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AN EMBEDDING THEOREM BETWEEN SPECIAL LINEAR GROUPS OVER ANY FIELDS

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Abstract

Abstract homomorphisms between subgroups of algebraic groups were studied in detail by A.Borel, J.Tits[1] and B.Weisfeiler[2] provided that the images of the homomorphisms are Zariski dense subsets and that the fields over which algebraic groups are defined are infinite. The purpose of this paper is to determine all embedding homomorphisms of $SL_n(k)$ into $SL_n(K)$ when k and K are any fields of the same characteristic, without assumption of Zariski density and infinitude of fields. The result in this paper generalizes a result of Chen Yu on homomorphisms of two dimensional linear groups[3].

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Let D and E be two division rings. Dicks and Hartley conjectured that all non-trivial homomorphisms of $SL_n(D)$ to $SL_n(E)$ arise from homomorphisms of D to E. The conjecture has been verified by themselves[4] when $n = 2$, char$D$ =char$E$, char$D \neq 2, 3, |D| \neq 5$ and E is finite dimensional over its center. Earlier, the conjecture had been verified by Chen Yu when $n = 2, D$ and E are fields and $|D| > 5$. Our contribution to this problem is the following result.

Theorem 1. *Let k and K be any fields of the same characteristic. Suppose that $\sigma : SL_n(k) \longrightarrow SL_n(K)$ is an embedding homomorphism, where $n \geq 3$. Then there exist $Q \in GL_n(K)$ and a homomorphism $\alpha : k \longrightarrow K$ such that for any $A \in SL_n(k)$,*

$$\sigma(A) = QA^\alpha Q^{-1} \text{ or } \sigma(A) = Q((A^\alpha)')^{-1}Q^{-1},$$

where $A^\alpha = (\alpha(a_{ij}))$ if $A = (a_{ij})$. Moreover, α is unique and Q is unique up to a scalar element of $GL_n(K)$.

For any field F and any positive integer $n \geq 2$, we first introduce the following subgroups of $SL_n(F)$. We write $U_+(F)$ and $U_-(F)$ for the subgroups of upper and lower triangular $n \times n$ matrices over F with ones on the diagonal, respectively. And for $1 \leq i, j \leq n, i \neq j$, we write $T_{ij}(F)$ for the root subgroup consisting of the matrices $I + aE_{ij}$, where $a \in F$. We have the following fundamental commutator relations if we denote by $T_{ij}(a)$ the matrix $I + aE_{ij}$ for any $a \in F$.

$$(T_{ij}(a) \quad T_{rs}(b)) = 1, \quad \text{if } j \neq r, i \neq s,$$

$$(T_{ij}(a) \quad T_{js}(b)) = T_{is}(ab), \quad \text{if } i \neq s.$$

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Lemma 1. *Let*

$$U_+(F) = U_+(F)^{(1)} \supset U_+(F)^{(2)} \supset \cdots \supset U_+(F)^{(n-1)} \supset U_+(F)^{(n)} = 1,$$

$$U_-(F) = U_-(F)^{(1)} \supset U_-(F)^{(2)} \supset \cdots \supset U_-(F)^{(n-1)} \supset U_-(F)^{(n)} = 1$$

denote the lower central series of $U_+(F)$ *and* $U_-(F)$. *Then for* $1 \le s \le n-1, U_+(F)^{(s)}$ *is generated by* $T_{ij}(F)$ *with* $j - i \ge s$ *and* $U_-(F)^{(s)}$ *is generated by* $T_{ij}(F)$ *with* $i - j \ge s$. *In particular,*

$$U_+(F)^{(n-1)} = T_{1n}(F), \qquad U_-(F)^{(n-1)} = T_{n1}(F).$$

Proof. By the fundamental commutator relations Lemma 1 is immediate.

We may characterize unipotent matrices in such a way. A matrix $A \in SL_n(F)$ is unipotent if and only if

i) when char $F = p > 0$, there exists a positive integer r such that $A^{p^r} = I$;

ii) when char $F = 0, A^k \neq I$ and there exists a positive integer r such that the matrices A^{k^r} are conjugate to each other in $SL_n(F)$, where $k = 1, 2, 3, \cdots$.

By using the characterization of unipotent matrices, it is easy to get

Lemma 2. *Let* k *and* K *be any fields with* char$k$ = char$K$. *Suppose that* $\sigma : SL_n(k) \longrightarrow SL_n(K)$ *is an embedding homomorphism. Then there exists a matrix* $Q \in SL_n(K)$ *such that*

$$Q\sigma(U_+(k))Q^{-1} \subseteq U_+(K).$$

Proof. For any $A \in U_+(k), \sigma(A)$ is a unipotent matrix by the above characterization, and then $\sigma(U_+(k))$ is a unipotent subgroup of $SL_n(K)$. Now the existence of the matrix Q is clear (see [5], Chapter 5, Theorem 2.1).

From now on we always assume that k and K are fields of the same characteristic, and F is any field. Let $H(F)$ denote the subgroup of diagonal matrices of $SL_n(F)$, and $B_\pm(F) = H(F)U_\pm(F)$. We have the following key lemma.

Lemma 3. *Suppose that* $\sigma : SL_n(k) \longrightarrow SL_n(K)$ *is an embedding homomorphism and* $\sigma(U_+(k)) \subseteq U_+(K)$. *Then there exists a matrix* $Q = DB$, *where* $B \in B_+(K), D$ *is a diagonal matrix in* $GL_n(K)$ *such that*

i) $Q\sigma(U_+(k))Q^{-1} \subseteq U_+(K), \quad Q\sigma(T_{1n}(1))Q^{-1} = T_{1n}(1)$;

ii) $Q\sigma(T_{n1}(k))Q^{-1} \subseteq T_{n1}(K)$.

Proof. It is clear that if we replace $U_+(k)$ and $U_+(K)$ by $U_-(k)$ and $U_-(K)$, the result of Lemma 2 is still true. Then there exists a matrix $A \in SL_n(K)$ such that

$$A\sigma(U_-(k))A^{-1} \subseteq U_-(K).$$

Lemma 1 implies that $A\sigma(T_{n1}(k))A^{-1} \subseteq T_{n1}(K)$. By the Bruhat decomposition A can be written as $A = CNB$, where $C \in U_-(K), B \in B_+(K)$ and N is a permutation matrix, so

$$B\sigma(T_{n1}(k))B^{-1} \subseteq N^{-1}C^{-1}T_{n1}(K)CN \subseteq T_{ij}(K)$$

for some root subgroup $T_{ij}(K)$. Since

$$B\sigma(U_+(k))B^{-1} \subseteq BU_+(K)B^{-1} \subseteq U_+(K)$$

and, by Lemma 1, $B\sigma(T_{1n}(k))B^{-1} \subseteq T_{1n}(K)$, we have

$$B\sigma((T_{1n}(1) \quad T_{n1}(1)) \quad T_{1n}(1))B^{-1} \in ((T_{1n}(K) \quad T_{ij}(K)) \quad T_{1n}(K)).$$

If $T_{ij}(K) \neq T_{n1}(K)$,

$$((T_{1n}(K) \quad T_{ij}(K)) \quad T_{1n}(K)) = I,$$

but $((T_{1n}(1) \quad T_{n1}(1)) \quad T_{1n}(1)) \neq I$, which is a contradiction. Thus

$$B\sigma(T_{n1}(k))B^{-1} \subseteq T_{n1}(K).$$

Assume that $B\sigma(T_{1n}(1))B^{-1} = T_{1n}(a)$ for some $a \in K^*$. Take

$$D = \mathrm{diag}(1, \cdots, 1, a),$$

then the matrix $Q = DB$ is required.

We hope that we can make induction on n to prove Theorem 1, so we need the following simple lemma, which is immediate by a direct computation.

Lemma 4. Let $A = (a_{ij}) \in SL_n(F)$, a be any element of F^*.

i) $AT_{1n}(a)A^{-1} \in T_{1n}(F)$ if and only if $a_{21} = \cdots = a_{n1} = \cdots = a_{nn-1} = 0$;

ii) $AT_{n1}(a)A^{-1} \in T_{n1}(F)$ if and only if $a_{12} = \cdots = a_{1n} = \cdots = a_{n-1n} = 0$.

For $n = 2$ we restate Theorem 1 in a slightly different way. The result is implicit in Chen Yu's work.

Lemma 5. *Let $\sigma : SL_2(k) \longrightarrow SL_2(K)$ be an embedding homomorphism. Assume that*

$$\sigma(T_{12}(k)) \subseteq T_{12}(K), \qquad \sigma(T_{21}(k)) \subseteq T_{21}(K)$$

and $\sigma(T_{12}(1)) = T_{12}(1)$. Then there exists a homomorphism $\alpha : k \longrightarrow K$ such that for any $A \in SL_2(k)$

$$\sigma(A) = A^{\alpha}.$$

Proof. First, it is clear that

$$\sigma(B_+(k)) \subseteq B_+(K) \text{ and } \sigma(B_-(k)) \subseteq B_-(K)$$

by Lemma 4. Then

$$\sigma(H(k)) = \sigma(B_+(k) \cap B_-(k)) \subseteq B_+(K) \cap B_-(K) \subseteq H(K).$$

It follows that $\sigma(-I) = -I$ since $(\sigma(-I))^2 = I$ and $\sigma(-I) \in H(K)$. Let

$$\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ u & v \end{pmatrix}, \qquad \sigma\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

If we apply σ to both sides of the following identity

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we have

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix},$$

which implies $x = 0$ and $u = -y^{-1}$. From the identity

$$\begin{pmatrix} 0 & y \\ -y^{-1} & v \end{pmatrix}^2 = \begin{pmatrix} -1 & yv \\ -y^{-1}v & v^2 - 1 \end{pmatrix} = \sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -I$$

it follows that $v = 0$. Then

$$\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

since we have the identity

$$\begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+s & 2+s \\ s & 1+s \end{pmatrix},$$

which is obtained by applying σ to the both sides of the identity

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We define maps α and β of k to K by

$$\sigma(T_{12}(x)) = T_{12}(\alpha(x)), \qquad \sigma(T_{21}(x)) = T_{21}(\beta(x))$$

for any $x \in k$. Obviously, α and β are well defined and both are homomorphisms of k to K, as additive groups. By applying σ to both sides of the identity

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we have

$$\begin{pmatrix} 0 & 1 \\ -1 & -\alpha(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\beta(x) \end{pmatrix},$$

which means that, in fact, $\alpha = \beta$. Moreover, it is easy to see that for all $x \in k^*$,

$$\sigma \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha(x^{-1}) \end{pmatrix}$$

if we apply σ to the identity

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, for $x, y \in k^*$,

$$\begin{pmatrix} \alpha(xy) & 0 \\ 0 & \alpha(xy)^{-1} \end{pmatrix} = \sigma \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \sigma \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} \alpha(x)\alpha(y) & 0 \\ 0 & \alpha(x^{-1})\alpha(y^{-1}) \end{pmatrix}.$$

Therefore, $\alpha(xy) = \alpha(x)\alpha(y)$ and α is a homomorphism of k^* to K^*, as multiplicative group. Since $SL_2(k)$ is generated by $T_{12}(k)$ and $T_{21}(k)$, for any $A \in SL_2(k), \sigma(A) = A^\alpha$.

Before we proceed to prove Theorem 1, we need one more lemma.

Lemma 6. *Suppose that $\sigma : SL_n(k) \longrightarrow SL_n(K)$ is an embedding homomorphism and, for $1 \le i \le n-1$,*

$$\sigma(T_{i\,i+1}(k)) \subseteq T_{i\,i+1}(K), \qquad \sigma(T_{i+1\,i}(k)) \subseteq T_{i+1\,i}(K).$$

Then there exist a diagonal matrix $D \in GL_n(K)$ and a homomorphism $\alpha : k \longrightarrow K$ such that

$$D\sigma(A)D^{-1} = A^\alpha$$

for any $A \in SL_n(k)$.

Proof. For $1 \le i \le n-1$, assume $\sigma(T_{i\,i+1}(1)) = T_{i\,i+1}(a_i)$ for some $a_i \in K^*$. Let

$$D = \mathrm{diag}(d_1, d_2, \cdots, d_n),$$

where $d_i = a_i a_{i+1} \cdots a_{n-1}, 1 \le i \le n-1, d_n = 1$. Then

$$D^{-1}\sigma(T_{i\,i+1}(1))D = T_{i\,i+1}(1).$$

By Lemma 5 we see that there exist homomorphisms $\alpha_i : k \longrightarrow K$ such that

$$D^{-1}\sigma(T_{i\,i+1}(x))D = T_{i\,i+1}(\alpha_i(x))$$

for all $x \in k$. By applying σ to the identity

$$T_{i\,i+2}(x) = (T_{i\,i+1}(1)\ T_{i+1\,i+2}(x)) = (T_{i\,i+1}(x)\ T_{i+1\,i+2}(1)),$$

it follows that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = \alpha.$$

Hence, $D^{-1}\sigma(A)D = A^\alpha$ for any $A \in SL_n(k)$ since $SL_n(k)$ is generated by

$$T_{i\,i+1}(k), \quad T_{i+1\,i}(k), \quad 1 \le i \le n-1.$$

Now we state the result which is slightly stronger than the statement of Theorem 1, so that we can make induction on n. For any $A \in SL_n(K)$, let $\gamma(A) = P(A')^{-1}P^{-1}$, where

$$P = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ \cdot & & & & \\ 1 & & & & \end{pmatrix},$$

γ is called a graph automorphism of $SL_n(K)$, and for any $x \in K$,

$$\gamma(T_{ij}(x)) = T_{n+1-j\,n+1-i}(x).$$

Theorem 2. *Suppose that $\sigma : SL_n(k) \longrightarrow SL_n(K)$ is an embedding homomorphism and $\sigma(U_+(k)) \subseteq U_+(K)$. Then there exist a matrix $Q \in GL_n(K)$, which is upper triangular, and a homomorphism $\alpha : k \longrightarrow K$ such that for any $A \in SL_n(k)$,*

$$Q\sigma(A)Q^{-1} = A^\alpha$$

or

$$Q\gamma(\sigma(A))Q^{-1} = A^\alpha.$$

Proof. Lemma 3 together with Lemma 5 implies that there exists an upper triangular matrix $B \in GL_n(K)$ such that

$$B\sigma(U_+(k))B^{-1} \subseteq B_+(K),$$
$$B\sigma(T_{1n}(1))B^{-1} = T_{1n}(1),$$
$$B\sigma(T_{n1}(1))B^{-1} = T_{n1}(1).$$

Denote by σ' the homomorphism, where $\sigma'(A) = B\sigma(A)B^{-1}$ for any $A \in SL_n(k)$.

1) When $n = 2$, by Lemma 5 Theorem 2 is clearly true.

2) Assume that $n = 3$. In this case, it is easy to see that

$$\sigma'(U_-(k)) \subseteq U_-(K)$$

by applying σ' to both sides of $U_-(k) = NU_+(k)N^{-1}$, where

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is fixed under the action of σ' by Lemma 5. For each $a \in k^*$, let

$$\sigma'(T_{12}(a)) = T_{23}(y)T_{12}(x)T_{13}(z).$$

Then

$$\sigma'(T_{32}(-a)) = \sigma'(NT_{12}(a)N^{-1}) = T_{21}(y)T_{32}(-x)T_{31}(-z).$$

Since $(T_{12}(k) \quad T_{32}(k)) = 1$, we have

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ -z & -x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ -z & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

which forces $z = 0, x = 0$ or $y = 0$. It implies that

$$\sigma'(T_{12}(a)) \in T_{12}(K) \text{ or } \sigma'(T_{12}(a)) \in T_{23}(K)$$

for any $a \in k^*$. For $T_{23}(a)$ we have the same conclusion. Since $(T_{12}(x) \quad T_{23}(y)) = T_{13}(xy)$, it follows that

$$\sigma'(T_{12}(k)) \subseteq T_{12}(K), \qquad \sigma'(T_{23}(k)) \subseteq T_{23}(K)$$

or

$$\sigma'(T_{12}(k)) \subseteq T_{23}(K), \qquad \sigma'(T_{23}(k)) \subseteq T_{12}(K).$$

If the latter case occurs,

$$D\gamma\sigma'(T_{12}(k))D^{-1} \subseteq T_{12}(K), \qquad D\gamma\sigma'(T_{23}(k))D^{-1} \subseteq T_{23}(K),$$

where $D = \mathrm{diag}(-1, 1, 1)$ and the aim of introducing D is to guarantee

$$D\gamma\sigma'(T_{13}(1))D^{-1} = T_{13}(1).$$

Thus the latter case can be reduced to the first one. If the first case occurs, then

$$\sigma'(T_{32}(k)) = \sigma'(NT_{12}(k)N^{-1}) \subseteq T_{32}(K)$$

and similarly $\sigma'(T_{21}(k)) \subseteq T_{21}(K)$. By Lemma 6, for $n = 3$ Theorem 2 follows.

3) For $n \geq 4$, let $\tilde{G}(F)$ and $G(F)$ denote the subgroups of $SL_n(F)$ consisting of all matrices

$$\begin{pmatrix} a_1 & & \\ & A & \\ & & a_n \end{pmatrix},$$

where $a_1, a_n \in F^*, A \in GL_{n-2}(F)$, and

$$\begin{pmatrix} 1 & & \\ & A & \\ & & 1 \end{pmatrix},$$

where $A \in SL_{n-2}(F)$, respectively. It is clear that $\widetilde{G}(F)' = G(F)$, where $\widetilde{G}(F)'$ is the derived subgroup of $\widetilde{G}(F)$, unless $|F| = 2$ and $n = 4$, and $\widetilde{G}(F) = G(F)$ when $|F| = 2$. By Lemma 4, $\sigma'(\widetilde{G}(k)) \subseteq \widetilde{G}(K)$, and

$$\sigma'(G(k)) = \sigma'(\widetilde{G}(K)') \subseteq G(K) \subseteq \widetilde{G}(K)$$

when $|k| > 2$ or $n > 4$. For $|k| = 2$ and $n = 4$, write

$$\sigma'(T_{32}(1)) = \begin{pmatrix} a_1 & & \\ & A & \\ & & a_4 \end{pmatrix}.$$

Since $\sigma'(T_{32}(1))^2 = 1$, it forces $a_1 = a_4 = 1$ and $\sigma'(T_{32}(1)) \in G(K)$. Then, in any way, we have $\sigma'(G(k)) \subseteq G(K)$. The restriction of σ' to $G(k)$ satisfies the assumption of Theorem

2, so the induction hypothesis can be applied. Thus, there exist an upper triangular matrix $C \in GL_n(K)$ and a homomorphism $\sigma'' : SL_n(k) \longrightarrow SL_n(K)$, where for any $A \in SL_n(k)$

$$\sigma''(A) = C\sigma'(A)C^{-1}$$

or when $n > 4$,

$$\sigma''(A) = C\gamma\sigma'(A)C^{-1},$$

such that the following conditions are satisfied:

i) $\sigma''(U_+(k)) \subseteq U_+(K)$,

ii) $\sigma''(T_{ij}(k)) \subseteq T_{ij}(K), \qquad \sigma''(T_{ij}(1)) = T_{ij}(1)$,

for $2 \le i, j \le n-1$, or $i = 1, j = n$, or $i = n, j = 1$. Let

$$N = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & \cdot & & & & \\ & -1 & & & & \\ -1 & & & & & \end{pmatrix}_{n\times n},$$

which is fixed under the action of σ''. Since

$$\sigma''(U_+^{(n-2)}(k)) \subseteq U_+^{(n-2)}(K)$$

by Lemma 1, we may write

$$\sigma''(T_{1\,n-1}(a)) = T_{2n}(y)T_{1\,n-1}(x)T_{1n}(z)$$

for $a \in k^*$, and

$$\sigma''(T_{n2}(-a)) = \sigma''(NT_{1\,n-1}(a)N^{-1}) = T_{n-1\,1}(-y)T_{n2}(-x)T_{n1}(-z).$$

It follows that $z = 0, x = 0$ or $y = 0$ as we see in the proof of 2) when $n = 3$. This means that

$$\sigma''(T_{1\,n-1}(a)) \in T_{1\,n-1}(K), \qquad \sigma''(T_{n2}(-a)) \in T_{n2}(K)$$

or

$$\sigma''(T_{1\,n-1}(a)) \in T_{2n}(K), \qquad \sigma''(T_{n2}(-a)) \in T_{n-1\,1}(K)$$

for $a \in k^*$. It implies that

$$\sigma''(T_{1\,n-1}(k)) \subseteq T_{1\,n-1}(K) \text{ or } \sigma''(T_{1\,n-1}(k)) \subseteq T_{2n}(K).$$

Otherwise, if there exist $a, b \in k$ such that

$$\sigma''(T_{1\,n-1}(a)) \in T_{1\,n-1}(K), \qquad \sigma''(T_{1\,n-1}(b)) \in T_{2n}(K),$$

we would have $\sigma''(T_{n2}(-b)) \in T_{n-1\,1}(K)$, which gives a contradiction since

$$(T_{1\,n-1}(a) \qquad T_{n2}(-b)) = 1$$

and

$$(\sigma''(T_{1\,n-1}(a)) \qquad \sigma''(T_{n2}(-b))) \neq 1.$$

By the same reason we have the same conclusion for $T_{2n}(k)$, that means

$$\sigma''(T_{2n}(k)) \subseteq T_{2n}(K) \text{ or } \sigma''(T_{2n}(k)) \subseteq T_{1\,n-1}(K).$$

If $\sigma''(T_{1\,n-1}(k))$ and $\sigma''(T_{2n}(k))$ are both contained in $T_{1\,n-1}(K)$, then $\sigma''(T_{n-1\,1}(k))$ and $\sigma''(T_{n2}(k))$ are both contained in $T_{n2}(K)$, which is impossible since the subgroups $T_{1\,n-1}(K)$ and $T_{n2}(K)$ are commutative elementwise, but $T_{1\,n-1}(k)$ and $T_{n-1\,1}(k)$ are not. Similarly, it is impossible that σ'' sends $T_{1\,n-1}(k)$ and $T_{2n}(k)$ into $T_{2n}(K)$. Thus, the case that can occurs is

$$\sigma''(T_{1\,n-1}(k)) \subseteq T_{1\,n-1}(K), \qquad \sigma''(T_{2n}(k)) \subseteq T_{2n}(K)$$

or

$$\sigma''(T_{1\,n-1}(k)) \subseteq T_{2n}(K), \qquad \sigma''(T_{2n}(k)) \subseteq T_{1\,n-1}(K).$$

Applying σ'' to both sides of the following relations

$$(T_{n1}(k)\ T_{1\,n-1}(k)) = T_{n\,n-1}(k), \qquad (T_{n1}(k)\ T_{2n}(k)) = T_{21}(k),$$

we have

$$\sigma''(T_{n\,n-1}(k)) \subseteq T_{n\,n-1}(K), \qquad \sigma''(T_{21}(k)) \subseteq T_{21}(K)$$

or

$$\sigma''(T_{n\,n-1}(k)) \subseteq T_{21}(K), \qquad \sigma''(T_{21}(k)) \subseteq T_{n\,n-1}(K),$$

which is equivalent to the fact that

$$\sigma''(T_{12}(k)) \subseteq T_{12}(K), \qquad \sigma''(T_{n-1\,n}(k)) \subseteq T_{n-1\,n}(K)$$

or

$$\sigma''(T_{12}(k)) \subseteq T_{n-1\,n}(K), \qquad \sigma''(T_{n-1\,n}(k)) \subseteq T_{12}(K)$$

since

$$NT_{n\,n-1}(k)N^{-1} = T_{12}(k), \quad NT_{21}(k)N^{-1} = T_{n-1\,n}(k).$$

Now Theorem 2 follows if the first case occurs. When $n = 4$, if the second case occurs, the homomorphism $\sigma''' : SL_4(k) \longrightarrow SL_4(K)$, defined by

$$\sigma'''(A) = D\gamma\sigma''(A)D^{-1}$$

for any $A \in SL_4(k)$, satisfies the same conditions which are satisfied by σ'', where

$$D = \mathrm{diag}(-1, -1, 1, 1).$$

Thus, the second case can be reduced to the first one for $n = 4$. When $n > 4$, the second case can not occur since $T_{12}(k)$ and $T_{n-2,n-1}(k)$ are commutative elementwise, but $T_{n-2\,n-1}(K)$ and $T_{n-1\,n}(K)$ are not.

Proof of Theorem 1. What we only need to do is to prove the uniqueness. Let α_1, α_2 be homomorphisms of fields from k to K, and $Q_1, Q_2 \in GL_n(K)$. If

$$Q_1 A^{\alpha_1} Q_1^{-1} = Q_2 A^{\alpha_2} Q_2^{-1}$$

or

$$Q_1((A^{\alpha_1})')^{-1} Q_1^{-1} = Q_2((A^{\alpha_2})')^{-1} Q_2^{-1}$$

for all $A \in SL_n(k)$, then $Q_1 T_{ij}(1) Q_1^{-1} = Q_2 T_{ij}(1) Q_2^{-1}$ for $1 \leq i, j \leq n$. This clearly means that $Q_2^{-1} Q_1$ is a scalar and therefore $\alpha_1 = \alpha_2$, which has proved the uniqueness if we can claim that it is impossible to have

$$Q_1 A^{\alpha_1} Q_1^{-1} = Q_2((A^{\alpha_2})')^{-1} Q_2^{-1}.$$

But it is obvious since, otherwise, we would have

$$Q_2^{-1}Q_1T_{ij}(1)Q_1^{-1}Q_2 = T_{ji}(-1).$$

And by an easy computation it is absurd when $n \geq 3$.

When k is a subfield of K, for any homomorphism $\alpha : k \longrightarrow K$, $SL_n(k)^\alpha$ can be viewed as a subgroup of $SL_n(K)$ in a natural way. We call subgroups G_1 and G_2 of $SL_n(K)$ equivalent if there exists an automorphism τ of $SL_n(K)$ such that $\tau(G_1) = G_2$. Theorem 1 implies

Corollary 1. *Let k be a subfield of K. Then any subgroup G of $SL_n(K)$ which is isomorphic to $SL_n(k)$ is equivalent to $SL_n(k)^\alpha$ for some homomorphism $\alpha : k \longrightarrow K$. In particular, if K is an algebraically closed field, then all subgroups of $SL_n(K)$ which are isomorphic to $SL_n(k)$ are equivalent.*

As an end, we generalise Theorem 1 to general linear groups.

Theorem 3. *If $\sigma : GL_n(k) \longrightarrow GL_n(K)$ is an embedding homomorphism, where $n \geq 3$, then σ is of the form*

$$\sigma(A) = QA^\alpha Q^{-1}\chi(\det A)I$$

or

$$\sigma(A) = Q((A^\alpha)')^{-1}Q^{-1}\chi(\det A)I$$

for all $A \in GL_n(k)$, where $Q \in GL_n(K)$, α is a homomorphism of fields from k to K and χ is a homomorphism of groups from k^* to K^*. Moreover, both α and χ are unique while Q is unique up to a scalar element of $GL_n(K)$.

Proof. By Lemma 2, σ sends unipotent subgroups of $GL_n(k)$ to unipotent subgroups of $GL_n(K)$, then the restriction of σ to $SL_n(k)$ is an embedding homomorphism of $SL_n(k)$ to $SL_n(K)$. There exist $Q \in GL_n(K)$ and a homomorphism $\alpha : k \longrightarrow K$ such that

$$\sigma'(A) = Q^{-1}\sigma(A)Q = A^\alpha$$

or

$$\sigma'(A) = Q^{-1}(\sigma(A)')^{-1}Q = A^\alpha$$

for any $A \in SL_n(k)$. We see easily that a matrix $D \in GL_n(K)$ is a diagonal matrix if and only if $DT_{ij}(1)D^{-1} \in T_{ij}(K)$ for $1 \leq i, j \leq n$. So it is clear that σ' maps diagonal matrices of $GL_n(k)$ into diagonal matrices of $GL_n(K)$. For any $a \in k^*$, suppose

$$\sigma'(\mathrm{diag}(1, \cdots, 1, a)) = \mathrm{diag}(b_1, b_2, \cdots, b_n),$$

where $b_1, b_2, \cdots, b_n \in K^*$. We obtain

$$b_1 = b_2 = \cdots = b_{n-1} = b$$

and $b_n = \alpha(a)b$ by applying σ' to the both sides of the following identities

$$DT_{i\,i+1}(1)D^{-1} = T_{i\,i+1}(1), 1 \leq i \leq n-2,$$

$$DT_{n-1\,n}(1)D^{-1} = T_{n-1\,n}(a^{-1}),$$

where $D = \mathrm{diag}(1, \cdots, 1, a) \in GL_n(k)$. Now define a map $\chi : k^* \longrightarrow K^*$ by

$$\sigma'(\mathrm{diag}(1, \cdots, 1, a)) = \mathrm{diag}(\chi(a), \cdots, \chi(a), \alpha(a)\chi(a))$$

for any $a \in k^*$. Obviously, χ is a homomorphism of groups from k^* to K^*. Since every matrix A can be written uniquely in the form

$$A = A_1 \mathrm{diag}(1, \cdots, 1, \det A)$$

for some $A_1 \in SL_n(k)$, and moreover

$$A^\alpha = A_1^\alpha \mathrm{diag}(1, \cdots, 1, \alpha(\det A)),$$

we have

$$\sigma'(A) = A_1^\alpha \mathrm{diag}(1, \cdots, 1, \alpha(\det A))I = A^\alpha \chi(\det A) I,$$

as requested. Finally, the proof of the uniqueness of Q, α, and χ is similar to that of Theorem 1.

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