immediate consequence of this lemma is

**Corollary 3.1.** *Let $\Omega \subset\subset \mathbf{C}^N$ be a domain with $C^2$ boundary, and $p \in \partial\Omega$ be a strongly pseudoconvex point. Then there exist a $\delta > 0$,*

$$\Omega_k = \{z \in \Omega \mid |z - p| < k\delta\}, \quad k = 1, 2,$$

*and a constant $K$ such that*

$$K_\Omega(z, w) \ge \frac{1}{2}\log\frac{1}{d(z)} - K, \quad \forall z \in U \cap \Omega_1, w \in \Omega \setminus \Omega_2.$$

This result was mentioned without proofs in the Remark after Theorem 2.3 of [8]. Here we give the main points for the proof.

(I) By Narasimhan's Theorem (see [11], Lemma 2.3), there are a defining function $\rho$ for $\Omega$, a neighborhood $U \subseteq \mathbf{C}^N$ of $p$, and a biholomorphic coordinate change $\phi$ on $U$,

$$\phi(z) = \Big(z_1 + \frac{1}{2}\sum_{j,k=1}^{N}\frac{\partial^2\rho}{\partial z_j\partial z_k}(p)z_jz_k, z_2, \cdots, z_N\Big),$$

such that $\phi(U \cap \partial\Omega) \subset \mathbf{C}^N$ is strongly convex.

(II) Choosing $U$ sufficiently small, for each point $q \in \phi(U \cap \partial\Omega)$, one can easily find a ball $B_q$ such that $\phi(U \cap \Omega) \subset B_q$ and $\phi(U \cap \partial\Omega)$ is internally tangent to $\partial B_q$ at $q$. These $B_q$'s can be chosen with uniformly bounded diameters. Let $\alpha_q$ be the unit outward normal of $\partial B_q$ at $q$. Define

$$g_A(z) = \exp\langle\phi(z) - q, \alpha\rangle, \quad z \in U \cap \Omega.$$

Then $g_A$ is a local peak function for $\Omega$ at the point

$$A = \phi^{-1}(q) \in \partial\Omega \cap U,$$

and

$$\frac{1}{4}|1 - g_A(z)| \le |z - A| \le 2|1 - |g_A(z)||^{\frac{1}{3}} \tag{3.2}$$

holds in a neighborhood of $A$.

(III) Choosing $\delta > 0$ sufficiently small, one can find a constant $C$ such that

$$\frac{1}{C}|1 - g_A(z)| \le |z - A| \le C|1 - |g_A(z)||^{\frac{1}{3}}, \quad \forall z \in \Omega_1, \quad A \in \partial\Omega_1 \cap \partial\Omega.$$

So Corollary 3.1 follows immediately from Lemma 3.1.

**Lemma 3.2** ([8], Proposition 2.5). *If $\Omega$ is a domain whose boundary $\partial\Omega$ is of class $C^{1+\varepsilon}(\varepsilon > 0)$ near a point $A \in \partial\Omega$, then there exist a neighborhood $U$ of $A$ and a constant $C$*

*such that for all* $z_0, z_1 \in \Omega \cap U$,

$$K_\Omega(z_0, z_1) \le \frac{1}{2}\sum_{j=0}^{1} \log \frac{1}{d(z_j)} - \frac{1}{2}\sum_{j=0}^{1} \log \frac{1}{d(z_j) + |z_0 - z_1|} + C,$$

*where* $d(z)$ *is the Euclidean distance from* $z$ *to* $\partial\Omega$.

This lemma implies directly the following

**Corollary 3.2.** *Let* $\Omega \subset \mathbf{C}^N$ *be a domain with* $C^{1+\varepsilon}$ *boundary. Then for any* $a \in \Omega, \exists K$ *such that*

$$K_\Omega(z, a) \le \frac{1}{2} \log \frac{1}{d(z)} + K, \quad \forall z \in \Omega.$$

Using Corollaries 3.1 and 3.2, a nearly line by line copy of the proof of Theorem 1.7 in [1] gives the following Proposition 3.1.

**Proposition 3.1.** *Let* $\Omega \subset\subset \mathbf{C}^N$ *be a domain with* $C^2$ *boundary which is strongly pseudoconvex at a point* $p \in \partial\Omega$. *Then for any* $a \in \Omega, R > 0$, *we have*

$$\overline{F_a(p, R)} \cap \partial\Omega = \{p\}.$$

**Proposition 3.2.** *Let* $\Omega \subset \mathbf{C}^2$ *be a weakly pseudoconvex domain with real-analytic boundary. Then for any* $a \in \Omega, p \in \partial\Omega, R > 0$, *we have*

$$\overline{F_a(p, R)} \cap \partial\Omega = \{p\}$$

*or*

$$\overline{F_a(p, R)} \cap \partial\Omega = \emptyset.$$

**Proof.** Denote by $A(\Omega)$ the family of all holomorphic functions which are continuous on $\bar{\Omega}$. Let

$$\mathcal{P}(A(\Omega)) = \{p \in \partial\Omega \mid \exists f_p \in A(\Omega), f_p(p) = 1, |f_p(z)| < 1, \forall z \in \bar{\Omega} \setminus \{p\}\}.$$

Bedford and Fornæss showed in 1978 that $\mathcal{P}(A(\Omega)) = \partial\Omega$ if $\Omega$ is a weakly pseudoconvex domain with real-analytic boundary (see [4], Theorem 3.1, and also [14], [9]). In [2], Basener proved that, if $\Omega$ is a Levi pseudoconvex domain with $C^\infty$ boundary, then $\mathcal{P}(A(\Omega))$ is contained in the closure of the strongly pseudoconvex points (see also [11]). So the set of strongly pseudoconvex points is dense in $\partial\Omega$. Now by Proposition 3.1, for each strongly pseudoconvex point $p \in \partial\Omega$, we have

$$\overline{F_a(p, R)} \cap \partial\Omega = \{p\}.$$

To prove Proposition 3.2 for general point $x \in \partial\Omega$, we first prove that

$$F_a(x, R) \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} F_a(x_n, R) \tag{3.3}$$

for any sequence $\{x_k\} \subset \partial\Omega$ with $x_k \to x$.

In fact, proving (3.3) is equivalent to proving that $\forall z \in F_a(x, R), \exists k = k(z)$ such that $z \in F_a(x_n, R)$ holds for all $n \ge k$. Suppose that this is not true, then $\exists k(j) \to \infty$ with $z \notin F_a(x_{k(j)}, R)$, that is,

$$\liminf_{w \to x_{k(j)}} K_\Omega(z, w) - K_\Omega(a, w) \ge \frac{1}{2} \log R. \tag{3.4}$$

Now, since $z \in F_a(x,R)$, we can find a sequence $w_j \to x$ with

$$\lim_{n\to\infty} K_\Omega(z,w_n) - K_\Omega(a,w_n) < \frac{1}{2}\log R. \tag{3.5}$$

Choose for each $k(j)$ a sequence $u_{ji} \to x_{k(j)}$ $(i \to \infty)$ with $u_{jj} = w_j$. Then by (3.4) we have

$$\liminf_{i\to\infty} K_\Omega(z,u_{ji}) - K_\Omega(a,u_{ji}) \geq \frac{1}{2}\log R, \quad \forall j = 1,2,\cdots.$$

So

$$\liminf_{j\to\infty} K_\Omega(z,u_{jj}) - K_\Omega(a,u_{jj}) \geq \frac{1}{2}\log R,$$

that is,

$$\lim_{j\to\infty} K_\Omega(z,w_j) - K_\Omega(a,w_j) \geq \frac{1}{2}\log R.$$

This contradicts (3.5), so we have proved (3.3).

To finish the proof, we need to prove that if $y \in \overline{F_a(x,R)} \cap \partial\Omega$, then $y = x$. To prove this, we suppose that $y \in \partial\Omega$, and $y \neq x$. Let $|x-y| = 2\varepsilon$, choose $\{x_n\} \subset \partial\Omega, x_n \to x$ and $|x_n - y| > \varepsilon$, where all the $x_n$'s are strongly pseudoconvex. By Proposition 3.1,

$$\overline{F_a(x_n,R)} \cap \partial\Omega = \{x_n\},$$

so we have

$$\overline{\bigcup_{k=1}^{\infty} F_a(x_k,R)} \cap \partial\Omega = \bigcup_{k=1}^{\infty}\{x_k\} \cup \{x\} \equiv E.$$

By the choice of $\{x_n\}$ we know that $\mathrm{dist}(E,y) > \varepsilon$. So $\exists\delta > 0$ such that

$$\Big(\bigcup_{k=1}^{\infty} F_a(x_n,R)\Big) \cap B(y,\delta) = \emptyset. \tag{3.6}$$

So by (3.3) and (3.6) we get $y \notin \overline{F_a(x,R)}$.

**Remark 3.1.** One can easily prove that

$$\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} F_a(x_n,R) \subset \overline{F_a(x,R)}.$$

So, by Proposition 3.2, if we can prove that

$$\overline{\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} F_a(x_n,R)} \cap \partial\Omega \neq \emptyset,$$

then we have

$$\overline{F_a(p,R)} \cap \partial\Omega = \{p\}$$

for any $p \in \Omega$.

**Lemma 3.3.** *Let $\Omega \subset \mathbf{C}^2$ be a weakly pseudoconvex domain with real-analytic boundary, $f \in H(\Omega,\bar\Omega)$. Then either $f \in H(\Omega,\Omega)$ or $f(z) \equiv \xi$ for some $\xi \in \partial\Omega$.*

**Proof.** It Suffices to show that if $f(a) = \xi \in \partial\Omega$ for some $a \in \Omega$, then $f(z) \equiv \xi$.

In fact, by [4] every boundary point of $\Omega$ is a peak point. So, if we denote by $A(\Omega)$ the set of all functions continuous on $\overline{\Omega}$ and holomorphic in $\Omega$, then for $\xi = f(a) \in \partial\Omega$ one can find a $g \in A(\Omega)$ with $g(\xi) = 1$, and

$$|g(z)| < 1, \forall z \in \overline{\Omega} - \{\xi\}.$$

Consider the holomorphic function $h = g \circ f$, then

$$h(a) = g(f(a)) = g(\xi) = 1,$$

and $|h(z)| \leq 1, \forall z \in \overline{\Omega}$. So the Maximum Modulus Theorem implies

$$h(z) \equiv 1, \text{ or } g(f(z)) \equiv 1.$$

This means $f(z) \equiv \xi$ by the choice of $g$.

Now we can prove our main theorem.

**Theorem 3.1.** *Let $\Omega \subset \mathbf{C}^2$ be a weakly pseudoconvex domain with real-analytic boundary, $f \in H(\Omega, \Omega)$. Then one of the following holds:*

(I) *The sequence $\{f^m\}$ converges to a point $\xi \in \overline{\Omega}$ uniformly on the compact subset of $\Omega$;*

(II) *There exists a unique holomorphic retraction $R_f \in \Gamma'(f)$ such that, for any $g \in \Gamma'(f), \exists T \in \mathrm{Aut}(V)$ with $g = T \circ R_f$, and $V = R_f(\Omega)$ is a submanifold of $\Omega$.*

**Proof.** If $f$ has a nonwandering point $p$, then the conclusion in Theorem 2.1 gives us the desired results.

If all points of $\Omega$ are wandering, then all limiting mappings map $\Omega$ into $\partial\Omega$. So by Lemma 3.3, any limiting map must be a constant map. Now, given $a \in \Omega$, since all limit points of $\{f^k\}$ lie in the boundary, we have

$$\lim_{m\to\infty} K_\Omega(a, f^m(a)) = \infty.$$

With no difficulty, one can choose a subsequence $\{m_j\}, m_j \to \infty$ and

$$K_\Omega(a, f^{m_j}(a)) < K_\Omega(a, f^{m_j+k}(a)), \qquad \forall k \in \mathbf{Z}^+. \tag{3.7}$$

By the normality of $\{f^m\}$, taking a subsequence if necessary, we can assume that $f^{m_j}$ is convergent. Let $f^{m_j} \to x \in \partial\Omega$. Then

$$\lim_{j\to\infty} f^{m_j}(a) = x,$$
$$\lim_{j\to\infty} f^{m_j+k}(a) = \lim_{j\to\infty} f^{m_j}(f^k(a)) = x, \quad \forall k = 1, 2, \cdots. \tag{3.8}$$

So given any $z \in E_a(x, R)$, for $k = 1, 2, \cdots$, by (3.7), (3.8) and the nonexpansivity of $f$, we have

$$\begin{aligned}
&\liminf_{w\to x}[K_\Omega(f^k(z), w) - K_\Omega(a, w)]\\
&\leq \liminf_{j\to\infty}[K_\Omega(f^k(z), f^{m_j+k}(a)) - K_\Omega(a, f^{m_j+k}(a))]\\
&\leq \liminf_{j\to\infty}[K_\Omega(z, f^{m_j}(a)) - K_\Omega(a, f^{m_j+k}(a))]\\
&\leq \liminf_{j\to\infty}[K_\Omega(z, f^{m_j}(a)) - K_\Omega(a, f^{m_j}(a))]\\
&\leq \limsup_{w\to x}[K_\Omega(z, w) - K_\Omega(a, w)]\\
&< \frac{1}{2}\log,
\end{aligned}$$

that is, $f^k(z) \in F_a(x, R)$. This means that

$$f^k(E_a(x, R)) \subset F_a(x, R).$$

So for any convergent subsequence $f^{k_j}$ we have

$$f^{k_j}(z) \to \chi \in \overline{F_a(x,R)} \cap \partial\Omega.$$

By Proposition 3.2, it must hold that $\chi = x$, that is, $f^k(z) \to x$. The theorem is proved.

When $\Omega$ is contractible, we have an even sharper result.

**Theorem 3.2.** *Let $\Omega \subset\subset \mathbf{C}^2$ be a contractible weakly pseudoconvex domain with real-analytic boundary and $f \in H(\Omega, \Omega)$. Then*

(I) *if* $\text{Fix}(f) = \emptyset$, *then* $f^k(z) \to \chi \in \partial\Omega$;

(II) *if* $\text{Fix}(f) \neq \emptyset$, *then there is a holomorphic retraction $R_f \in \Gamma'(f)$ such that for any $g \in \Gamma'(f), \exists T \in \text{Aut}(V)$ with $g = T \circ R_f$, and $V = R_f(\Omega)$ is a submanifold of $\Omega$.*

**Proof.** By Theorem 3.1, we need only to prove that Conclusion (II) of Theorem 3.1 ensures that $\text{Fix}(f) \neq \emptyset$. The proof for this is essentially the same as that of Theorems 6 and 7 of [13]. Here we omit it.

**Remark 3.2.** In the proof of Theorem 3.1, we have used the result of Theorem 2.1, which is proved only for domains in $\mathbf{C}^2$ and may not be true for domains in general $\mathbf{C}^N$. But the existence of peak functions for $A(\Omega)$ implies that $\Omega$ is complete in the Kobayashi metric. So the method used in [17] is available here for the proof of Theorem 3.1. The same idea allows us to prove the result of Theorem 3.1 for any weakly pseudoconvex domain in $\mathbf{C}^N$, which is of $C^\infty$ boundary and satisfies $\mathcal{P}(A(\Omega)) = \partial\Omega$.

**Remark 3.3.** Proposition 3.2 is true for any smoothly weakly pseudoconvex domain $\Omega$ in $\mathbf{C^2}$. To prove this, the main point is that the set of strongly pseudoconvex points is dense on $\partial\Omega$. In fact, the strongly pseudoconvex points are generic on the boundaries of such domains. This can be proved by using the notion of finite type (see [9] for the definition) and the Foliation Theorem (see [11], p.274), for the details of this see [12].

Recall that a domain $\Omega \subset\subset \mathbf{C}^N$ is said to be of simple boundary if all holomorphic mappings $h : \{z \in \mathbf{C} \mid |z| < 1\} \to \partial\Omega$ are constants ( see [3] ). Especially when $\Omega \subset\subset \mathbf{C}^2$ is a smooth pseudoconvex domain of finite type, it is proved in [9] that at each point $p \in \partial\Omega$ there exists a peak function $f \in A(\Omega)$. Hence the proof of Lemma 3.3 shows that $\partial\Omega$ is simple. So by Remarks 3.2 and 3.3, using the proof of Theorems 3.1 and 3.2 we have

**Theorem 3.3.** *When $\Omega \subset \mathbf{C}^2$ is a weakly pseudoconvex domain with smoothly simple boundary, especially, when $\Omega \subset\subset \mathbf{C}^2$ is a smooth pseudoconvex domain of finite type, and $f \in H(\Omega, \Omega)$, the results of Theorem* 3.1 *hold.*

**Theorem 3.4.** *When $\Omega \subset\subset \mathbf{C}^2$ is a contractible weakly pseudoconvex domain with smoothly simple boundary, especially, when $\Omega \subset\subset \mathbf{C}^2$ is a contractible smooth pseudoconvex domain of finite type, and $f \in H(\Omega, \Omega)$, the results of Theorem* 3.2 *hold.*

## References


[1] Abate, M., Horspheres and iterates of Holomorphic maps, *Math. Z.,* **198**(1988), 225-238.

[2] Basener, R., Peak points, bariers, and pseudoconvex boundary points, *Proc. Amer. Math.Soc.,* **65**(1977), 89-92.

[3] Bedford, E., On the automorphism group of a stein manifold, *Math. Ann.,* **266**(1983), 215-227.

[4] Bedford, E. & Fornæss, J. E., A construction of peak functions on weakly pseudoconvex domains, *Ann. of Math.,* **107**(1978), 555-568.

[5] Cartan, H., Sur les functions de plusieurs variables complexes, *Math. Z.,* **35**(1932), 760-773.

[6] Cartan, H., Sur kes rétractions d'une vari eté,*C.R. Acad. Sc. Paris,* **303**(1986), 715-716.

[7] Diederich, K. & Fornæss, J. E., Pseudoconvex domains with real analytic boundaries, *Ann. of Math.,* **107**(1978), 371-384.

[8] Forstneric, F. & Rosay, J. P., Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings, *Math. Ann.,* **279**(1987), 239-252.

[9] Fornæss, J. E. & Sibony, N., Construction of p.s.h. functions on weakly pseudoconvex domains, *Duke Math. J.,* **58**(1989), 633-655.

[10] Hriljac, P., Analytic dynamics in some pseudoconvex domains, *Nonlinear Anal. Theory, Methods,* & *Appl.,* **19**(1992), 717-730.

[11] Krantz, S., Function theory of SCV, John Wiley & Sons, New York, 1982.

[12] Krantz, S. G., Recent progress and future directions in SCV, *Lecture Notes in Math.,* **1268**(1987), 1-23.

[13] Ma, D., On iterates of holomorphic maps, *Math. Z.,* **207**(1991), 417-428.

[14] Sibony, N., Some aspects of weakly pseudoconvex domains, *Proceedings of Symposia in Pure Math.,* **52**(1991), Part1, 199-231.

[15] Vigué, J. P., Fixed points of holomorphic mappings, *Lecture Notes in Math.,* **1422** (1990), 101-106.

[16] Wallace, A., The structure of topological semigroup, *Bull. Amer. Math. Soc.,* **61**(1955), 95-112.

[17] Zhang, W. J., Ren, F. Y. & Qiao, J. Y., On iterations of holomorphic self-maps on strongly pseudoconvex domains, *Chinese Sci. Bulletin*, **38**(1993), 881-883.