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ALMOST PERIODIC SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONAUTONOMOUS OPERATOR**

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Abstract

This paper presents some existence theorems on almost periodic solutions for almost periodic neutral functional differential equation and its perturbed systems by means of Liapunov functional, and extends the corresponding results in [2,4,5,10,11].

Keywords Almost periodic solution, Neutral functional differential equation, Nonautonomous operator.
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The existence on almost periodic solutions has been widely investigated. There have been many theorems on the existence of almost periodic solutions to almost periodic functional differential equations by means of Liapunov functional (see [2, 4-6, 8, 10-12]). In [11], the author established some theorems on the existence of almost periodic solutions to NFDE, neutral functional differential equations, by Liapunov functional (where the operator is autonomous). Up to now, the author has not seen any existence theorems on almost periodic solutions to NFDE with nonautonomous operator. Using the inherence of stable operator and the same idea as [11], we can still establish some existence theorems on almost periodic solutions to NFDE with nonautonomous operator by means of Liapunov functional. Now, all the theorems are generalization of the theorems of [2, 4, 5, 10, 11].

Consider NFDE

$$\frac{d}{dt}D(t)x_t = f(t, x_t) \tag{1}$$

and its product systems

$$\frac{d}{dt}D(t)x_t = f(t, x_t), \qquad \frac{d}{dt}D(t)y_t = f(t, y_t). \tag{1*}$$

We also consider the perturbed systems

$$\frac{d}{dt}D(t)x_t = f(t, x_t) + h(t), \tag{2}$$

$$\frac{d}{dt}D(t)x_t = f(t, x_t) + \eta g(t, x_t) \tag{3}$$

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and its product systems

$$\frac{d}{dt}D(t)x_t = f(t,x_t) + h(t), \qquad \frac{d}{dt}D(t)y_t = f(t,y_t) + h(t), \tag{2*}$$

$$\frac{d}{dt}D(t)x_t = f(t,x_t) + \eta g(t,x_t), \qquad \frac{d}{dt}D(t)y_t = f(t,y_t) + \eta g(t,y_t), \tag{3*}$$

where $h : R \to R^n$ is continuous; $f, g : R \times C \to R^n$ are continuous and local Lipschitzian in ϕ; $C = C([-r,0], R^n)$. We define $x_t \in C$ by $x_t(\theta) = x(t+\theta), \theta \in [-r,0]$ for a given continuous function $x : R \to R^n$. For any given norm $|\cdot|$ in R^n, $(C, ||\cdot||)$ is a Banach space with $||\phi|| = \sup_{\theta\in[-r,0]} |\phi(\theta)|$. $D : R \to C^*$, where C^* denotes the dual space of C with the operator norm. For any $t \in R$, we define linear operator $D(t) : C \to R^n$ by

$$D(t)\phi = \phi(0) - \int_{-r}^{0} [d_\theta \mu(t+\theta)]\phi(\theta), \qquad \forall \phi \in C,$$

where $\mu(t,\theta)$ is an $n \times n$ bounded variation matrix function, and there exists a continuous nondecreasing function $l(s), s \in [0,r], l(0) = 0$, such that

$$\Big|\int_{-s}^{0} [d_\theta \mu(t,\theta)]\Big| \le l(s) \sup_{-s\le\theta\le 0} |\phi(\theta)|, \qquad \forall \phi \in C.$$

We always suppose that $D : R \to C^*$ is almost periodic (see [7]); the Frechét derivative of D exists on R (denoted by $\dot{D}(t)$) and is uniformly continuous on R; $h(t)$ is almost periodic; $f, g : R \times C_{H^*} \to R^n$ are almost periodic in t uniformly for $\phi \in C_{H^*}$ (see [7,9]) and for $\alpha > 0$, there is $n(\alpha) > 0$ such that

$$|f(t,\phi)| \le n(\alpha), \qquad ||\phi|| < \alpha \text{ and } (t,\phi) \in R \times C_{H^*},$$

with $C_{H^*} = \{\phi \in C; ||\phi|| < H^*\}$.

Clearly, for each $t \in R$, $D(t) : C_{H^*} \to R^n$ is a continuous linear operator. Under the above hypotheses, it is well known that $D : R \to C^*$ is uniformly bounded on R and $\dot{D}(t)$ is also almost periodic (see [7]). So, $\dot{D}(t)$ is a bounded operator. Thus, there exists $\overline{L} > 0$ such that

$$||D(t_2) - D(t_1)|| \le \overline{L}|t_2 - t_1|.$$

It is also well known that there is a unique solution $x_t(\sigma,\phi)$ of (1) through every initial value $(\sigma,\phi) \in R \times C_{H^*}$, the solution $x_t(\sigma,\phi)$ is continuous in (t,σ,ϕ), and $x_t(\sigma,\phi)$ exists on $[0,+\infty)$ whenever $|x_t(\sigma,\phi)| \le H < H^*$ (see [1,3]).

Let $C([\tau,+\infty), R^n)$ be the set that consists of continuous functions $H : [\tau,+\infty) \to R^n$, where τ is a fixed number. For $H \in C([\tau,+\infty), R^n)$, we consider

$$\begin{cases} D(t)x_t = D(\sigma)\phi + H(t) - H(\sigma), & t \ge \sigma \ge \tau, \\ x_\sigma = \phi. \end{cases} \tag{4}$$

Let $x(\sigma,\phi,H)(t), t \ge \sigma$, denote a solution of (4).

Definition 1[1]. *Suppose that $\mathcal{H} \subset C([\tau,\infty), R^n)$. We say that the operator $D(t)$ is uniformly stable with respect to $\mathcal{H}$ if there are constants K, M such that for any $\phi \in C, \sigma \in [\tau,+\infty)$ and $H \in \mathcal{H}$, the solution $x(\sigma,\phi,H)$ of (4) satisfies*

$$||x_t(\sigma,\phi,H)|| \le K||\phi|| + M \sup_{\sigma\le u\le t} |H(u) - H(\sigma)|. \tag{5}$$

Lemma 1[1]. *If $D(t)$ is a uniformly stable operator with respect to $C([\tau,\infty),R^n)$, then there are positive constants a,b,c,d such that for any $\bar{h}\in C([\tau,+\infty),R^n),\sigma\in[\tau,+\infty)$, the solution $x(\sigma,\phi,\bar{h})$ of the equation*

$$D(t)x_t=\bar{h}(t),\qquad t\geq\sigma,x_\sigma=\phi$$

satisfies

$$||x_t(\sigma,\phi,\bar{h})||\leq e^{-a(t-\sigma)}(b||\phi||+c\cdot\sup_{\sigma\leq u\leq t}|\bar{h}(u)|)+d\cdot\sup_{\sigma\leq u\leq t}|\bar{h}(u)|,\qquad t\geq\sigma. \tag{6}$$

Furthermore, the constants a,b,c,d can be chosen so that for any $s\in[\sigma,\infty)$,

$$||x_t(\sigma,\phi,\bar{h})||\leq e^{-a(t-s)}(b||\phi||+c\cdot\sup_{\sigma\leq u\leq t}|\bar{h}(u)|)+d\cdot\sup_{s\leq\sigma\leq t}|\bar{h}(u)| \tag{7}$$

for $t\geq s+r$.

Remark 1. Clearly, if the operator $D(t)$ is uniformly stable with respect to $C([\tau,\infty),R^n)$, then for any $\sigma\geq 0$, the operator $D(t+\sigma)$ is also uniformly stable with respect to $C([\tau,\infty),R^n)$ with the same numbers K,M,a,b,c,d as Definition 1 and Lemma 1.

Lemma 2[13]. *Suppose that $D(t)$ is a uniformly stable operator with respect to $\mathcal{H}$. If $x(t)$ is a solution of Equation (1), $||x_t||\leq\alpha,t\in R$, and $\int_0^t f(s,x_s)ds\in\mathcal{H}$, then for any $t_1,t_2\in R$, we have*

$$||x_{t_2}-x_{t_1}||\leq M\cdot[n(\alpha)+\bar{L}\alpha]\cdot|t_2-t_1|.$$

Here M is the number in (5).

Lemma 3. *Suppose that the operator $D(t)$ is uniformly stable with respect to $C((-\infty,+\infty);R^n),u:R\to R^n$ is a solution of Equation (1) and $||u_t||\leq H<H^*$ for $t\in R^+$. Then the closure $cl\{u_t;t\in R^+\}$ is a compact set in C.*

Proof. By Lemma 1, u_t is uniformly continuous on R^+. Take any sequence $\{u_{t_k}\},t_k\geq 0$. Clearly, the sequence of function $\{u(t_k+\theta)\}$ is uniformly bounded and equicontinuous on $[-r,0]$. Hence, $cl\{u_t;t\in R^+\}$ is a compact set in C by Arzela-Ascoli theorem.

Let $AP=\{\phi\in C(R,R^n);\phi(t)$ is an almost periodic function $\}$. For $\beta>0$ and $N>0$, we set

$$B_{\beta,N}=\{\phi\in AP||\phi(t)|\leq\beta,t\in R;|\phi(t_1)-\phi(t_2)|\leq N|t_1-t_2| \text{ for } t_1,t_2\in R;\text{ and } \mathrm{mod}(\phi)\subset\mathrm{mod}(D,f,g)\}.$$

Similar to the proof in [11], we can obtain

Lemma 4. *For $\beta>0$ and $N>0$, $B_{\beta,N}$ is a compact set in Banach space $C_0(R,R^n)$, where $C_0(R,R^n)$ is the set that consists of continuously bounded functions from R into R^n with supremum norm $||\cdot||^\infty$. Furthermore, if $\phi\in B_{\beta,N}$ and $t\in R$, then $g(t,\phi_t)\in AP$ and is bounded uniformly on $\phi\in B_{\beta,N}$ and $t\in R$.*

From the condition that g is local Lipschitzian in ϕ and Lemma 3, it follows that there is a $\overline{K}>0$, such that

$$|g(t,\phi_t)-g(t,\psi_t)|\leq\overline{K}||\phi-\psi||^\infty$$

for any $\phi,\psi\in B_{\beta,N}$ and $t\in R$.

Similar to the proof in [11], we can obtain

Theorem 1. *Suppose that $D(t)$ is uniformly stable with respect to $C((-\infty,+\infty);R^n)$ and Equation (1) has a solution $\xi(t), |\xi_t| \le H < H^*, t \ge 0$. If $\xi(t)$ is asymptotically almost periodic (see [2, 5]), then Equation (1) has an almost periodic solution.*

Suppose that $V : R^+ \times C_{H^*} \times C_{H^*} \to R^+$ is continuous. We define the derivative of V along the solution of Equation (1*) by

$$V'_{(1^*)}(t,\phi,\psi) = \limsup_{h\to o^+} \frac{1}{h}[V(t+h, x_{t+h}(t,\phi), y_{t+h}(t,\psi)) - V(t,\phi,\psi)],$$

where $(x(t,\phi), y(t,\psi))$ is a solution of Equation (1*) through $(t,(\phi,\psi)), \phi,\psi \in C_{H^*}$.

Theorem 2. *Suppose that $D(t)$ is uniformly stable with respect to $C((-\infty,+\infty);R^n)$, and there is $V : R^+ \times C_{H^*} \times C_{H^*} \to R^+$ such that*

(i) $u(|D(t)\phi - D(t)\psi|) \le V(t,\phi,\psi) \le v(||\phi-\psi||), u, v \in CIP, v(0) = 0$;

(ii) $|V(t,\phi_1,\psi_1) - V(t,\phi_2,\psi_2)| \le L[|D(t)\phi_1 - D(t)\phi_2| + |D(t)\psi_1 - D(t)\psi_2|]$;

(iii) $V'_{(1^*)}(t,\phi,\psi) \le -c_0 V(t,\phi,\psi)$, *for some constant $c_0 > 0$.*

If there is a solution $\xi(t)$ of Equation (1) such that $|\xi_t| \le H < H^*$ for $t \ge t_0 \ge 0$, then Equation (1) has a unique almost periodic solution $p(t)$ which is uniformly asymptotically stable, and $\mathrm{mod}(p) \subset \mathrm{mod}(D,f)$. Furthermore, if $D(t+\omega) = D(t)$ and $f(t+\omega,\phi) = f(t,\phi)$, then Equation (1) has an ω-periodic solution.

Proof. Let a, b, c, d be the numbers in Lemma 2. We complete the proof by dividing it into four steps.

(1) Prove that Equation (1) has an almost periodic solution.

We can assume $t_0 \ge 0$. Put $W_1 = cl\{\xi_t; t \ge 0\}$. It follows from Lemma 3 that W_1 is a compact set in C. Let $\alpha = \{\alpha_n\}, \alpha_n \to \infty$ as $n \to \infty$, be any sequence. We can suppose that $\{\alpha_n\}$ is increase (if necessary, take subsequence). Since $D(t)$ and $\dot{D}(t)$ are almost periodic and $f(t,\phi)$ is almost periodic in t uniformly for $\phi \in C_{H^*}$, we can suppose that $D(t+\alpha_n)$ and $\dot{D}(t+\alpha_n)$ converge uniformly on R as $n \to \infty$ and $f(t+\alpha_n,\phi)$ converges uniformly on $R \times W_1$ as $n \to \infty$ (if necessary, take subsequence). Thus, for any $\epsilon > 0$ ($\epsilon < H$), there exists $l_0 = l_0(\epsilon, W_1)$ such that when $m \ge k \ge l_0$, we have $\alpha_k \ge r$,

$$2He^{-a\alpha_k}\{b + [l(r)+1]c\} < \epsilon/2,$$

$$|f(t+\alpha_k,\phi) - f(t+\alpha_m,\phi)| \le \frac{u(\epsilon/2d)c_0}{6L}, \quad (t,\phi) \in R \times W_1,$$

$$||D(t+\alpha_k) - D(t+\alpha_m)|| \le \frac{u(\epsilon/2d)c_0}{6LM[n(H) + \overline{L}H]}, \qquad t \in R,$$

and

$$||\dot{D}(t+\alpha_k) - \dot{D}(t+\alpha_m)|| \le \frac{u(\epsilon/2d)c_0}{6LH}, \quad t \in R.$$

Now, we consider functional $V(t, \xi_t, \xi_{t+\alpha_m-\alpha_k}), t \ge 0, m \ge k \ge l_0$.

$$\begin{aligned}
&V'(t,\xi_t,\xi_{t+\alpha_m-\alpha_k}) \\
&= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, \xi_{t+h}, \xi_{t+\alpha_m-\alpha_k+h}) - V(t,\xi_t,\xi_{t+\alpha_m-\alpha_k})] \\
&\le \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x_{t+h}(t,\xi_t), y_{t+h}(t,\xi_{t+\alpha_m-\alpha_k})) - V(t,\xi_t,\xi_{t+\alpha_m-\alpha_k})] \\
&\quad + \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, \xi_{t+h}, \xi_{t+\alpha_m-\alpha_k+h}) - V(t+h, x_{t+h}(t,\xi_t), y_{t+h}(t,\xi_{t+\alpha_m-\alpha_k}))]
\end{aligned}$$

$$
\begin{aligned}
&\le -c_0 V(t,\xi_t,\xi_{t+\alpha_m-\alpha_k}) + \limsup_{h\to 0^+}\frac{L}{h}[|D(t+h)(\xi_{t+h}-x_{t+h}(t,\xi_t))| \\
&\quad + |D(t+h)(\xi_{t+\alpha_m-\alpha_k+h}-y_{t+h}(t,\xi_{t+\alpha_m-\alpha_k}))|] \\
&= -c_0 V(t,\xi_t,\xi_{t+\alpha_m-\alpha_k}) \\
&\quad + \limsup_{h\to 0^+}\frac{L}{h}|[D(t+\alpha_m-\alpha_k+h)\xi_{t+\alpha_m-\alpha_k+h} - D(t+\alpha_m-\alpha_k)\xi_{t+\alpha_m-\alpha_k}] \\
&\quad - [D(t+h)y_{t+h}(t,\xi_{t+\alpha_m-\alpha_k}) - D(t)\xi_{t+\alpha_m-\alpha_k}] \\
&\quad + [D(t+h)-D(t+\alpha_m-\alpha_k+h)](\xi_{t+\alpha_m-\alpha_k+h}-\xi_{t+\alpha_m-\alpha_k}) \\
&\quad + \{[D(t+h)-D(t)]-[D(t+\alpha_m-\alpha_k+h)-D(t+\alpha_m-\alpha_k)]\}\xi_{t+\alpha_m-\alpha_k}|. \qquad (8)
\end{aligned}
$$

Since $\xi(t)$ is a solution of Equation (1), $\xi(t+\alpha_m-\alpha_k)$ is a solution of the system

$$\frac{d}{dt}D(t+\alpha_m-\alpha_k)w_t = f(t+\alpha_m-\alpha_k, w_t). \qquad (9)$$

Using Remark 1 and Lemma 2, we obtain

$$||\xi_{t+\alpha_m-\alpha_k+h}-\xi_{t+\alpha_m-\alpha_k}|| \le M[n(H)+\overline{L}H]h.$$

From (8), it follows that

$$V'(t,\xi_t,\xi_{t+\alpha_m-\alpha_k}) \le -c_0 V(t,\xi_t,\xi_{t+\alpha_m-\alpha_k}) + \frac{u(\epsilon/2d)c_0}{2}.$$

By using differential inequality, we have

$$V(t,\xi_t,\xi_{t+\alpha_m-\alpha_k}) \le e^{-c_0 t}\Big[V(0,\xi_0,\xi_{\alpha_m-\alpha_k}) - \frac{u(\epsilon/2d)}{2}\Big] + \frac{u(\epsilon/2d)}{2}, \qquad t\ge 0.$$

For the above $\epsilon>0$, there exists $T>0$ such that when $t>T$, we have

$$e^{-c_0 t}v(2H) \le \frac{u(\epsilon/2d)}{2}.$$

Thus, when $t\ge T$, we have

$$V(t,\xi_t,\xi_{t+\alpha_m-\alpha_k}) \le u(\epsilon/2d).$$

It implies that

$$|D(t)(\xi_t-\xi_{t+\alpha_m-\alpha_k})| \le \epsilon/2d, \qquad t\ge T.$$

It follows from (7) that

$$
\begin{aligned}
&||\xi_t-\xi_{t+\alpha_m-\alpha_k}|| \\
&\le e^{-a\alpha_k}\{2Hb+2H[l(r)+1]c\} + d\sup_{t\le u\le t+\alpha_k}|D(u)(\xi_u-\xi_{u+\alpha_m-\alpha_k})| \\
&\le \epsilon, \qquad \forall t\ge T, m\ge k\ge l_0.
\end{aligned}
$$

In this time, there is an l_1 such that for any $t\in R^+$ we have $t+\alpha_{l_1}\ge T$. So, we obtain

$$||\xi_{t+\alpha_{l_1}+\alpha_k}-\xi_{t+\alpha_{l_1}+\alpha_m}|| \le \epsilon, \qquad \forall t\in R^+, m\ge k\ge l_0.$$

This implies that $\eta(t)=\xi(t+\alpha_{l_1})$ is an asymptotically almost periodic function (see [2,5]). Clearly, $\xi(t)$ is also an asymptotically almost periodic function. It follows from Theorem 1 that Equation (1) has an almost periodic solution $p(t)$.

(2) Show that $p(t)$ is uniformly stable for $t\ge t_0 (t_0\in R)$.

Put

$$\hat{H} = \begin{cases} H+1, & H^*=\infty, \\ \frac{H+H^*}{2}, & H^*<\infty. \end{cases}$$

For any $\epsilon > 0$ ($\epsilon < \hat{H}$), we take $\delta : b\delta < \epsilon/2$ and $v(\delta) < \frac{1}{2}u(\epsilon/2(c+d))$. For any $t_0 \in R,\ \phi \in C_{H^*} : ||\phi - p_{t_0}|| < \delta$, let $x(t) = x(t_0, \phi)(t)$ be a solution of Equation (1) through (t_0, ϕ). Let $\beta = \sup\{t; t \geq t_0 \text{ and } \{x_s; s \in [t_0, t)\} \subset C_{\hat{H}}\}$. Then

$$W_2 = cl\{x_t; t \in [t_0, \beta)\} \cup cl\{p_t; t \in R\}$$

is a compact set in C by Lemma 3. Since $D(t)$ and $\dot{D}(t)$ are almost periodic and $f(t, \phi)$ is almost periodic in t uniformly for $\phi \in C_{H^*}$, there is $\tau_1, \tau_1 + t_0 \geq 0$, such that

$$|f(t+\tau_1, \phi) - f(t, \phi)| \leq \frac{u(\epsilon/2(c+d))c_0}{12L}, \qquad \forall (t, \phi) \in R \times W_2,$$

$$||D(t+\tau_1) - D(t)|| \leq \frac{u(\epsilon/2(c+d))c_0}{12LM[n(\hat{H}) + \overline{L}\hat{H}]}, \qquad t \in R,$$

and

$$||\dot{D}(t+\tau_1) - \dot{D}(t)|| \leq \frac{u(\epsilon/2(c+d))c_0}{12L\hat{H}}, \qquad t \in R.$$

Now

$$\begin{aligned}
&V'(t+\tau_1, x_t, p_t)\\
&= \limsup_{h \to 0^+} \frac{1}{h}[V(t+\tau_1+h, x_{t+h}, p_{t+h}) - V(t+\tau_1, x_t, p_t)]\\
&\leq \limsup_{h \to 0^+} \frac{1}{h}|[D(t+h)x_{t+h}(t, x_t) - D(t)x_t]\\
&\quad - [D(t+\tau_1+h)x_{t+\tau_1+h}(t+\tau_1, x_t) - D(t+\tau_1)x_{t+\tau_1}(t+\tau_1, x_t)|\\
&\quad + \limsup_{h \to 0^+} \frac{1}{h}|[D(t+\tau_1+h) - D(t+h)](x_{t+h} - x_t)|\\
&\quad + \limsup_{h \to 0^+} \frac{1}{h}|\{[D(t+\tau_1+h) - D(t+\tau_1)] - [D(t+h) - D(t)]\}x_t|.
\end{aligned}$$

Since $x(t)$ is a solution of Equation (1), we have

$$||x_{t+h} - x_t|| \leq M[n(\hat{H}) + \overline{L}\hat{H}]h, \qquad t \in [t_0, \beta).$$

Thus, we have

$$\limsup_{h \to 0^+} \frac{1}{h}|D(t+\tau_1+h)[x_{t+h} - x_{t+\tau_1+h}(t+\tau_1, x_t)]| \leq \frac{u(\epsilon/2(c+d))c_0}{4L}.$$

Similarly, we obtain

$$\limsup_{h \to 0^+} \frac{1}{h}|D(t+\tau_1+h)[p_{t+h} - y_{t+\tau_1+h}(t+\tau_1, p_t)]| \leq \frac{u(\epsilon/2(c+d))c_0}{4L}.$$

It follows that

$$V'(t+\tau_1, x_t, p_t) \leq -c_0 V(t+\tau_1, x_t, p_t) + \frac{u(\epsilon/2(c+d))c_0}{2}. \tag{10}$$

Using this differential inequality, we obtain

$$\begin{aligned}
&V(t+\tau_1, x_t, p_t)\\
&\leq e^{-c_0(t-t_0)}\Big[V(t_0+\tau_1, x_{t_0}, p_{t_0}) - \frac{u(\epsilon/2(c+d))}{2}\Big] + \frac{u(\epsilon/2(c+d))}{2}\\
&\leq v(\delta) + \frac{u(\epsilon/2(c+d))}{2} \leq u(\epsilon/2(c+d)), \qquad \forall t \in [t_0, \beta).
\end{aligned}$$

It imples that

$$|D(t+\tau_1)(x_t - p_t)| < \epsilon/2(c+d), \qquad \forall t \in [t_0, \beta).$$

By (6) in Lemma 1, we obtain

$$||x_t - p_t|| \le e^{-a(t-t_0)}[b||x_{t_0} - p_{t_0}|| + c \cdot \sup_{t_0 \le u \le t} |D(u+\tau_1)(x_u - p_u)|]$$
$$+ d \cdot \sup_{t_0 \le u \le t} |D(u+\tau_1)(x_u - p_u)| < \epsilon, \qquad \forall t \in [t_0, \beta).$$

So, it is easy to know that $\beta = +\infty$. Thus, we have

$$|x(t) - p(t)| < \epsilon, \qquad t \ge t_0.$$

This implies that $p(t)$ is uniformly stable.

(3) Show that $p(t)$ is quasi-uniformly asymptotically stable for $t \ge t_0 (t_0 \in R)$.

Since $p(t)$ is uniformly stable, there is a $\delta_0 > 0$ such that when $||\phi - p_{t_0}|| < \delta_0$,

$$||x_t(t_0, \phi) - p_t|| < \overline{H}(< H^* - H) \text{ for } t \ge t_0$$

by the second step. Put $x_t = x_t(t_0, \phi)$. Thus,

$$W_3 = cl\{x_t; t \in [t_0, \infty)\} \cup cl\{p_t; t \in R\}$$

is a compact set in C by lemma. For any $\epsilon > 0$ ($\epsilon < \overline{H}$), there is $\tau_2, \tau_2 + t \ge 0$, such that

$$|f(t+\tau_2, \phi) - f(t, \phi)| \le \frac{u(\epsilon/2d)c_0}{12L}, \qquad \forall (t, \phi) \in R \times W_3,$$
$$||D(t+\tau_2) - D(t)|| \le \frac{u(\epsilon/2d)c_0}{12LM[n(\overline{H} + H) + \overline{L}(\overline{H} + H)]}, \qquad t \in R,$$

and

$$||\dot{D}(t+\tau_2) - \dot{D}(t)|| \le \frac{u(\epsilon/2d)c_0}{12L(\overline{H} + H)}, \qquad t \in R.$$

Similar to (10), we can obtain

$$V'(t+\tau_2, x_t, p_t) \le -c_0 V(t+\tau_2, x_t, p_t) + \frac{u(\epsilon/2d)}{2}, \qquad t \ge t_0.$$

Using this differential inequality, we have

$$V(t+\tau_2, x_t, p_t) \le e^{-c_0(t-t_0)}\Big[V(t_0+\tau_2, x_{t_0}, p_{t_0}) - \frac{u(\epsilon/2d)}{2}\Big] + \frac{u(\epsilon/2d)}{2}, \quad \forall t \ge t_0.$$

Take a $T_1 > 0$ such that it satisfies

$$e^{-c_0 T_1} v(\delta_0) < \frac{u(\epsilon/2d)}{2}$$

and

$$e^{-aT_1}[b\delta_0 + c(l(r)+1)\overline{H}] < \epsilon/2.$$

Then, when $t \ge t_0 + T_1$, we have

$$|D(t+\tau_2)(x_t - p_t)| < \epsilon/2d.$$

Let $T = 2T_1$. Hence, when $t \ge t_0 + T$, we have

$$||x_t - p_t|| \le e^{-aT_1}(b||x_{t_0} - p_{t_0}|| + c \cdot \sup_{t_0 \le u \le t} |D(u+\tau_2)(x_u - p_u)|)$$
$$+ d \cdot \sup_{t_0 + T_1 \le u \le t} |D(u+\tau_2)(x_u - p_u)| < \epsilon, \qquad \forall t \ge t_0 + T.$$

It implies that

$$|x(t) - p(t)| < \epsilon, \qquad t \ge t_0 + T,$$

i.e., $p(t)$ is quasi-uniformly asymptotically stable.

(4) From the third step proof, it follows that for any almost periodic solution $\bar{p}(t)$ of Equation (1), $t \in R$, we have

$$|p(t) - \bar{p}(t)| \to 0, \qquad \text{as } t \to \infty.$$

Using the almost periodicity, we obtain

$$p(t) \equiv \bar{p}(t), \qquad \forall t \in R.$$

This means that the almost periodic solution of Equation (1) is unique. Now, we want to show

$$\text{mod}(p) \subset \text{mod}(D, f).$$

Suppose $\{\gamma_k\}$ is any sequence and satisfies the condition that, for any compact W in C_{H^*}, $f(t+\gamma_k, \phi)$ converges to $f(t, \phi)$ uniformly on $R \times W$ as $k \to \infty$ and $D(t+\gamma_k)$ converges to $D(t)$ uniformly on R as $k \to \infty$. Since $p(t)$ is an almost periodic solution, there exists $\{\gamma_{k_j}\} \subset \{\gamma_k\}$ such that $p(t + \gamma_{k_j})$ is uniformly convergent on R as $j \to \infty$. Put

$$Q(t) = \lim_{j \to \infty} p(t + \gamma_{k_j}).$$

Then, $Q(t)$ is an almost periodic function (see [2, 5]). It is easy to show that $Q(t)$ is a solution of Equation (1) . We obtain $Q(t) = p(t)$ by the uniqueness. It implies that $\text{mod}(p) \subset \text{mod}(D, f)$ (see [2, 5]).

Clearly, when $D(t + \omega) = D(t)$ and $f(t + \omega, \phi) = f(t, \phi)$ for $(t, \phi) \in R \times C_{H^*}$, we can show that Equation (1) has an ω-periodic solution.

Theorem 3. *Suppose that $D(t)$ is uniformly stable with respect to $C((-\infty, \infty); R^n)$, and there is $V : R^+ \times C_{H^*} \times C_{H^*} \to R^+$ such that it satisfies the conditions* (i) *and* (iii) *of Theorem* 2 *and*

(ii)' $|V(t, \phi_1, \psi_1) - V(t, \phi_2, \psi_2)| \leq L|[D(t)\phi_1 - D(t)\phi_2] - [D(t)\psi_1 - D(t)\psi_2]|$, *for some constant $L > 0$ on $R^+ \times C_{H^*} \times C_{H^*}$.*
Assume that there is a solution $\xi(t)$ of Equation (1) *such that $||\xi_t|| \leq H < H^*$ for all $t \geq t_0 \geq 0$. If $u^{-1}(LK/c_0)(c+d) + H \leq H_1 < H^*$ (where $|h(t)| \leq K, t \in R$, and c, d are the numbers in Lemma* 2 *), then Equation* (2) *has a unique almost periodic solution $p(t)$ which is uniformly asymptotically stable and* $\text{mod}(p) \subset \text{mod}(D, f, h)$. *Furthermore, if $D(t), f(t, \phi)$, and $h(t)$ are ω-periodic in t, then Equation* (2) *has an ω-periodic solution.*

Proof. Let a, b, c, d be the numbers in Lemma 2. Suppose that $(x_t^{(1)}(t_0, \phi), y_t^{(1)}(t_0, \psi))$ is a solution of Equation (1^*) through $(t_0, (\phi, \psi))$ and $(x_t^{(2)}(t_0, \phi), y_t^{(2)}(t_0, \psi))$ is a solution of Equation (2^*) through $(t_0, (\phi, \psi))$. By the condition (ii)', we have

$$\begin{aligned} & V'_{(2^*)}(t, \phi, \psi) \\ &= \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x_{t+h}^{(2)}(t, \phi), y_{t+h}^{(2)}(t, \psi)) - V(t, \phi, \psi)] \\ &\leq \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x_{t+h}^{(1)}(t, \phi), y_{t+h}^{(1)}(t, \psi)) - V(t, \phi, \psi)] \\ &\quad + \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x_{t+h}^{(2)}(t, \phi), y_{t+h}^{(2)}(t, \psi)) - V(t+h, x_{t+h}^{(1)}(t, \phi), y_{t+h}^{(1)}(t, \psi))] \\ &\leq -c_0 V(t, \phi, \psi), \qquad \forall (t, \phi, \psi) \in R^+ \times C_{H^*} \times C_{H^*}. \end{aligned} \tag{11}$$

It suffices to show that Equation (2) has a solution $\eta(t), |\eta_t| \le H < H^*$, on $[t_0, \infty)$ by Theorem 2. Let $\phi = \xi_{t_0}, \eta_t = x_t^{(2)}(t_0, \phi)$ be the solution of Equation (2) through (t_0, ϕ) and $[t_0, \beta)$ be its maximal existence interval to the right.

$$\begin{aligned} V'(t, \xi_t, \eta_t) &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, \xi_{t+h}, \eta_{t+h}) - V(t, \xi_t, \eta_t)] \\ &\le \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, \xi_{t+h}, x_{t+h}^{(1)}(t, \eta_t)) - V(t, \xi_t, \eta_t)] \\ &\quad + \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, \xi_{t+h}, \eta_{t+h}) - V(t+h, \xi_{t+h}, x_{t+h}^{(1)}(t, \eta_t))] \\ &\le -c_0 V(t, \xi_t, \eta_t) + LK, \qquad \forall t \in [t_0, \beta). \end{aligned}$$

By this differential inequality, we obtain

$$V(t, \xi_t, \eta_t) \le e^{-c_0(t-t_0)}\Big[V(t_0, \xi_{t_0}, \eta_{t_0}) - \frac{LK}{c_0}\Big] + \frac{LK}{c_0} \le \frac{LK}{c_0}, \quad \forall t \in [t_0, \beta).$$

Thus, we have

$$u(|D(t)\xi_t - D(t)\eta_t|) \le \frac{LK}{c_0}, \qquad \forall t \in [t_0, \beta).$$

It follows that

$$\begin{aligned} ||\xi_t - \eta_t|| &\le e^{-a(t-t_0)}[b||\xi_{t_0} - \eta_{t_0}|| + c \cdot \sup_{t_0 \le u \le t} |D(u)(\xi_u - \eta_u)|] \\ &\quad + d \cdot \sup_{t_0 \le u \le t} |D(u)(\xi_u - \eta_u)| \\ &\le (c+d) \sup_{t_0 \le u \le t} |D(u)(\xi_u - \eta_u)| \\ &\le u^{-1}(\frac{LK}{c_0})(c+d), \qquad \forall t \in [t_0, \beta). \end{aligned}$$

It implies

$$||\eta_t|| \le u^{-1}\Big(\frac{LK}{c_0}\Big)(c+d) + H \le H_1 < H^*, \qquad \forall t \in [t_0, \beta).$$

Clearly, $\beta = +\infty$.

Theorem 4. *Suppose that $D(t)$ is uniformly stable with respect to $C((-\infty, \infty), R^n)$, and there is $V : R^+ \times C_{H^*} \times C_{H^*} \to R^+$ such that it satisfies the condition* (ii)' *of Theorem* 3, *the condition* (iii) *of Theorem* 2, *and*

(i)' $M_0|D(t)\phi - D(t)\psi| \le V(t, \phi, \psi) \le v(||\phi - \psi||), v \in CIP, v(0) = 0$, *for some constant* $M_0 > 0$.

If Equation (1) *has a solution $\xi(t), ||\xi_t|| \le H < H^*, t \ge t_0$, then for any $\beta : H < \beta < H^*$ and $N > (n(H) + \overline{L})M$ (here M stands for the number in definition), there exists $\eta_0 > 0$ such that when $0 \le \eta < \eta_0$, Equation* (3) *has a unique solution in $B_{\beta,N}$. Furthermore, if $D(t), f(t, \phi)$, and $g(t, \phi)$ are ω-periodic in t, then when $0 \le \eta < \eta_0$ Equation* (3) *has an ω-periodic solution.*

Proof. For $\beta > H$ and $N > M[n(H) + \overline{L}]$, we first show that there is an $\eta_2 > 0$ such that when $0 \le \eta < \eta_2$, for any $\phi \in B_{\beta,N}$, the system

$$\frac{d}{dt}D(t)x_t = f(t, x_t) + \eta g(t, \phi_t) \tag{12}$$

has a unique solution in $B_{\beta,N}$.

Put $C_1 = \sup\{|g(t,\phi_t)|; t \in R, \phi \in B_{\beta,N}\}$. It follows from Lemma 4 that $C_1 < +\infty$. Suppose that $x(t)$ is a solution of Equation (12) which satisfies $x_{t_0} = \xi_{t_0}$, and its maximal existence interval to the right is $[t_0, \alpha)$. We have

$$\begin{aligned} V'(t,x_t,\xi_t) &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x_{t+h},\xi_{t+h}) - V(t,x_t,\xi_t)] \\ &\le -c_0 V(t,x_t,\xi_t) + \limsup_{h\to 0^+} \frac{1}{h} L\big|D(t+h)x_{t+h} - D(t+h)x^{(1)}_{t+h}(t,x_t)\big| \\ &\le -c_0 V(t,x_t,\xi_t) + L\eta C_1 \qquad \forall t \in [t_0,\alpha). \end{aligned} \tag{13}$$

By this differential inequality, we have

$$V(t,x_t,\xi_t) \le e^{-c_0(t-t_0)}\Big[V(t_0,x_{t_0},\xi_{t_0}) - \frac{L\eta C_1}{c_0}\Big] + \frac{L\eta C_1}{c_0} \le \frac{L\eta C_1}{c_0}, \quad \forall t \in [t_0,\alpha).$$

It follows from the condition (i)' that

$$|D(t)x_t - D(t)\xi_t| \le \frac{L\eta C_1}{c_0 M_0}, \qquad \forall t \in [t_0,\alpha).$$

By Lemma 2, we obtain

$$\begin{aligned} ||x_t - \xi_t|| &\le e^{-a(t-t_0)}[b||x_{t_0} - \xi_{t_0}|| + c\cdot \sup_{t_0\le u\le t}|D(u)(x_u-\xi_u)|] \\ &\quad + d\cdot \sup_{t_0\le u\le t}|D(u)(x_u-\xi_u)| \\ &\le (c+d)\sup_{t_0\le u\le t}|D(u)(x_u-\xi_u)| \le (c+d)\frac{L\eta C_1}{c_0M_0}, \forall t\in[t_0,\alpha). \end{aligned}$$

It implies that

$$||x_t|| \le (c+d)\frac{L\eta C_1}{c_0 M_0} + H.$$

By the assumptions, we know that there is an $\eta_1 > 0$ such that when $0 \le \eta < \eta_1$,

$$(c+d)\frac{\eta L C_1}{c_0 M_0} + H < \beta < H^*.$$

Thus, $x(t)$ is infinite continuation to the right, i.e., $\alpha = +\infty$. Assume that (12^*) denotes the product system of Equation (12). Similar to (11), we have

$$V'_{(12^*)}(t,\phi,\psi) \le -c_0 V(t,\phi,\psi).$$

It follows from Theorem 2 that Equation (12) has a unique almost periodic solution $p(t)$ that is uniformly asymptotically stable, $\mathrm{mod}(p) \subset \mathrm{mod}(D,f,g)$, and

$$||p_t|| \le (c+d)\frac{\eta L C_1}{c_0 M_0} + H. \tag{14}$$

For any $t_1, t_2 \in R$,

$$\begin{aligned} |D(t_2)p_{t_2} - D(t_1)p_{t_1}| &= \Big|\int_{t_1}^{t_2} [f(s,p_s) + \eta g(s,p_s)]ds\Big| \\ &\le \Big[n\Big(\frac{\eta L C_1}{c_0 M_0}(c+d) + H\Big) + \eta C_1\Big]|t_2 - t_1|. \end{aligned}$$

It follows that

$$\begin{aligned} |D(t_2)(p_{t_2} - p_{t_1})| &\le |D(t_2)p_{t_2} - D(t_1)p_{t_1}| + |[D(t_2) - D(t_1)]p_{t_1}| \\ &\le \Big[n\Big(\frac{\eta L C_1}{c_0 M_0}(c+d) + H\Big) + \eta C_1 + \overline{L}\Big]|t_2 - t_1|. \end{aligned}$$

Thus, we obtain

$$||p_{t_2} - p_{t_1}|| \leq M\Big[n\Big(\frac{\eta L C_1}{c_0 M_0}(c+d) + H\Big) + \eta C_1 + \overline{L}\Big]|t_2 - t_1|. \tag{15}$$

Since $\beta > H, N > M[n(H) + \overline{L}]$, we can take an η_2 such that, when $0 \leq \eta < \eta_2$, Equation (12) has a unique solution in $B_{\beta,N}$.

For any $\phi \in B_{\beta,N}$, let $T\phi$ stand for a unique solution of Equation (12) in $B_{\beta,N}$. Thus, T is a mapping from $B_{\beta,N}$ into $B_{\beta,N}$. For any $\phi, \psi \in B_{\beta,N}$, we have

$$\begin{aligned}&V'(t, (T\phi)_t, (T\psi)_t)\\ &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, (T\phi)_{t+h}, (T\psi)_{t+h}) - V(t, (T\phi)_t, (T\psi)_t)]\\ &\leq -c_0 V(t, (T\phi)_t, (T\psi)_t) + L\eta\overline{K}||\phi - \psi||^\infty.\end{aligned}$$

By this differential inequality, we obtain

$$\begin{aligned}&V(t, (T\phi)_t, (T\psi)_t)\\ &\leq e^{-c_0 t}\Big[V(0, (T\phi)_0, (T\psi)_0) - \frac{L\eta\overline{K}}{c_0}||\phi - \psi||^\infty\Big] + \frac{L\eta\overline{K}}{c_0}||\phi - \psi||^\infty.\end{aligned}$$

It follows from the condition (i)' that

$$\begin{aligned}&|D(t)(T\phi)_t - D(t)(T\psi)_t|\\ &\leq \frac{e^{-c_0 t} v(||(T\phi)_0 - (T\psi)_0||)}{M_0} + \frac{L\eta\overline{K}}{c_0 M_0}||\phi - \psi||^\infty.\end{aligned} \tag{16}$$

Since $T\phi, T\psi \in B_{\beta,N}$ and $D(t)$ is almost periodic, there is a sequence $\{\alpha_n\}, \alpha_n \to \infty$ as $n \to \infty$, such that

$$T\phi(t+\alpha_n) - T\psi(t+\alpha_n) \to T\phi(t) - T\psi(t)$$

uniformly on R as $n \to \infty$ and $D(t+\alpha_n) \to D(t)$ uniformly on R as $n \to \infty$. Replacing t by $t + \alpha_n$ in (16), we obtain

$$|D(t)(T\phi)_t - D(t)(T\psi)_t| \leq \frac{L\eta\overline{K}}{c_0 M_0}||\phi - \psi||^\infty.$$

It follows from Lemma 2 that

$$\begin{aligned}&||(T\phi)_t - (T\psi)_t||\\ &\leq e^{-at}[b||(T\phi)_0 - (T\psi)_0|| + c\sup_{0\leq u\leq t}|D(u)(T\phi)_u - D(u)(T\psi)_u|\\ &\quad + d\cdot\sup_{0\leq u\leq t}|D(u)(T\phi)_u - D(u)(T\psi)_u|\\ &\leq e^{-at}\Big[b||(T\phi)_0 - (T\psi)_0|| + c\frac{L\eta\overline{K}}{c_0 M_0}||\phi - \psi||^\infty\Big] + d\frac{L\eta\overline{K}}{c_0 M_0}||\phi - \psi||^\infty.\end{aligned}$$

If we replace t by $t + \alpha_n$ on the above, and $n \to \infty$, we can obtain

$$||(T\phi)_t - (T\psi)_t|| \leq d\frac{L\eta\overline{K}}{c_0 M_0}||\phi - \psi||^\infty, \qquad t \in R.$$

Let $\eta_0 = \min\ \{\eta_1, \eta_2, C_0 M_0/dL\overline{K}\}$. Then, when $0 \leq \eta < \eta_0, T$ is a contraction mapping from $B_{\beta,N}$ into $B_{\beta,N}$. Hence, T has a unique fixed point in $B_{\beta,N}$.

Clearly, if D, f and g are ω-periodic in t, there is an η_0 such that when $0 \leq \eta < \eta_0$, Equation (3) has an ω-periodic solution.

Remark 2. Clearly, the theorems in [11] are special cases of the above theorems when $D(t) = D$ and the theorems in [4,10] are special cases of the above theorems when $D(t)\phi = \phi(0)$ for $\phi \in C$.

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