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VIRTUAL SYSTEM METHOD FOR SYSTEM TREE**

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Abstract

A machine, or other type of "system", can often be divided into several subsystems (components) and these subsystems again can be divided into sevreral subsystems (second generation). This process forms a system tree. To assess the reliability of the machine based on data from the trials of components of the machine, virtual system method is employed. It is proved in the paper that the lower confident limit of the reliability of the machine set by the virtual system method is level consistent and asymptotically optimal while the one set by Lindstrom-Maddens method is not.

Keywords Lower confidence limit, Reliability, Virtual system method, System tree.
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§1. Introduction and Main result

In practice, a system usually is divided into several subsystems and subsystems agian can be also divided into the second generation subsystems, $\cdots$. Finally, a system tree is formed. An example of system tree is explained in Fig. 1.

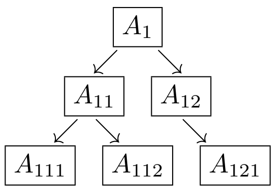

Fig.1 An Example of System Tree

In Fig.1, system A_1 is divided into subsystems A_{11} and A_{12} where subsystem A_{11} is again divided into A_{111} and A_{112} and A_{12} has only one system A_{121} as its subsystem. In this paper, we denote the system tree by $\{A_m : m \in M\}$ where M is a finite set of indices satisfying

(i) $m = (1) \in M$,

(ii) $m = (i_1, \cdots, i_l) \in M \Longrightarrow i_1 = 1$, where $i_1, \cdots, i_l$ are positivre integers,

(iii) $(i_1, \cdots, i_l) \in M \Longrightarrow (i_1, \cdots, i_{l-1}) \in M,\ (i_1, \cdots, i_{l-1}, i) \in M, \quad i = 1, \cdots, i_{l-1}$.

Difinition 1.1. *$A_{\widetilde{m}}$ is said to be the subsystem of A_m, if $m = (i_1, \cdots, i_k)$ and $\widetilde{m} = (i_1, \cdots, i_k, i_{k+1}, \cdots i_l)$. $A_{\widetilde{m}}$ is said to be the first generation subsystem of A_m, if $m =$*

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$(i_1, \cdots, i_k)$, $\widetilde{m} = (i_1, \cdots, i_k, i_{k+1})$. *System* A_m *is said to be the last generation subsystem if no system in the tree* $\{A_m : m \in M\}$ *is a subsystem of* A_m. *Subsystem* A_m *is said to be the next to the last generation subsystem if all the subystems of* A_m *are the last generation subsystems.*

In Fig. 1, A_{111}, A_{112} and A_{121} are the last generation subsystems, A_{11} and A_{12} are next to last generation systems. Denote

$$M_0 = \{m \in M : A_m \text{ is the last generation subsystem}\},$$
$$M(m) = \{\widetilde{m} : A_{\widetilde{m}} \text{ is the first generation subsystem of } A_m\}. \tag{1.1}$$

Let R_m be the reliability of A_m, i.e., R_m is the probability that A_m will work perfectly. For the relation between R_m's, we have

Assumption 1.1. Suppose that $m \notin M_0$ and $M(m) = \{m_1, \cdots, m_l\}$. Then

$$R_m = R_m(R_{m_1}, \cdots, R_{m_l}\}, \tag{1.2}$$

where the function $R_m(R_{m_1}, \cdots, R_{m_l})$ is a known function with continuous partial drivatives.

If the system structure is the series structure, then $R_m = \prod_{i=1}^{l} R_{m_i}$. It is easy to know that among the parameters R_m, $m \in M$, the independent parameters are $\{R_m, m \in M_0\}$. Suppose that for every subsystem, A_m, we have n_m trials with S_m successes. The joint distribution of the experiment is

$$f(s, r) = \prod_{m \in M} \binom{n_m}{S_m} R_m^{S_m} (1 - R_m)^{n_m - S_m}, \tag{1.3}$$

where $s = (S_m, m \in M)$, $r = (R_m, m \in M_0)$. The informaton matrix of this model is

$$I(r) = \sum_{m \in M} \frac{n_m}{R_m(1 - R_m)} \frac{\partial R_m}{\partial r} \frac{\partial R_m}{\partial r^\tau}. \tag{1.4}$$

To estimate the reliability $R_{(1)}$ of the system $A_{(1)}$, it is well known from the estimation theorem that the efficient estimator $\widehat{R}_{(1)}$ should be asymptotically normal with asymptotic distribution

$$N\Big(R_{(1)}, \frac{\partial R_{(1)}}{\partial r^\tau} I(r)^{-1} \frac{\partial R_{(1)}}{\partial r}\Big). \tag{1.5}$$

In this paper we present an iterative method to calculate the asymptotic variance. Let $\{A_{m_1}, \cdots, A_{m_k}\} = \{A_m, m \in M_0\}$ and A_{m_0} be a subsystem next to the last generation subsystems with its offsprings $\{A_{m_{l+1}}, \cdots, A_{m_k}\} = \{A_m : m \in M(m_0)\}$. Denote

$$\left.\begin{aligned} &M^{(1)} = M \backslash \{m_{l+1}, \cdots, m_k\}, \\ &r^{(1)} = (R_{m_1}, \cdots, R_{m_l}, R_{m_0})^\tau = (R_m, m \in M_0^{(1)})^\tau, \\ &n_m^{(1)} = \begin{cases} n_m, & m \neq m_0, \\ n_{m_0} + \tilde{n}_{m_0}, & m = m_0, \end{cases} \end{aligned}\right\} \tag{1.6}$$

where

$$\tilde{n}_{m_0} = \frac{R_{m_0}(1 - R_{m_0})}{\sum\limits_{m \in M(m_0)} \frac{R_m(1-R_m)}{n_m} \Big(\frac{\partial R_{m_0}}{\partial R_m}\Big)^2}. \tag{1.7}$$

For the new system $\{A_m, m \in M^{(1)}\}$, which is called a virtual system tree, we define

$$I(r^{(1)}) = \sum_{m\in M^{(1)}} \frac{n_m^{(1)}}{R_m(1-R_m)} \frac{\partial R_m}{\partial r^{(1)}} \frac{\partial R_m}{\partial r^{(1)\tau}}. \tag{1.8}$$

Theorem 1.1. *For $\{A_m, m \in M\}$ and its virtual system $\{A_m, m \in M^{(1)}\}$, we have*

$$\frac{\partial R_{(1)}}{\partial r^{\tau}} I(r)^{-1} \frac{\partial R_{(1)}}{\partial r} = \frac{\partial R_{(1)}}{\partial r^{(1)\tau}} I(r^{(1)})^{-1} \frac{\partial R_{(1)}}{\partial r^{(1)}}. \tag{1.9}$$

The proof will be given in Section 2.

Remark. For the virtual system $\{A_m, m \in M^{(1)}\}$ we rewrite the vector $r^{(1)}$ into the form

$$r^{(1)} = \{R_{m_1^{(1)}}, \cdots, R_{m_k^{(1)}}\}, \tag{1.10}$$

where the number $k = l + 1$ in (1.6). Similarly, there exists a system $A_{m_0^{(1)}}$ such that $M^{(1)}(m_0^{(1)}) \subset M_0^{(1)} = \{m_1^{(1)}, \cdots, m_k^{(1)}\}$. Without loss of generality, we may let $M^{(1)}(m_0^{(1)}) = \{m_{l+1}^{(1)}, \cdots, m_k^{(1)}\}$.

Furthermore we define $M^{(2)}, r^{(2)}, n_m^{(2)}$ by

$$M^{(2)} = M^{(1)} \backslash M^{(1)}(m_0^{(1)}), \quad r^{(2)} = (R_{m_1^{(1)})}, \cdots, R_{m_l^{(1)}}, R_{m_0^{(1)}})^{\tau},$$

$$n_m^{(2)} = \begin{cases} n_m^{(1)}, & m \neq m_0^{(1)}, \\ n_m^{(1)} + \tilde{n}_m^{(1)}, & m = m_0^{(1)}, \end{cases}$$

where

$$\tilde{n}_{m_0^{(1)}}^{(1)} = \frac{R_{m_0^{(1)}}(1-R_{m_0^{(1)}})}{\sum\limits_{i=l+1}^{k} \frac{R_{m_i^{(1)}}(1-R_{m_i^{(1)}})}{n_{m_i^{(1)}}} \left(\frac{\partial R_{m_0^{(1)}}}{\partial R_{m_i^{(1)}}}\right)^2}.$$

By Theorem 1.1, we obtain

$$\frac{\partial R_{(1)}}{\partial r^{(1)\tau}} I(r^{(1)})^{-1} \frac{\partial R_{(1)}}{\partial r^{(1)}} = \frac{R_{(1)}}{\partial r^{(2)\tau}} I(r^{(2)})^{-1} \frac{\partial R_{(1)}}{\partial r^{(2)}}.$$

Using Theorem 1.1 repeatedly, we obtain

$$\frac{\partial R_{(1)}}{\partial r^{\tau}} I(r)^{-1} \frac{\partial R_{(1)}}{\partial r} = \cdots = \frac{R_{(1)}(1-R_{(1)})}{n_{(1)}^{(u)}}, \tag{1.11}$$

where u is an integer such that $M^{(u)} = \{(1)\}$.

From practical point of view, we need to set a lower confidence limit for $R_{(1)}$, the reliability of $A_{(1)}$. Here we present a virtual system method. Define N_m recursively. Let

$$N_m = \begin{cases} n_m, & m \in M_0, \\ n_m + \tilde{n}_m, & \text{otherwise}, \end{cases} \tag{1.12}$$

where

$$\tilde{n}_m = \frac{R_m(1-R_m)}{\sum\limits_{\widetilde{m}\in M(m)} \frac{R_{\widetilde{m}}(1-R_{\widetilde{m}})}{N_{\widetilde{m}}} \left(\frac{\partial R_m}{\partial R_{\widetilde{m}}}\right)^2}.$$

For the last generation subsystem $m \in M_0, \widehat{N}_m, \widehat{R}_m$, and $\widehat{S}_m$ are defined by

$$\widehat{N}_m = n_m, \quad \widehat{S}_m = S_m, \quad \widehat{R}_m = S_m/n_m. \tag{1.13}$$

Now suppose that m is not a last generation system and that, for all its sons $\widetilde{m} \in M(m)$, the corresponding $\widehat{N}_{\widetilde{m}}, \widehat{R}_{\widetilde{m}}$, and $\widehat{S}_{\widetilde{m}}$ have been defined. Without loss of generality, let $M(m) = \{\widetilde{m}_1, \cdots, \widetilde{m}_l\}$. $\widehat{N}_m, \widehat{R}_m$, and $\widehat{S}_m$ are defined by

$$\widehat{R}_m = R(\widehat{R}_{\widetilde{m}_1}, \cdots, \widehat{R}_{\widetilde{m}_l}), \tag{1.14}$$

$$\widehat{N}_m = n_m + \hat{\hat{n}}_m = n_m + \frac{\widehat{R}_m(1-\widehat{R}_m)}{\sum\limits_{\widetilde{m}\in M(m)} \frac{\widehat{R}_{\widetilde{m}}(1-\widehat{R}_{\widetilde{m}})}{\widehat{N}_{\widetilde{m}}} \Big(\widehat{\frac{\partial R_m}{\partial R_{\widetilde{m}}}}\Big)^2}, \tag{1.15}$$

$$\widehat{S}_m = s_m + \hat{\hat{n}}_m \widehat{R}_m, \tag{1.16}$$

where $R_M(\cdot)$ and $\widehat{\frac{\partial R_m}{\partial R_{\widetilde{m}}}}$ stand for the functions of the arguments $\widehat{R}_{\widetilde{m}_1}, \cdots, \widehat{R}_{\widetilde{m}_l}$. By induction, for all the subsystems $m \in M$, $\widehat{N}_m, \widehat{R}_m$, and $\widehat{S}_m$ are defined. Let $\underline{R}_{(1)}$ be the solution of the following equation

$$\sum_{i=\widehat{S}_{(1)}}^{\widehat{N}_{(1)}} \binom{\widehat{N}_{(1)}}{i} \underline{R}_{(1)}^i (1-\underline{R}_{(1)})^{\widehat{N}_{(1)}-i} = \alpha. \tag{1.17}$$

Actually, when $\widehat{N}_{(1)}$ ans $\widehat{S}_{(1)}$ are not integers, we solve the following

$$\frac{\int_0^{\underline{R}_{(1)}} t^{\widehat{S}_{(1)}-1}(1-t)^{\widehat{N}_{(1)}-\widehat{S}_{(1)}} dt}{\int_0^1 t^{\widehat{S}_{(1)}-1}(1-t)^{\widehat{N}_{(1)}-\widehat{S}_{(1)}} dt} = \alpha \tag{1.18}$$

instead of (1.17).

Theorem 1.2. *For the system tree $\{A_m, m \in M\}$, we have*

$$\lim_{\min\{n_m, m\in M\}\to\infty} \Pr\{R_{(1)} \geq \underline{R}_{(1)}\} = 1-\alpha. \tag{1.19}$$

Theorem 1.3. *Under the condition of Theorem* 1.2

$$\Big(\frac{\partial R_{(1)}}{\partial r^\tau} I(r)^{-1} \frac{\partial R_{(1)}}{\partial r}\Big)^{-\frac{1}{2}} (\underline{R}_{(1)} - R_{(1)}) \xrightarrow{d} N(-u_{1-\alpha}, 1), \tag{1.20}$$

where $u_{1-\alpha}$ is the $(1-\alpha)$ quantile of the standardized normal distribution function Φ.

According to the point estimation theory, when $\alpha = \frac{1}{2}$, $\underline{R}_{(1)}$ is the efficient estimator of $R_{(1)}$.

§2. Proofs

Proof of Theorem 1.1. Let

$$r = \binom{r_1}{r_2},$$

where

$$r_1 = \begin{pmatrix} R_{m_1} \\ \vdots \\ R_{m_l} \end{pmatrix}, \quad r_2 = \begin{pmatrix} R_{m_{l+1}} \\ \vdots \\ R_{m_k} \end{pmatrix},$$

and $\{m_{l+1}, \cdots, m_k\} = M(m_0)$. By using the notation of partitioned matrix, $I(r)$ can be rewritten into the form of

$$I(r) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{2.1}$$

where

$$D_{11}=\sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m}{R_m(1-R_m)}\frac{\partial R_m}{\partial r_1}\frac{\partial R_m}{\partial r_1^\tau},$$

$$D_{12}=\sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m}{R_m(1-R_m)}\frac{\partial R_m}{\partial r_1}\frac{\partial R_m}{\partial R_{m_0}}\frac{\partial R_{m_0}}{\partial r_2^\tau},$$

$$D_{21}=D_{12}^\tau,$$

$$D_{22}=\sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m}{R_m(1-R_m)}\Big(\frac{\partial R_m}{\partial R_{m_0}}\Big)^2\frac{\partial R_{m_0}}{\partial r_2}\frac{\partial R_{m_0}}{\partial r_2^\tau}+\frac{n_{m_0}}{R_{m_0}(1-R_{m_0})}\frac{\partial R_{m_0}}{\partial r_2}\frac{\partial R_{m_0}}{\partial r_2^\tau}+\operatorname{diag}\Big(\frac{n_{m_{l+1}}}{R_{m_{l+1}}(1-R_{m_{l+1}})},\cdots,\frac{n_{m_k}}{R_{m_k}(1-R_{m_k})}\Big).$$

Similarly, the information matrix $I(r^{(1)})$ is partitioned into

$$I(r^{(1)})=\begin{pmatrix}D_{11}^{(1)} & D_{12}^{(1)}\\ D_{21}^{(1)} & D_{22}^{(1)}\end{pmatrix},\tag{2.2}$$

where

$$D_{11}^{(1)}=D_{11},\qquad D_{12}^{(1)}=\sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m^{(1)}}{R_m(1-R_m)}\frac{\partial R_m}{\partial r_1}\frac{\partial R_m}{\partial R_{m_0}},$$

$$D_{21}^{(1)}=D_{21}^{(1)^\tau},\quad D_{22}^{(1)}=\sum_{m\in M^{(1)}}\frac{n_m^{(1)}}{R_m(1-R_m)}\Big(\frac{\partial R_m}{\partial R_{m_0}}\Big)^2.$$

By the inverse formula of partitioned matrix

$$I^{-1}=\begin{pmatrix}D_{11}^{-1}+D_{11}^{-1}D_{12}\Delta^{-1}D_{21}D_{11}^{-1} & -D_{11}^{-1}D_{12}\Delta^{-1}\\ -\Delta^{-1}D_{12}D_{11}^{-1} & \Delta^{-1}\end{pmatrix},$$

where $\Delta=(D_{22}-D_{21}D_{11}^{-1}D_{12})$, we obtain

$$\begin{aligned}\frac{\partial R_{(1)}}{\partial r^\tau}(I^{-1}(r))\frac{\partial R_{(1)}}{\partial r}&=\Big(\frac{\partial R_{(1)}}{\partial r_1^\tau},\frac{\partial R_{(1)}}{\partial R_{m_0}}\frac{\partial R_{m_0}}{\partial r_2^\tau}\Big)\cdot\\ &\cdot\begin{pmatrix}D_{11}^{-1}+D_{11}^{-1}D_{12}\Delta^{-1}D_{21}D_{11}^{-1} & -D_{11}^{-1}D_{12}\Delta^{-1}\\ -\Delta^{-1}D_{12}D_{11}^{-1} & \Delta^{-1}\end{pmatrix}\begin{pmatrix}\frac{\partial R_{(1)}}{\partial r_1}\\ \frac{\partial R_{m_0}}{\partial r_2}\frac{\partial R_m}{\partial R_{m_0}}\end{pmatrix}\\ &=\Big(\frac{\partial R_{(1)}}{\partial r_1^\tau},\frac{\partial R_{(1)}}{\partial R_{m_0}}\Big)\cdot\\ &\cdot\begin{pmatrix}D_{11}^{-1}+D_{11}^{-1}D_{12}\Delta^{-1}D_{21}D_{11}^{-1} & -D_{11}^{-1}D_{12}\Delta^{-1}\frac{\partial R_{m_0}}{\partial r_2}\\ -\frac{\partial R_{m_0}}{\partial r_2^\tau}\Delta^{-1}D_{12}D_{11}^{-1} & \frac{\partial R_{m_0}}{\partial r_2^\tau}\Delta^{-1}\frac{\partial R_{m_0}}{\partial r_2}\end{pmatrix}\begin{pmatrix}\frac{\partial R_{(1)}}{\partial r_1}\\ \frac{\partial R_{(1)}}{\partial R_{m_0}}\end{pmatrix}.\end{aligned}\tag{2.3}$$

From the definition of D_{ij}, we know that

$$D_{11}^{(1)}=D_{11},\quad D_{12}=D_{12}^{(1)}\frac{\partial R_{m_0}}{\partial r_2^\tau}.$$

Subtituting these formulas into (2.3) we obtain

$$\frac{\partial R_{(1)}}{\partial r^\tau}(I^{-1}(r))\frac{\partial R_{(1)}}{\partial r} = \frac{\partial R_{(1)}}{\partial r^{(1)^\tau}}\cdot$$
$$\cdot\begin{pmatrix} D_{11}^{(1)^{-1}} + D_{11}^{(1)^{-1}} D_{12}^{(1)} \frac{\partial R_{m_0}}{\partial r_2^\tau}\Delta^{-1}\frac{\partial R_{m_0}}{\partial r_2} D_{21}^{(1)} D_{11}^{(1)^{-1}} & -D_{11}^{(1)^{-1}} D_{12}^{(1)} \frac{\partial R_{m_0}}{\partial r_2^\tau}\Delta^{-1}\frac{\partial R_{m_0}}{\partial r_2} \\ -\frac{\partial R_{m_0}}{\partial r_2^\tau}\Delta^{-1}\frac{\partial R_{m_0}}{\partial r_2} D_{12}^{(1)} D_{11}^{(1)^{-1}} & \frac{\partial R_{m_0}}{\partial r_2^\tau}\Delta^{-1}\frac{\partial R_{m_0}}{\partial r_2} \end{pmatrix}\cdot$$
$$\cdot\frac{\partial R_{(1)}}{\partial r^{(1}},$$

from which we know that to prove (1.9) it sufficies to prove the following equation

$$\frac{\partial R_{m_0}}{\partial r_2^\tau}\Delta^{-1}\frac{\partial R_{m_0}}{\partial r_2} = (D_{22}^{(1)} - D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)})^{-1} \widehat{=} \Delta^{(1)^{-1}}. \tag{2.4}$$

By using the formula for matrices

$$(A+UV^\tau)^{-1} = A^{-1} - \frac{(A^{-1}U)V^\tau A^{-1}}{1+V^\tau A^{-1}U},$$

where V and U are vectors, we have

$$\frac{\partial R_{m_0}}{\partial r_2^\tau}\Delta^{-1}\frac{\partial R_{m_0}}{\partial r_2} = \frac{\partial R_{m_0}}{\partial r_2^\tau}\Big(\mathrm{diag}\Big(\frac{n_{m_{l+1}}}{R_{m_{l+1}}(1-R_{m_{l+1}})},\cdots,\frac{n_{m_k}}{R_{m_k}(1-R_{m_k})}\Big)$$
$$+\sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m}{R_m(1-R_m)}\Big(\frac{\partial R_m}{\partial R_{m_0}}\Big)^2\frac{\partial R_{m_0}}{\partial r_2}\frac{\partial R_{m_0}}{\partial r_2^\tau} + \frac{n_{m_0}}{R_{m_0}(1-R_{m_0})}\frac{\partial R_{m_0}}{\partial r_2}\frac{\partial R_{m_0}}{\partial r_2^\tau}$$
$$-D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)}\frac{\partial R_{m_0}}{\partial r_2}\frac{\partial R_{m_0}}{\partial r_2^\tau}\Big)^{-1}\frac{\partial R_{m_0}}{\partial r_2}$$
$$= \frac{\partial R_{m_0}}{\partial r_2^\tau}\Big(\mathrm{diag}\Big(\frac{R_{m_{l+1}}(1-R_{m_{l+1}})}{n_{m_{l+1}}},\cdots,\frac{R_{m_k}(1-R_{m_k})}{n_{m_k}}\Big)$$
$$-\mathrm{diag}\Big(\frac{R_{m_{l+1}}(1-R_{m_{l+1}})}{n_{m_{l+1}}},\cdots,\frac{R_{m_k}(1-R_{m_k})}{n_{m_k}}\Big)\frac{\partial R_{m_0}}{\partial r_2}\cdot$$
$$\cdot\frac{\partial R_{m_0}}{\partial r_2^\tau}\mathrm{diag}\Big(\frac{R_{m_{l+1}}(1-R_{m_{l+1}})}{n_{m_{l+1}}},\cdots,\frac{R_{m_k}(1-R_{m_k})}{n_{m_k}}\Big)\cdot$$
$$\cdot\Big(\frac{n_{m_0}}{R_{m_0}(1-R_{m_0})} + \sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m}{R_m(1-R_m)}\Big(\frac{\partial R_m}{\partial R_{m_0}}\Big)^2 - D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)}\Big)$$
$$\Big(1+\Big(\frac{n_{m_0}}{R_{m_0}(1-R_{m_0})} + \sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m}{R_m(1-R_m)}\Big(\frac{\partial R_m}{\partial R_{m_0}}\Big)^2$$
$$-D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)}\Big)\frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}}\Big)^{-1}\Big)\frac{\partial R_{m_0}}{\partial r_2}$$
$$= \frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}} - \Big(\frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}}\Big)^2\Big(\frac{n_{m_0}}{R_{m_0}(1-R_{m_0})}$$
$$+\sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m}{R_m(1-R_m)}\Big(\frac{\partial R_m}{\partial R_{m_0}}\Big)^2 - D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)}\Big)\cdot$$
$$\cdot\Big[1+\Big(\frac{n_{m_0}}{R_{m_0}(1-R_{m_0})} + \sum_{m\in M^{(1)}\setminus\{m_0\}}\frac{n_m}{R_m(1-R_m)}\Big(\frac{\partial R_m}{\partial R_{m_0}}\Big)^2$$
$$-D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)}\Big)\frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}}\Big]^{-1}$$

$$= \frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}} \cdot$$
$$\cdot \left(1 - \frac{\frac{n_{m_0}}{\widetilde{n}_{m_0}} + \frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}} \Big(\sum\limits_{m \in M^{(1)} \setminus \{m_0\}} \frac{n_m}{R_m(1-R_m)} \Big(\frac{\partial R_m}{\partial R_{m_0}} \Big)^2 - D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)} \Big)}{1 + \frac{n_{m_0}}{\widetilde{n}_{m_0}} + \frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}} \Big(\sum\limits_{m \in M^{(1)} \setminus \{m_0\}} \frac{n_m}{R_m(1-R_m)} \Big(\frac{\partial R_m}{\partial R_{m_0}} \Big)^2 - D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)} \Big)} \right)$$
$$= \frac{\frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}}}{1 + \frac{n_{m_0}}{\widetilde{n}_{m_0}} + \frac{R_{m_0}(1-R_{m_0})}{\widetilde{n}_{m_0}} \Big(\sum\limits_{m \in M^{(1)} \setminus \{m_0\}} \frac{n_m}{R_m(1-R_m)} \Big(\frac{\partial R_m}{\partial R_{m_0}} \Big)^2 - D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)} \Big)}$$
$$= \Big(\frac{n_{m_0} + \widetilde{n}_{m_0}}{R_{m_0}(1-R_{m_0})} + \sum_{m \in M^{(1)} \setminus \{m_0\}} \frac{n_m}{R_m(1-R_m)} \Big(\frac{\partial R_m}{\partial R_{m_0}} \Big)^2 - D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)} \Big)^{-1}$$
$$= (D_{22}^{(1)} - D_{21}^{(1)} D_{11}^{(1)^{-1}} D_{12}^{(1)})^{-1},$$

which shows that (2.4) holds.

For the proof of Theorem 1.2 and Theorem 1.3, we need some preparations. Suppose that there exists a series of iid trials for every $m \in M$ such that, for every n_m, S_m is the number of successes of the first n_m trials in this series.

Lemma 2.1. *When* $\min\{n_m, m \in M\} \to \infty$,

$$\frac{\widehat{R}_m}{R_m} \to 1, \quad wp1, \tag{2.5}$$
$$\frac{\widehat{S}_m}{\widehat{N}_m} \to R_m, \quad wp1, \tag{2.6}$$
$$\frac{\widehat{N}_m}{N_m} \to 1, \quad wp1, \tag{2.7}$$
$$\frac{\widehat{S}_m - \widehat{N}_n R_m}{\sqrt{\widehat{N}_m R_m (1-R_m)}} \xrightarrow{d} N(0,1), \tag{2.8}$$
$$\frac{\widehat{R}_m - R_m}{\sqrt{\frac{R_m(1-R_m)}{\widetilde{n}_m}}} \xrightarrow{d} N(0,1), \quad m \notin M_0. \tag{2.9}$$

Proof. We prove this lemma by induction. When $m \in M_0$, according to the definition of $\widehat{R}_m, \widehat{S}_m$,

$$\widehat{R}_m = \frac{S_m}{n_m}, \quad \widehat{S}_m = S_m, \quad \widehat{N}_m = N_m = n_m.$$

So (2.5)-(2.8) hold for $m \in M_0$. Suppose that (2.5)-(2.8) hold for all $\widetilde{m} \in M(m)$. Let $M(m) = \{\widetilde{m}_1, \cdots, \widetilde{m}_l\}$. By induction we know that

$$\widehat{R}_m = R_m \Big(\frac{\widehat{S}_{\widetilde{m}_1}}{\widehat{N}_{\widetilde{m}_1}}, \cdots, \frac{\widehat{S}_{\widetilde{m}_l}}{\widehat{N}_{\widetilde{m}_l}} \Big) \to R_m(R_{\widetilde{m}_1}, \cdots, R_{\widetilde{m}_l}) = R_m, \ wp1,$$

i.e., (2.5) holds for m. From the following expressions

$$\widehat{N}_m = n_m + \frac{\widehat{R}_m(1-\widehat{R}_m)}{\sum\limits_{\widetilde{m} \in M(m)} \frac{\widehat{R}_{\widetilde{m}}(1-\widehat{R}_{\widetilde{m}})}{\widehat{N}_{\widetilde{m}}} \Big(\frac{\partial \widehat{R}_m}{\partial R_{\widetilde{m}}} \Big)^2},$$

$$N_m = n_m + \frac{R_m(1-R_m)}{\sum\limits_{\widetilde{m}\in M(m)} \frac{R_{\widetilde{m}}(1-R_{\widetilde{m}})}{N_{\widetilde{m}}}\Big(\frac{\partial R_m}{\partial R_{\widetilde{m}}}\Big)^2},$$

where

$$\frac{\partial \widehat{R}_m}{\partial R_{\widetilde{m}}} \triangleq \frac{\partial R_m}{\partial R_{\widetilde{m}}}\Big(\frac{\widehat{S}_{\widetilde{m}_1}}{\widehat{N}_{\widetilde{m}_1}},\cdots,\frac{\widehat{S}_{\widetilde{m}_l}}{\widehat{N}_{\widetilde{m}_l}}\Big) \to \frac{\partial R_m}{\partial R_{\widetilde{m}}}(R_{\widetilde{m}_1},\cdots,R_{\widetilde{m}_l}),$$

we obtain (2.7). According to the definition of $\widehat{S}_m$, $\widehat{S}_m = S_m + \widehat{\widehat{n}}_m \widehat{R}_m$, we have

$$\frac{\widehat{S}_m}{\widehat{N}_m} = \frac{S_m + \widehat{\widehat{n}}_m \widehat{R}_m}{n_m + \widehat{\widehat{n}}_m} \to R_m,$$

i.e., (2.6) holds for m. By Taylor expansion

$$\begin{aligned}\widehat{R}_m - R_m &= R_m\Big(\frac{\widehat{S}_{\widetilde{m}_1}}{\widehat{N}_{\widetilde{m}_1}},\cdots,\frac{\widehat{S}_{\widetilde{m}_l}}{\widehat{N}_{\widetilde{m}_l}}\Big) - R_m(R_{\widetilde{m}_1},\cdots,R_{\widetilde{m}_l})\\ &= \sum_{i=1}^{l} \frac{\partial R_m}{\partial R_{\widetilde{m}_i}}\Big(\frac{\widehat{S}_{\widetilde{m}_i}}{\widehat{N}_{\widetilde{m}_i}} - R_{\widetilde{m}_i}\Big)(1+o_p(1))\\ &= \sum_{i=1}^{l} \frac{\partial R_m}{\partial R_{\widetilde{m}_i}}\sqrt{\frac{R_{\widetilde{m}_i}(1-R_{\widetilde{m}_i})}{\widehat{N}_{\widetilde{m}_i}}}\,\frac{\Big(\frac{\widehat{S}_{\widetilde{m}_i}}{\widehat{N}_{\widetilde{m}_i}} - R_{\widetilde{m}_i}\Big)}{\sqrt{\frac{R_{\widetilde{m}_i}(1-R_{\widetilde{m}_i})}{\widehat{N}_{\widetilde{m}_i}}}}(1+o_p(1)).\end{aligned}$$

Since $\frac{\widehat{S}_{\widetilde{m}_i}}{\widehat{N}_{\widetilde{m}_i}}$, $i=1,\cdots,l$, are independently distributed and

$$\frac{\Big(\frac{\widehat{S}_{\widetilde{m}_i}}{\widehat{N}_{\widetilde{m}_i}} - R_{\widetilde{m}_i}\Big)}{\sqrt{\frac{R_{\widetilde{m}_i}(1-R_{\widetilde{m}_i})}{\widehat{N}_{\widetilde{m}_i}}}} \xrightarrow{d} N(0,1),$$

we obtain

$$\frac{\widehat{R}_m - R_m}{\sqrt{\frac{R_m(1-R_m)}{\widehat{n}_{\widetilde{m}}}}} \xrightarrow{d} N(0,1).$$

By the same reason,

$$\begin{aligned}&\frac{\widehat{S}_m - \widehat{N}_m R_m}{\sqrt{\widehat{N}_m R_m(1-R_m)}}\\ &= \frac{S_m - n_m R_m}{\sqrt{(n_m+\widehat{\widehat{n}}_m)R_m(1-R_m)}} + \frac{\widehat{\widehat{n}}(\widehat{R}_m - R_m)}{\sqrt{(n_m+\widehat{\widehat{n}}_m)R_m(1-R_m)}} \xrightarrow{d} N(0,1).\end{aligned}$$

Proof of Teorem 1.2. Without loss of generality, we assume that $\widehat{N}_m$ and $\widehat{S}_m$ are integers so that $\underline{R}_{(1)}$ is the solution of (1.17). Let

$$D = \sup\Big\{s : \sum_{i=s}^{\widehat{N}_{(1)}} \binom{\widehat{N}_{(1)}}{i} R_{(1)}^i (1-R_{(1)})^{\widehat{N}_{(1)}-i} \ge \alpha\Big\}.$$

It is easy to see that $P\{R_{(1)} \ge \underline{R}_{(1)}\} = P\{\widehat{S}_{(1)} \le D\}$. By Central Limit Theorem, we know that

$$D = \widehat{N}_{(1)}R_{(1)} + u_{1-\alpha+o(1)}\sqrt{\widehat{N}_{(1)}R_{(1)}(1-R_{(1)})},$$

where u_α is the α quantile of the standardized normal distribution and $o(1) \to 0$ as $\widehat{N}_{(1)} \to \infty$. Hence by Lemma 2.1.

$$P\{R_{(1)} \geq \underline{R}_{(1)}\} = P\left\{\frac{\widehat{S}_{(1)} - \widehat{N}_{(1)}R_{(1)}}{\sqrt{\widehat{N}_{(1)}R_{(1)}(1-R_{(1)})}} \leq u_{1-\alpha+o(1)}\right\} \to 1-\alpha.$$

Proof of Theorem 1.3. Let $Z_i,\ i = 1, \cdots, \widehat{N}_{(1)}$, be a sequence of iid random variables with

$$P\{Z_i = 1\} = \underline{R}_{(1)}, \quad P\{Z_i = 0\} = 1 - \underline{R}_{(1)}.$$

From (1.17) we know that

$$P\left\{\frac{\sum_{i=1}^{\widehat{N}_{(1)}} Z_i - \widehat{N}_{(1)}\underline{R}_{(1)}}{\sqrt{\widehat{N}_{(1)}\underline{R}_{(1)}(1-\underline{R}_{(1)})}} \geq \frac{\widehat{S}_{(1)} - \widehat{N}_{(1)}\underline{R}_{(1)}}{\sqrt{\widehat{N}_{(1)}\underline{R}_{(1)}(1-\underline{R}_{(1)})}}\right\} = \alpha, \tag{2.10}$$

where $P\{A\}$ is the conditional probability of A for given $\widehat{S}_{(1)}, \widehat{N}_{(1)}$. From (2.6) and (1.17) we know that $\underline{R}_{(1)} \to R_{(1)}, wp1$. By Berry Essen Theorem, it is easy to show that for almost all sequence of trials, the sequence of conditional distributions of

$$\frac{\sum_{i=1}^{\widehat{N}_{(1)}} Z_i - \widehat{N}_{(1)}\underline{R}_{(1)}}{\sqrt{\widehat{N}_{(1)}\underline{R}_{(1)}(1-\underline{R}_{(1)})}}$$

tends to standardized normal distribution function $\Phi(x)$. Therefore by (2.10)

$$\frac{\widehat{S}_{(1)} - \widehat{N}_{(1)}\underline{R}_{(1)}}{\sqrt{\widehat{N}_{(1)}\underline{R}_{(1)}(1-\underline{R}_{(1)})}} \to u_{1-\alpha}, wp1$$

or

$$u_{1-\alpha} + \frac{\sqrt{\widehat{N}_{(1)}}(\underline{R}_{(1)} - R_{(1)})}{\sqrt{\underline{R}_{(1)}(1-\underline{R}_{(1)})}} - \frac{\widehat{S}_{(1)} - \widehat{N}_{(1)}R_{(1)}}{\sqrt{\widehat{N}_{(1)}\underline{R}_{(1)}(1-\underline{R}_{(1)})}} \to 0, wp1,$$

by which, combined with (2.5), (2.8), the following holds

$$\frac{\underline{R}_{(1)} - R_{(1)}}{\sqrt{\frac{R_{(1)}(1-R_{(1)})}{\widehat{N}_{(1)}}}} \to N(-u_{1-\alpha}, 1),$$

or (1.20) holds.

Remark. In practice, the commonly used method is Lindstrom and Maddens method[1] for series structure. In [2],, we discussed the asymptotic behaviour of Lindstrom and Maddens method. Let $\{A_{(1)}, A_{(11)}, A_{(12)}\}$ be a system tree. It is proved in [2] that $\underline{R}_{LM}$, the lower confidence limit of $R_{(1)}$ set by Lindstrom and Maddens method is not level consistence, i.e.,

$$\lim_{\substack{n_{(11)}\to\infty \\ n_{(12)}\to\infty}} P\{R_{(1)} \geq \underline{R}_{LM}\} \begin{cases} > 1-\alpha, & \alpha < \dfrac{1}{2}, \\ < 1-\alpha, & \alpha > \dfrac{1}{2}. \end{cases}$$

The method offered in this paper is an improvement of Lindstrom and Maddens method.

Example. Let a system tree be expressed in Fig.2,

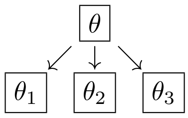

Fig.2 An Example of System Tree

where $\theta, \theta_1, \theta_2, \theta_3$ are the reliabilities of $m_{(1)}, m_{(1,1)}, m_{(1,2)}$ and $m_{(1,3)}$. The structure of the system is connected in series, i.e., $\theta = \theta_1 \cdot \theta_2 \cdot \theta_3$. In the system $n_{(1)} = 0,\ n_{(1,1)} = 2,\ n_{(1,2)} = 3,\ n_{(1,3)} = 4$. Let $\underline{\theta}_{VS}$ be the lower confidence limit of θ by virtual system method. Let $\alpha = 0.3$. The actual level of $\underline{\theta}_{VS}$ is given by

$$P\{\theta \geq \underline{\theta}_{VS}\}, \tag{2.11}$$

which is a function of parameters $\theta_1, \theta_2, \theta_3$. As a comparison, let $\underline{\theta}_{LM}$ be the lower confidence limit of θ by the traditional Lindstrom and Maddens method. The actual level of $\underline{\theta}_{LM}$ is $P\{\theta \geq \underline{\theta}_{LM}\}$. For simplicity, let $\theta_1 = 0.9, \theta_2 = 0.9$. In Fig.3, it is shown that $\underline{\theta}_{VS}$ is better than $\underline{\theta}_{LM}$, since the actual level of $\underline{\theta}_{VS}$ is smaller than that of $\underline{\theta}_{LM}$.

Fig.3 Lower Bound of the Reliability of Binary System Connected in Series $\theta_1 = (0.9),\ \theta_2 = (0.9),\ n = (2, 3, 4)$

References

[1] O'Neill, T. S., System reliability assessment from its components, *Appl. Stat. JRSS, Ser. C,* (1978).

[2] Zheng Zhongguo & Jiang Jiming, Asymptotic property of Lindstrom-Maddens' confidence lower limit, *Tech. Report Beijing Univ.,* (1991).

be

$$\begin{pmatrix} \frac{\partial R_{(1)}}{\partial r_1} \\ \frac{\partial R_{(1)}}{\partial R_{m_0}} \end{pmatrix}. \quad n_{(1)}^{(u)} \quad u \quad M^{(u)} \quad N_m = \quad -u_{1-\alpha}, \quad u_{1-\alpha}$$

$$\frac{\partial R_{(1)}}{\partial R_{m_0}} \quad D_{11}^{(1)^{-1}} \quad + \quad (D_{22}^{(1)} \quad P$$