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OPTIMIZATION AND DUALITY OF CONE-D.C. PROGRAMMING**

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Abstract

This paper gives the definitions and some properties of ε-directional derivate and ε-subgradients of cone-convex function. From them, the optimality conditions of local and global optimal point of unconstrained cone-d.c. programming are gained. At last, the duality theorems of this programming are presented.

Keywords ε-directional derivate, ε-subgradient, Cone-d.c. function, Optimality condition, Duality theorem.
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§1. Introduction

As well known, convex analysis plays more and more important role in the theory of optimization with a single objective since 1950s. Along with it, the analysis and optimization of convex function have obtained a great deal of results[1]. As the generalization of the definition and properties of convex function, tremendous achievements have been made on the analysis and optimization of the nonconvex function in the last two decades, especially for the class of so called d.c. function[2]. Because of the particular structure of d.c. function, the tools and techniques received from convex analysis are quite helpful[3,4,5].

But a lot remains unsolved in this area, such as for the vector-valued functions. In this paper, we intend to give a discussion about the optimization and duality of vector-valued d.c. function. Since the analysis of the vector-valued convex function has not widely carried out, first of all, we should extend some results about convex function to cone-convex function, as the tools for the discussion of cone-d.c. function.

This paper is organized as follows. Section 2 investigates the concepts of ε-directional derivate and ε-subgradients of cone-convex function, which can be regarded as the preliminary of the following sections. Section 3 is devoted to the necessary and sufficient conditions of local and global optimal point of unconstrained cone-d.c. programming. The duality theorems of this programming are given in Section 4.

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§2. ε-Directional Derivate and ε-Subgradients of Cone-Convex Function

Assume that all spaces considered in this paper are finite dimensional Euclidean spaces, and that D is a closed pointed convex cone with nonempty interior. We adopt the definitions of positive polar cone D° of D, conical orders, (weak) efficient point and ideal point in ordered vector space $(\Re^p, D)$, and D-convex function in [6]. Clearly, $f: \Re^n \to \Re^p$ is a D-convex function if and only if its D-epigraph $D\text{-epi}f = \{(x, y) \in \Re^n \times \Re^p | f(x) \leq_D y\}$ is a convex set in $\Re^n \times \Re^p$.

We call $f: \Re^n \to \Re^p$ a D-sublinear function, if $f(\lambda x) = \lambda f(x)$, for any $\lambda \in [0, +\infty)$; and $f(x^1 + x^2) \leq_D f(x^1) + f(x^2)$, for any $x^1, x^2 \in \Re^n$.

Similar to the scalar sublinear function, we can easily prove the following lemma from the generalized Hahn-Banach Theorem presented in [7].

Lemma 2.1. *Suppose that $f: \Re^n \to \Re^p$ is a D-sublinear function. Then for every $x^0 \in \Re^n$, there exists $T \in \Re^{n \times p}$ such that*

$$T^T x^0 = f(x^0) \text{ and } T^T x \leq_D f(x), \quad \forall x \in \Re^n.$$

For a set $Y \subset \Re^p$, $y^* \in \Re^p$ is called an order upper-bound of the order upper-bounded set Y, if $y \leq_D y^*$ for every $y \in Y$. y^* is called order-supremum of Y, if it is an order upper-bound of Y, and for every order upper-bound y' of Y, $y^* \leq_D y'$. For a set Y without order upper-bound we define its order-supremum as $+\infty$.

The definitions of order lower-bound, order lower-bounded set and order-infimum are similar.

The order-supremum is something different from the maximal ideal point of the set Y, which is denoted as IMax Y, since the order-supremum is not required to be in the set whereas the other is.

Obviously, the order-supremum and order-infimum are unique if they exist, and denoted by sup$Y$ and inf$Y$ respectively. But sometimes, even for the order upper(lower)-bounded set, the order-supremum (infimum) does not exist. The ordered vector space $(\Re^p, D)$ is called the order-complete vector space, if for every order upper(lower)-bounded set in $(\Re^p, D)$ there exists the order-supremum (infimum) of this set.

The space $(\Re^p, D)$ is assumed to be the order-complete vector space in this paper.

In [8], we proved that D-convex function f is the locally D-Lipschitz function, which is defined as: for every $x \in \Re^n$, there exist a ball neighbourhood $B(x; \delta)$ with radius $\delta > 0$ and $d \in D$ such that

$$-\|x^1 - x^2\| d \leq_D f(x^1) - f(x^2) \leq_D \|x^1 - x^2\| d, \quad \forall x^1, x^2 \in B(x; \delta);$$

d is called the Lipschitz constant.

In the remainder of this section, we make some discussion on the ε-directional derivate and ε-subgradients of cone-convex function.

Definition 2.1. *For a D-convex function $f: \Re^n \to \Re^p$ and $\varepsilon \in \Re^p$, its ε-directional derivate $f'_\varepsilon: \Re^n \times \Re^n \to \Re^p \cup \{+\infty, -\infty\}$ is defined by*

$$f'_\varepsilon(x; v) = \inf_{t>0} \frac{f(x + tv) - f(x) + \varepsilon}{t}.$$

Obviously, Definition 2.1 is the extension of the direction derivate $f': \Re^n \times \Re^n \to \Re^p$ for

D-convex function f, which is presented in [8] as

$$f'(x;v) = \inf_{t>0} \frac{f(x+tv)-f(x)}{t}.$$

In [8], it was proved that the direction derivate can be equivalently defined by

$$f'(x;v) = \lim_{t\to 0^+} \frac{f(x+tv)-f(x)}{t}; \tag{2.1}$$

but it does not hold for the ε-directional derivate.

Theorem 2.1. *Let $f\colon \Re^n \to \Re^p$ be a D-convex function, then for every pair $(x;v) \in \Re^n \times \Re^n$, f'_ε exists in $\Re^p$ when $\varepsilon \in D$; and $f'_\varepsilon(x;v) = -\infty$ otherwise.*

Proof. i) $\varepsilon \in D$. By the assumption that $(\Re^p, D)$ is an order-complete vector space, it suffices to show that $\left\{\dfrac{f(x+tv)-f(x)+\varepsilon}{t} | t>0\right\}$ is order lower-bounded. This is immediately from

$$\frac{f(x+tv)-f(x)+\varepsilon}{t} \geqq_D \frac{f(x+tv)-f(x)}{t} \geqq_D f'(x;v), \quad \forall t>0,$$

where $f'(x;v)$ is in $\Re^p$.

ii)$\varepsilon \notin D$. Suppose to the contrary that there is $y \in \Re^p$, which is the order lower-bound of $\left\{\dfrac{f(x+tv)-f(x)+\varepsilon}{t} | t>0\right\}$. It follows that

$$y \leq_D \frac{f(x+tv)-f(x)}{t} + \frac{1}{t}\varepsilon, \quad \forall t>0. \tag{2.2}$$

Since $\varepsilon \notin D$ and D is a closed convex cone, there is $\lambda \in D^{\circ}$ with $\langle \lambda, \varepsilon \rangle < 0$. Taking inner product of (2.2) with λ, we have

$$\langle \lambda, y \rangle \leq \langle \lambda, \frac{f(x+tv)-f(x)}{t} \rangle + \frac{1}{t}\langle \lambda, \varepsilon \rangle. \tag{2.3}$$

When $t \to 0$, $\langle \lambda, \dfrac{f(x+tv)-f(x)}{t} \rangle \to \langle \lambda, f'(x;v) \rangle \in \Re$. Meanwhile $\dfrac{1}{t}\langle \lambda, \varepsilon \rangle \to -\infty$ when $t \to 0$. Thus

$$\langle \lambda, \frac{f(x+tv)-f(x)}{t} \rangle + \frac{1}{t}\langle \lambda, \varepsilon \rangle \to -\infty.$$

It contradicts (2.3).

The proof is completed.

Definition 2.2. *For a D-convex function $f\colon \Re^n \to \Re^p$, $T \in \Re^{n\times p}$ is called ε-strong subgradient (resp. ε-subgradient, ε-weak subgradient) of f at x, if $T^T v \leq_D f'_\varepsilon(x;v)$ (resp. $T^T v \ngeq_D f'_\varepsilon(x;v)$, $T^T v \not>_D f'_\varepsilon(x;v)$). ε-$\partial_s f(x)$ (resp. ε-$\partial f(x)$, ε-$\partial_w f(x)$) denotes the set of ε-strong subgradients (resp. ε-subgradients, ε-weak subgradients) at x, and is called the ε-strong subdifferential (resp. ε-subdifferential, ε-weak subdifferential) of f at x.*

Clearly, Definition 2.2 is the extension of the subdifferential $\partial_s f(x)$, $\partial f(x)$ and $\partial_w f(x)$ of D-convex function f, which are defined by $T^T v \leq_D f'(x;v)$, $T^T v \ngeq_D f'(x;v)$ and $T^T v \not>_D f'(x;v)$ respectively (see [8]).

Immediately from the definition, we have the following theorem.

Theorem 2.2. *For a D-convex function $f\colon \Re^n \to \Re^p$,*

$T \in \varepsilon$-$\partial_s f(x^0)$ if and only if $T^T(x-x^0) \leq_D f(x) - f(x^0) + \varepsilon$, $\forall x \in \Re^n$;

if $T \in \varepsilon$-$\partial f(x^0)$, then $T^T(x-x^0) \ngeq_D f(x) - f(x^0) + \varepsilon$, $\forall x \in \Re^n$;

if $T \in \varepsilon\text{-}\partial_w f(x^0)$, *then* $T^T(x - x^0) \not\geqq_D f(x) - f(x^0) + \varepsilon, \ \forall x \in \Re^n$.

Next, we discuss the relation between the ε-directional derivate and ε-subdifferentials.

Suppose that $h: \Re^n \to \Re^p$ is a D-convex function satisfying $h(0) \geqq_D 0$. Let $H = \bigcup\limits_{\lambda>0} \dfrac{D\text{-epi}h}{\lambda} \cup \{(0,0)\}$, and $g: \Re^n \to \Re^p$ be defined by $g(x) = \inf\{\mu|(x,\mu) \in H\}$. The following lemma holds.

Lemma 2.2. *$g(0) = 0$, and $g(x) = \inf\limits_{\lambda>0} \dfrac{h(\lambda x)}{\lambda}$ for $x \in \Re^n \backslash \{0\}$.*

Proof. i) If $x = 0$, $(0,\mu) \in \bigcup\limits_{\lambda>0} \dfrac{D\text{-epi}h}{\lambda}$ implies that there is $\lambda > 0$ such that $(0, \lambda\mu) \in D\text{-epi}h$, i.e.,

$$\mu \geqq_D \frac{h(0)}{\lambda} \geqq_D 0.$$

It follows from $(0,0) \in H$ that $g(0) = 0$.

ii) If $x \in \Re^n \backslash \{0\}$, $(x,\mu) \in H$ implies that there is $\lambda > 0$ such that $(\lambda x, \lambda\mu) \in D\text{-epi}h$, that is, $\mu \geqq_D h(\lambda x)/\lambda$. Thus

$$g(x) \geqq_D \inf_{\lambda>0} \frac{h(\lambda x)}{\lambda}.$$

On the other hand, $(\lambda x, h(\lambda x)) \in D\text{-epi}h$ for any $\lambda > 0$. Then $h(\lambda x)/\lambda \in \{\mu|(x,\mu) \in H\}$. This means that

$$\inf_{\lambda>0} \frac{h(\lambda x)}{\lambda} \geqq_D g(x).$$

Therefore $g(x) = \inf\limits_{\lambda>0} \dfrac{h(\lambda x)}{\lambda}$.

By Lemma 2.2, $g(x)$ is a positively homogeneous function. Let $H_x = \{\mu|(x,\mu) \in H\}$. Since H is a convex cone, $H_{x^1} + H_{x^2} \subset H_{x^1+x^2}$. So

$$\inf H_{x^1} + \inf H_{x^2} \geqq_D \inf H_{x^1+x^2}.$$

It means that g is a D-sublinear function.

Theorem 2.3. *Let $f: \Re^n \to \Re^p$ be a D-convex function. Then*

$$f'_\varepsilon(x^0; v^0) = \mathrm{IMax}\{T^T v^0 | T \in \varepsilon\text{-}\partial_s f(x^0)\}, \quad f'_\varepsilon(x^0; v^0) \in \mathrm{Max}\{T^T v^0 | T \in \varepsilon\text{-}\partial f(x^0)\},$$

and $f'_\varepsilon(x^0; v^0) \in \mathrm{WMax}\,\{T^T v^0 | T \in \varepsilon\text{-}\partial_w f(x^0)\}$ for every pair $(x^0; v^0) \in \Re^n \times \Re^n$ and $\varepsilon \in D$.

Proof. Let $h(v) = f(x^0 + v) - f(x^0) + \varepsilon$. Then h is a D-convex function with $h(0) \geqq_D 0$. It follows that $f'_\varepsilon(x^0; v) = \inf\limits_{\lambda>0} \dfrac{h(\lambda v)}{\lambda}$ and $f'_\varepsilon(x^0; 0) = 0$. From the discussion above, $f'_\varepsilon(x^0; \cdot)$ is a D-sublinear function.

Then from Lemma 2.1 there exists $T^0 \in \Re^{n \times p}$ such that

$$(T^0)^T v^0 = f'_\varepsilon(x^0; v^0) \text{ and } (T^0)^T v \leq_D f'_\varepsilon(x^0; v), \quad \forall v \in \Re^n.$$

It means that $T^0 \in \varepsilon\text{-}\partial_s f(x^0)$. Since $T^T v^0 \leq_D f'_\varepsilon(x^0; v^0)$ for any $T \in \varepsilon\text{-}\partial_s f(x^0)$, $f'_\varepsilon(x^0; v^0) = \mathrm{IMax}\{T^T v^0 | T \in \varepsilon\text{-}\partial_s f(x^0)\}$.

Notice that $\varepsilon\text{-}\partial_s f(x) \subset \varepsilon\text{-}\partial f(x)$ and $T^T v^0 \ngeq_D f'_\varepsilon(x^0; v^0)$ when $T \in \varepsilon\text{-}\partial f(x)$. It is immediate that $f'_\varepsilon(x^0; v^0) \in \mathrm{Max}\{T^T v^0 | T \in \varepsilon\text{-}\partial f(x^0)\}$.

Analogously, $f'_\varepsilon(x^0; v^0) \in \mathrm{WMax}\{T^T v^0 | T \in \varepsilon\text{-}\partial_w f(x^0)\}$.

The following corollary can be got by taking $\varepsilon = 0$ in Theorem 2.3.

Corollary 2.1. *Let $f:\Re^n \to \Re^p$ be a D-convex function. Then*

$$f'(x^0; v^0) = \mathrm{IMax}\{T^T v^0 | T \in \partial_s f(x^0)\}, \quad f'(x^0; v^0) \in \mathrm{Max}\{T^T v^0 | T \in \partial f(x^0)\},$$

and $f'(x^0; v^0) \in \mathrm{WMax}\ \{T^T v^0 | T \in \partial_w f(x^0)\}$ for every pair $(x^0; v^0) \in \Re^n \times \Re^n$.

It is known from Theorem 2.1 that

$$\varepsilon\text{-}\partial_s f(x) = \varepsilon\text{-}\partial f(x) = \varepsilon\text{-}\partial_w f(x) = \emptyset$$

when $\varepsilon \notin D$. Theorem 2.3 shows that ε-$\partial_s f(x)$, ε-$\partial f(x)$ and ε-$\partial_w f(x)$ are all nonempty sets, for $\varepsilon \in D$.

The succeeding theorem shows the relation of the function value and ε-directional derivate.

Theorem 2.4. *let $f:\Re^n \to \Re^p$ be a D-convex function. Then*

$$f(x+v) - f(x) = \sup_{\varepsilon \in D} (f'_\varepsilon(x; v) - \varepsilon).$$

In order to prove this theorem, the D-convex function $f:\Re^n \to \Re^p \cup \{+\infty\}$ is concerned, and $\mathrm{dom} f \subset \Re^n$ denotes the effective domain of f, i.e., $\{x \in \Re^n | f(x) <_D +\infty\}$. The concept of strong conjugate function of f is given below.

Definition 2.3. *For a D-convex function $f:\Re^n \to \Re^p \cup \{+\infty\}$, its strong conjugate function $f_s^*:\Re^{n\times p} \to \Re^p \cup \{+\infty\}$ is defined by*

$$f_s^*(T) = \sup\{T^T x - f(x) | x \in \Re^n\}.$$

The strong conjugate function of f_s^*, $f_s^{**}:\Re^n \to \Re^p \cup \{+\infty\}$, defined by

$$f_s^{**}(x) = \sup\{T^T x - f_s^*(T) | T \in \Re^{n\times p}\}$$

is called the strong biconjugate function of f.

Similar to the scalar function, we can also prove that

$$f_s^{**}(s) = f(x) \quad \text{for every } x \in \mathrm{int}(\mathrm{dom} f). \tag{2.4}$$

Proof of Theorem 2.4. Define $\sigma_v : \Re^1 \to \Re^p \cup \{+\infty\}$ by

$$\sigma_v(\mu) = \begin{cases} \mu(f(x+v/\mu) - f(x)), & \mu > 0, \\ \lim\limits_{\delta\to 0^+} \delta(f(x+v/\delta) - f(x)), & \mu = 0, \\ +\infty, & \mu < 0. \end{cases}$$

For any $\mu^1, \mu^2 > 0$ and $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned} f\Big(x + \frac{v}{\lambda_1\mu^1 + \lambda_2\mu^2}\Big) &= f\Big(\frac{\lambda_1\mu^1}{\lambda_1\mu^1 + \lambda_2\mu^2}\Big(x + \frac{v}{\mu^1}\Big) + \frac{\lambda_2\mu^2}{\lambda_1\mu^1 + \lambda_2\mu^2}\Big(x + \frac{v}{\mu^2}\Big)\Big) \\ &\le_D \frac{\lambda_1\mu^1}{\lambda_1\mu^1 + \lambda_2\mu^2} f\Big(x + \frac{v}{\mu^1}\Big) + \frac{\lambda_2\mu^2}{\lambda_1\mu^1 + \lambda_2\mu^2} f\Big(x + \frac{v}{\mu^2}\Big), \end{aligned}$$

thus,

$$\begin{aligned} &(\lambda_1\mu^1 + \lambda_2\mu^2)\Big(f\Big(x + \frac{v}{\lambda_1\mu^1 + \lambda_2\mu^2}\Big) - f(x)\Big) \\ \le_D\ & \lambda_1\mu^1\Big(f\Big(x + \frac{v}{\mu^1}\Big) - f(x)\Big) + \lambda_2\mu^2\Big(f\Big(x + \frac{v}{\mu^2}\Big) - f(x)\Big). \end{aligned}$$

Since D is a closed cone, σ_v is a D-convex function.

$$\begin{aligned}\sigma_v^*(\bar\varepsilon) &= \sup\{\mu\bar\varepsilon - \sigma_v(\mu)|\mu\in\Re\} = \sup_{\mu>0}\{\mu\bar\varepsilon-\sigma_v(\mu)\}\\ &= \sup_{\mu>0}\{\mu(\bar\varepsilon - f(x+v/\mu)+f(x))\}\\ &= -\inf_{\mu>0}\{\mu(f(x+v/\mu)-f(x)-\bar\varepsilon)\} = -f'_{-\bar\varepsilon}(x;v).\end{aligned}$$

Obviously $1\in \mathrm{int}(\mathrm{dom}\sigma_v)$. By (2.4), one has $\sigma_v(1)=\sigma_v^{**}(1)$. So

$$f(x+v)-f(x)=\sup\{\bar\varepsilon-(-f'_{-\bar\varepsilon}(x;v)|\bar\varepsilon\in\Re^p\}.$$

Let $\varepsilon=-\bar\varepsilon$, $f(x+v)-f(x)=\sup\{f'_\varepsilon(x;v)-\varepsilon|\varepsilon\in\Re^p\}$. It follows from Theorem 2.1 that

$$f(x+v)-f(x)=\sup_{\varepsilon\in D}\{f'_\varepsilon(x;v)-\varepsilon\}.$$

§3. Optimality Condition for Unconstrained Cone-D.C. Programming

Definition 3.1. *$f:\Re^n\to\Re^p$ is called a D-d.c. function (difference of two D-convex functions), if there exist D-convex functions $g,h:\Re^n\to\Re^p$ such that*

$$f(x)=g(x)-h(x),\quad \forall x\in\Re^n.$$

D-d.c. function is a large class of the vector functions. For example, D-convex function and D-concave function are D-d.c. functions; and $f=(f_1,\cdots,f_p)^T:\Re^n\to\Re^p$ is $\Re^p_+$-d.c. function, where f_i is an indefinite quadratic function. In this section, we discuss the necessary and sufficient conditions for local and global optimal points of unconstrained D-d.c. programming.

As mentioned above, the D-convex function is a locally D-Lipschitz function. Then it is easy to verify that D-d.c. function is also a D-Lipschitz function.

Before we go further, some definitions are presented. First, from (2.1) it can be derived that the directional derivate of D-d.c. function is well-defined, and

$$f'(x;v)=g'(x;v)-h'(x;v),\quad \forall(x;v)\in\Re^n\times\Re^n.$$

Next, we give another kind of the concepts of subgradient, ε-subgradient and conjugate mapping of D-convex function.

Definition 3.2. *For a D-convex function $f:\Re^n\to\Re^p$, $T\in\Re^{n\times p}$ is called the subgradient of f at x^0, if $T^T(x-x^0)\not\geq_D f(x)-f(x^0)$ for any $x\in\Re^n$. The set of all subgradients is denoted as $\tilde\partial f(x^0)$.*

Of course, $\tilde\partial_s f(x^0)$, $\tilde\partial_w f(x^0)$ can be defined as well. But it is easy to verify that they are equal to $\partial_s f(x^0)$, $\partial_w f(x^0)$ respectively; and $\partial f(x^0)\subset\tilde\partial f(x^0)$.

Definition 3.3. *$T\in\Re^{n\times p}$ is called the ε-subgradient (ε-weak subgradient) of a D-convex function $f:\Re^n\to\Re^p$ at x^0, if*

$$T^T(x-x^0)\not\geq_D f(x)-f(x^0)+\varepsilon(T^T(x-x^0)\not>_D f(x)-f(x^0)+\varepsilon)$$

for any $x\in\Re^n$. The set of them is denoted by ε-$\tilde\partial f(x^0)$ (ε-$\tilde\partial_w f(x^0)$).

Immediately from Theorem 2.2, ε-$\partial f(x^0)\subset\varepsilon$-$\tilde\partial f(x^0)$ and ε-$\partial_w f(x^0)\subset\varepsilon$-$\tilde\partial_w f(x^0)$.

Definition 3.4. *The conjugate mapping $f^*:\Re^n\twoheadrightarrow\Re^p$ (weak conjugate mapping $f^*_w:\Re^n\twoheadrightarrow\Re^p$) of D-convex function f is a set-valued mapping defined by $f^*(T)=\mathrm{Max}\{T^Tx-f(x)|x\in\Re^n\}$ ($f^*_w(T)=\mathrm{WMax}\{T^Tx-f(x)|x\in\Re^n\}$).*

Apparently, $T \in \tilde{\partial} f(x^0)$ if and only if $T^T x^0 - f(x^0) \in f^*(T)$, and $T \in \partial_w f(x^0)$ if and only if $T^T x^0 - f(x^0) \in f_w^*(T)$.

Definition 3.5. *Let A, B be two subsets in $\Re^{n\times p}$, their star-difference $A \underline{*} B$ is defined by $A \underline{*} B = \{T \in \Re^{n\times p} | T + B \subset A\}$.*

Now we consider the optimization of the following unconstrained D-d.c. programming:

$$\text{(P):} \quad \begin{array}{l} D\text{-min } f(x) - g(x) \\ \text{s.t.} \quad x \in \Re^n; \end{array}$$

where $f, g\colon \Re^n \to \Re^p$ are D-convex functions.

Theorem 3.1 (Necessary Condition of Local Efficient Solution). *If x^0 is a local efficient solution of* (P)*, then $0 \in \tilde{\partial} f(x^0) \underline{*} \partial_s g(x^0)$.*

Proof. If x^0 is a local efficient solution of (P), then there is a neighbourhood $B(x^0, \varepsilon)$ such that

$$f(x) - f(x^0) \not\leq_D g(x) - g(x^0), \quad \forall x \in B(x^0, \varepsilon).$$

For any $T \in \partial_s g(x^0)$, i.e., $T^T(x - x^0) \leq_D g(x) - g(x^0)$, it is immediate that

$$f(x) - f(x^0) \not\leq_D T^T(x - x^0), \quad \forall x \in B(x^0, \varepsilon).$$

Notice that, for every $x \in \Re^n$, there is $\delta > 0$ sufficiently small such that $\delta x + (1 - \delta)x^0 \in B(x^0, \varepsilon)$. It is easy to verify that $f(x) - f(x^0) \not\leq_D T^T(x - x^0)$ by the D-convexity of f. Thus $T \in \tilde{\partial} f(x^0)$.

It means that $0 \in \tilde{\partial} f(x^0) \underline{*} \partial_s g(x^0)$.

Obviously, for the local weak efficient solution of x^0, the necessary condition is: $0 \in \partial_w f(x^0) \underline{*} \partial_s g(x^0)$. Then by Corollary 2.1, $f'(x^0; v) - g'(x^0; v) \not<_D 0$ for any $v \in \Re^n \backslash \{0\}$, which has been got in [9].

Theorem 3.2 (Sufficient Condition of Local Efficient Solution). *If $0 \in \operatorname{int}[\partial f(x^0) \underline{*} \partial_s g(x^0)]$, then x^0 is a local strict efficient solution of* (P)*, that is, $f(x^0) - g(x^0) \ngeqq_D f(x) - g(x)$, $\forall x \in B(x^0, \varepsilon) \backslash \{x^0\}$ for some $\varepsilon > 0$.*

Proof. From the assumption, there exists $\delta > 0$ such that $B(0, \delta) + \partial_s g(x^0) \subset \partial f(x^0)$.

We claim that $g'(x^0; v) \ngeqq_D f'(x^0; v)$ for every $v \in \Re^n \backslash \{0\}$.

In fact, if there is $v \in \Re^n \backslash \{0\}$ such that $g'(x^0; v) \geqq_D f'(x^0; v)$, by Corollary 2.1 there is $T \in \partial_s g(x^0)$ with $T^T v = g'(x^0; v)$. Take $d \in D \backslash \{0\}$. Let $\tilde{T} = v d^T$. Then $\widetilde{T}^T v \in D \backslash \{0\}$. When $\lambda > 0$ is sufficiently small, $T' = \lambda \widetilde{T} \in B(0, \delta)$ and $(T')^T v \geq_D 0$. It follows that

$$T + T' \in \partial f(x^0) \text{ and } (T + T')^T v \geq_D f'(x^0; v).$$

This contradicts $f'(x^0; v) \in \operatorname{Max}\{T^T v | T \in \partial f(x^0)\}$.

Therefore $(f - g)'(x^0; v) \not\leq_D 0$, for any $v \in \Re^n \backslash \{0\}$.

If the conclusion fails, then there is a sequence $\{x^i | i = 1, 2, \cdots\}$ with $x^i \neq x^0$ such that $f(x^i) - g(x^i) \leq_D f(x^0) - g(x^0)$ and $x^i \to x^0$.

When i is sufficiently large, $\dfrac{(f - g)(x^i) - (f - g)(x^0)}{\|x^i - x^0\|} \in (d - D) \cap (-d + D)$ where d is the Lipschitz constant of $(f - g)$. Since $\left\| \dfrac{x^i - x^0}{\|x^i - x^0\|} \right\| = 1$ and $(d - D) \cap (-d + D)$ is a bounded set, assume without loss of generality that $\dfrac{x^i - x^0}{\|x^i - x^0\|} \to \bar{v} \neq 0$ and $\dfrac{(f - g)(x^i) - (f - g)(x^0)}{\|x^i - x^0\|}$

converges to a point in $-D$. Notice that

$$-\left\|\bar{v}-\frac{(x^i-x^0)}{\|x^i-x^0\|}\right\| d \leq_D \frac{(f-g)(x^0+\|x^i-x^0\|\bar{v})-(f-g)(x^i)}{\|x^i-x^0\|}$$

$$\leq_D \left\|\bar{v}-\frac{(x^i-x^0)}{\|x^i-x^0\|}\right\| d.$$

One has

$$(f-g)'(x^0;\bar{v}) = \lim_{i\to\infty} \frac{(f-g)(x^0+\|x^i-x^0\|\bar{v})-(f-g)(x^0)}{\|x^i-x^0\|}$$

$$= \lim_{i\to\infty} \frac{(f-g)(x^i)-(f-g)(x^0)}{\|x^i-x^0\|} \leq_D 0,$$

a contradiction. The proof is completed.

Theorems 3.1 and 3.2 are the extensions of the corresponding ones of the scalar d.c. function, they are the same as the optimality conditions in [2] when $p=1$. Next, we give the necessary and sufficient conditions for the global optimal point.

Theorem 3.3. (Necessary Condition of Global Efficient Solution). *If x^0 is an efficient solution of* (P), *then* $0 \in \varepsilon\text{-}\tilde{\partial} f(x^0) \underline{*} \varepsilon\text{-}\partial_s g(x^0)$ *for any* $\varepsilon \in D$.

Proof. From the assumption, $f(x)-f(x^0)+\varepsilon \not\leq_D g(x)-g(x^0)+\varepsilon$, $\forall x \in \Re^n$.

For any $T \in \varepsilon\text{-}\partial_s g(x^0)$, $T^T(x-x^0) \leq_D g(x)-g(x^0)+\varepsilon$. Thus,

$$T^T(x-x^0) \not\leq_D f(x)-f(x^0)+\varepsilon, \quad \forall x \in \Re^n.$$

It means $T \in \varepsilon\text{-}\tilde{\partial} f(x^0)$.

Then $0 \in \varepsilon\text{-}\tilde{\partial} f(x^0) \underline{*} \varepsilon\text{-}\partial_s g(x^0)$.

Theorem 3.4 (Sufficient Condition of Global Efficient Solution). *If for every $v \in \Re^n$ there is $z_v \not\geq_D 0$ such that $g'_\varepsilon(x^0;v)-f'_\varepsilon(x^0;v) \in z_v - D$ for any $\varepsilon \in D$, then x^0 is an efficient solution of* (P).

Proof. By the assumption, it is immediate that $\sup\limits_{\varepsilon\in D}(g'_\varepsilon(x^0;v)-f'_\varepsilon(x^0;v)) \leq_D z_v$. Since

$$\sup_{\varepsilon\in D}(g'_\varepsilon(x^0;v)-\varepsilon) - \sup_{\varepsilon\in D}(f'_\varepsilon(x^0;v)-\varepsilon) \leq_D \sup_{\varepsilon\in D}(g'_\varepsilon(x^0;v)-\varepsilon-f'_\varepsilon(x^0;v)+\varepsilon) \leq_D z_v$$

and $z_v \not\geq_D 0$,

$$\sup_{\varepsilon\in D}(g'_\varepsilon(x^0;v)-\varepsilon) \not\geq_D \sup_{\varepsilon\in D}(f'_\varepsilon(x^0;v)-\varepsilon).$$

By Theorem 2.4, it implies that

$$f(x^0)-g(x^0) \not\geq_D f(x^0+v)-g(x^0+v), \quad \forall v \in \Re^n.$$

Thus x^0 is the efficient solution.

Theorem 3.3 guarantees that $0 \in \varepsilon\text{-}\tilde{\partial} f(x^0) \underline{*} \varepsilon\text{-}\partial_s g(x^0)$ for the efficient solution. If stronger condition holds: $0 \in \varepsilon\text{-}\partial f(x^0) \underline{*} \varepsilon\text{-}\partial_s g(x^0)$, it can be concluded that

$$g'_\varepsilon(x^0;v) \not\geq_D f'_\varepsilon(x^0;v), \quad \forall \varepsilon \in D, v \in \Re^n, \tag{3.1}$$

by Theorem 2.3. On the other hand, the assumption of Theorem 3.4 implies that

$$\sup_{\varepsilon\in D}(g'_\varepsilon(x^0;v)-f'_\varepsilon(x^0;v)) \not\geq_D 0, \quad \forall v \in \Re^n,$$

which is stronger than (3.1). But in scalar case they are equivalent. So

$$\varepsilon\text{-}\partial g(x^0) \subset \varepsilon\text{-}\partial f(x^0), \quad \forall \varepsilon \geqq 0$$

is the necessary and sufficient condition of the optimal solution of the unconstrained d.c. programming. It is the main result of [3].

It seems that the assumption of Theorem 3.4 is too strong. But it does have unconstrained cone-d.c. programming satisfying the assumption. Here is an example.

Example. The cone-d.c. programming is

$$\Re^2_+\text{-}\min f(x)-g(x)=\begin{pmatrix}(x)^2+2x\\-2x\end{pmatrix}-\begin{pmatrix}|x|\\|x|\end{pmatrix}.$$

Consider $x^0=0$. It is clear that $g'_\varepsilon(0;v)=(|v|,|v|)^T$ and

$$f'_\varepsilon(0;v)=\begin{pmatrix}2x+2\sqrt{\varepsilon_1}|v|\\-2v\end{pmatrix}.$$

Consequently,

$$g'_\varepsilon(0;v)-f'_\varepsilon(0;v)=\begin{pmatrix}|v|-2v-2\sqrt{\varepsilon_1}|v|\\|v|+2v\end{pmatrix}\leq_D\begin{pmatrix}|v|-2v\\|v|+2v\end{pmatrix},$$

where $\begin{pmatrix}|v|-2v\\|v|+2v\end{pmatrix}\not\geq_D 0$.

It satisfies the assumption of Theorem 3.4 for $z_v=\begin{pmatrix}|v|-2v\\|v|+2v\end{pmatrix}$, therefore $x^0=0$ is an efficient solution.

§4. Duality of Unconstrained Cone-D.C. Programming

Toland has developed the Toland's duality for the unconstrained scalar d.c. programming in [4,5], etc. In this section, we extend it to cone-d.c. programming.

Theorem 4.1. *If x^0 is a local efficient solution of* (P)*, then*

$$f(x^0)-g(x^0)\in \mathrm{Min}\{g^*_s(T)-f^*(T)|T\in\partial_s g(x^0)\}.$$

Proof. i) For $T\in\partial_s g(x^0)$, $g^*_s(T)=T^Tx^0-g(x^0)$. By the assumption and Theorem 3.1, $T\in\tilde{\partial}f(x^0)$. Thus $-T^Tx^0+f(x^0)\in -f^*(T)$. It implies

$$f(x^0)-g(x^0)\in g^*_s(T)-f^*(T),\quad \forall T\in\partial_s g(x^0).$$

ii) Suppose that there exists $T\in\partial_s g(x^0)$ and $y=T^Tx-f(x)\in f^*(T)$ such that

$$f(x^0)-g(x^0)\geq_D g^*_s(T)-y.$$

Since $-g(x^0)=g^*_s(T)-T^Tx^0$, $T^Tx^0-f(x^0)\leq_D y$. This contradicts

$$T^Tx^0-f(x^0)\in f^*(T)=\mathrm{Max}\{T^Tx-f(x)|x\in\Re^n\}.$$

In sum, $f(x^0)-g(x^0)\in \mathrm{Min}\{g^*_s(T)-f^*(T)|T\in\partial_s g(x^0)\}$.

Theorem 4.2. *If x^0 is an efficient solution of* (P)*, then*

$$f(x^0)-g(x^0)\in \mathrm{Min}\{g^*_s(T)-f^*(T)|T\in\Re^{n\times p}\},$$

and $T\in\partial_s g(x^0)$ is the efficient solution of the dual programming.

Proof. i) By part i) in the proof of Theorem 4.1,

$$f(x^0)-g(x^0)\in \{g^*_s(T)-f^*(T)|T\in\Re^{n\times p}\}.$$

ii) For any $T\in\Re^{n\times p}$,

$$f(x^0)-g(x^0)\not\geq_D f(x)-g(x)=f(x)-T^Tx+T^Tx-g(x),\quad \forall x\in\Re^n.$$

Hence $g_s^*(T) - (f(x^0) - g(x^0)) \not\leq_D T^T x - f(x)$. Thus

$$g_s^*(T) - (f(x^0) - g(x^0)) \not\leq_D y, \quad \forall y \in f^*(T) = \mathrm{Max}\{T^T x - f(x) | x \in \Re^n\}.$$

i) and ii) imply that $f(x^0) - g(x^0) \in \mathrm{Min}\{g_s^*(T) - f^*(T) | T \in \Re^{n \times p}\}$.

Finally, from part i) in the proof of Theorem 4.1, we draw the conclusion that $T \in \partial_s g(x^0)$ is the efficient solution of the dual programming.

Obviously, for the local weak efficient solution x^0, we have

$$f(x^0) - g(x^0) \in \mathrm{WMin}\{g_s^*(T) - f_w^*(T) | T \in \partial_s g(x^0)\};$$

and for the global weak efficient solution x^0,

$$f(x^0) - g(x^0) \in \mathrm{WMin}\{g_s^*(T) - f_w^*(T) | T \in \Re^{n \times p}\}.$$

If $p = 1$, Theorem 4.2 is exactly the Toland's duality for the unconstrained scalar d.c. programming.

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