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REGULARITY ESTIMATES FOR THE OBLIQUE DERIVATIVE PROBLEM ON NON-SMOOTH DOMAINS (II)**

Guan Pengfei* E. Sawyer*

Abstract

This is a continuation of the previous paper [6]. The authors prove Hölder and L^p regularity of operators constructed from the oblique derivative problem in [6] by establishing estimates of pseudodifferential operators with parameters.

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§1. Some Special Operators

In the previous paper [6] in this journal, we considered the oblique derivative problem on nonsmooth domains, and reduced the question of regularity of the solutions to the boundedness properties of certain pseudodifferential operators with limited smoothness. In this paper we will apply the technique of symbol splitting to these operators, and ultimately reduce their boundedness on Sobolev spaces to weighted norm inequalities for the Hardy operator. For example, one of the basic operators we consider in the third subsection below is of the form

$$Ku(x,t)=(2\pi)^{-\frac{n}{2}}\int_{t_0}^{t}\int_{|\xi|\geq 1}e^{ix\cdot\xi}K(x,t,t',\xi)\widetilde{u^*}\,(\xi,t')d\xi dt'.$$

One obvious feature of this operator is that it is a pseudodifferential operator in the x-variables, parameterized by t and t'. So we will also consider extending the rough ψdo calculus to pseudodifferential operators with parameters in the second subsection below. But first, we collect the necessary material on rough symbols.

1.1. Rough Pseudodifferential Operators

In this subsection we will define the symbol classes $\mathcal{C}^dS^m_{\gamma,\delta}$, recall the mapping properties of their associated operators in the Hölder, L^p Sobolev and Besov scale of spaces, and describe the symbol splitting which permits a calculus for the composition of two such operators. In particular, this calculus is crucial for the estimates in the remainder of this section and in section 2.

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*Department of Mathematics and Statistics McMaster University, Hamilton, Ontario L8S 4K1, Canada.
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Definition 1.1. *A symbol* $\sigma : R^n \times R^n \to R$ *belongs to the symbol class* $\mathcal{C}^M S^m_{\rho,\delta}$ (*where* m *is real,* $0 \le \rho, \delta \le 1$, *and* M *is a nonnegative integer*) *if for all multi-indices* α *and* β *with* $|\alpha| \le M$, *there are constants* $C_{\alpha,\beta}$ *such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|^2)^{\frac{1}{2}(m+\delta|\alpha|-\rho|\beta|)}. \tag{1.1}$$

If $0 < \mu < 1$, *then* σ *belongs to* $\mathcal{C}^{M+\mu} S^m_{\rho,\delta}$ *if in addition to* (1.1), *we have*

$$\left|\partial_\xi^\beta \sigma(x+h,\xi) - \sum_{\ell=0}^{M} \frac{(h\cdot\nabla_x)^\ell}{\ell!}\partial_\xi^\beta\sigma(x,\xi)\right| \le C_{M,\beta}\,|h|^{M+\mu}\,(1+|\xi|^2)^{\frac{1}{2}(m+\delta(M+\mu)-\rho|\beta|)}. \tag{1.2}$$

These symbol classes are treated for example in M. Taylor's book [13]. Here $\mathcal{C}^d$ denotes the usual Lipschitz space of continuous functions whose derivatives of order $\lfloor d \rfloor$ are bounded (when d is an integer), or satisfy a Hölder condition of order $d-[d]$ (when d is not an integer). Let Λ^s be the usual Hölder space, denoted by C^s_* in [13]. Of course $\Lambda^s = \mathcal{C}^s$ for s not an integer, and consists of those f with $D^\alpha f$ in the Zygmund class for $|\alpha| = s-1$, when s is a positive integer.

Definition 1.2. *A symbol* $\sigma : R^n \times R^n \to R$ *belongs to* O^m_I *if its associated operator* $\sigma(f)(x) = \int_{R^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi$ *admits a bounded extension from* $H^{s+m}_{p,comp}$ *to* $H^s_{p,loc}$ (*respectively* Λ^{s+m}_{comp} *to* Λ^s_{loc}) *for all* s *in the interval* I (*respectively* $I \cap (0,\infty)$) *and all* $1 < p < \infty$. *The symbol* σ *belongs to* $\overline{O}^m_I$ *if in addition, its associated operator is bounded from* Λ^{t+m}_{comp} *to* Λ^t_{loc} *where* t *is the right endpoint of the interval* I.

The symbol classes arising naturally in the oblique derivative problem are $\mathcal{C}^d S^m_{1,\delta}$ (see subsection 1.2 of the introduction) and so we restrict attention to the classes $\mathcal{C}^d S^m_{1,\delta}$ for the remainder of this section. In [2], R. Coifman and Y. Meyer showed that $\mathcal{C}^d S^m_{1,0} \subset O^m_{\{0\}}$ for $d > 0$ (a special case of Proposition 9, p. 38), and G. Bourdaud [1, Bou] then extended this to the following result (see e.g. section 2.1 of [13]).

Theorem 1.1. $\mathcal{C}^d S^m_{1,\delta} \subset \overline{O}^m_{(-(1-\delta)d,d)}$ *for all* m *real, all* $d > 0$ *and* $0 \le \delta < 1$.

We now recall the technique of symbol splitting (see (1.3.21) and (1.3.15) of [13]).

Proposition 1.1. *Given* $\tau \in \mathcal{C}^d S^m_{1,\delta}$, *and* $\delta < r < 1$, *we can write* $\tau = \tau^\sharp + \tau^\flat$ *with* $\tau^\sharp \in S^m_{1,\gamma}$ *and* $\tau^\flat \in \mathcal{C}^d S^{m+d(\delta-\gamma)}_{1,\gamma}$. *Moreover, there are the following improved estimates:*

$$\nabla_x^\ell \tau^\sharp \in \begin{cases} S^{m+|\ell|\delta}_{1,\gamma} & for\ 0 \le |\ell| \le d, \\ S^{m+\gamma(|\ell|-d)+\delta d}_{1,\gamma} & for\ |\ell| > d, \end{cases} \tag{1.3}$$

This symbol splitting permits the use of $\tau^\sharp$ in the classical asymptotic formula for the composition of two smooth pseudodifferential operators, namely

$$\sigma\circ\tau - \sum_{\ell=0}^{M} \frac{1}{i^\ell \ell!}\nabla_\xi^\ell\sigma\cdot\nabla_x^\ell\tau \ \in \mathcal{C}^d S^{m_1+m_2+(M+1)(\delta-1)}_{1,\delta} \quad \text{for all } M \ge 0, \tag{1.4}$$

where $\sigma \in \mathcal{C}^d S^{m_1}_{1,\delta}$ and $\tau \in S^{m_2}_{1,\delta}$ with $0 \le \delta < 1$ and $d > 0$. Note that the standard proofs in the smooth case apply even when σ is rough and τ is smooth (see e.g. section 3 of Chapter VI of [12]). As a consequence of (1.4) and Proposition 1.1 we obtain

Proposition 1.2. *Let* $\sigma \in \mathcal{C}^d S^{m_1}_{1,\delta}$ *and* $\tau \in \mathcal{C}^{M+d} S^{m_2}_{1,\delta}$, *where* $0 \le \delta < 1$, $d > 0$ *and*

$m_1 < M + d$. *Suppose* $d > 1$. *Then with* $\gamma = 1 - \frac{(d-1)(1-\delta)}{M+d}$, *we have*

$$\sigma \circ \tau - \sum_{\ell=0}^{M} \frac{1}{i^\ell \ell!} \nabla_\xi^\ell \sigma \cdot \nabla_x^\ell \tau \ \in \overline{\mathcal{O}}^{m_1+m_2+(M+1)(\delta-1)}_{(-d(1-\gamma),\min\{d,M+d-m_1\})}. \tag{1.5}$$

Suppose $0 < d \le 1$. *Then with* $\gamma = 1 - \frac{\epsilon}{M+1}$ *for any* $\epsilon > 0$, *we have*

$$\sigma \circ \tau - \sum_{\ell=0}^{M} \frac{1}{i^\ell \ell!} \nabla_\xi^\ell \sigma \cdot \nabla_x^\ell \tau \ \in \mathcal{O}^{m_1+m_2+(M+1)(\delta-1)+(1-d)(\gamma-\delta)+\epsilon}_{(-d(1-\gamma),\min\{d,M+d-m_1\})}. \tag{1.6}$$

1.2. Pseudodifferential Operators with Parameters

Define $t_0 = -1$ if $\overrightarrow{\ell}$ satisfies case (I), and $t_0 = 0$ if $\overrightarrow{\ell}$ satisfies case (II) (see [6]). We now define our symbol classes for operators with parameters. We will be working exclusively with the cases $\rho = 1$ and $0 \le \delta < 1$.

Definition 1.3. *Let* g *be a nonnegative function on* $(0, 1)$. *We say that the symbol* $\sigma(x, t, t', \xi) \in \mathcal{C}^M \mathcal{S}^m_{1,\delta,g}$ *(where* $M \ge 0$ *is an integer) if for all multi-indices* α, β *and non-negative integers* s, s', *with* $|\alpha| + s + s' \le M$, *there are constants* $C_{\alpha,\beta,\gamma}$ *such that*

$$\left|\partial_t^s \partial_{t'}^{s'} \partial_x^\alpha \partial_\xi^\beta \sigma(x, t, t', \xi)\right| \le C_{\alpha,\beta,s,s'}\ g\left(|t - t'|\right)\ (1 + |\xi|)^{m+\delta|\alpha|-|\beta|+s+s'} \tag{1.7}$$

for $|\xi| \ge 1$, $x \in R^n$ *and* $t, t' \in (-1, 1)$. *If* $0 < \mu < 1$, *then* σ *belongs to* $\mathcal{C}^{M+\mu}\mathcal{S}^m_{1,\delta,g}$ *if in addition to (1.7), we have for* $0 \le s + s' \le M$,

$$\left|\partial_t^s \partial_{t'}^{s'} \partial_\xi^\beta \sigma(x + h, t, t', \xi) - \sum_{\ell=0}^{M-s-s'} \frac{(h \cdot \nabla_x)^\ell}{\ell!} \partial_t^s \partial_{t'}^{s'} \partial_\xi^\beta \sigma(x, t, t', \xi)\right|$$
$$\le C_{M,\beta,s,s'}\ g\left(|t - t'|\right)\ |h|^{M-s-s'+\mu}\ (1 + |\xi|^2)^{\frac{1}{2}(m+\delta(M+\mu)-|\beta|)}.$$

Definition 1.4. *We say that the symbol* $\sigma(x, t, t', \xi) \in \mathcal{O}^m_{I,g}$ *if its associated operator,*

$$\sigma f(x, t, t') = \int_{R^n} e^{ix\cdot\xi} \sigma(x, t, t', \xi) \widetilde{f}(\xi, t') d\xi$$

satisfies

$$\|\sigma f(\cdot, t, t')\|_{H^s_p(I^n)} \le C\ g\left(|t - t'|\right)\ \|f(\cdot, t, t')\|_{H^{s+m}_p(I^n)}$$

for all $f(\cdot, t, t') \in H^{s+m}_p(I^n)$, $s \in I$ *and* $t, t' \in (-1, 1)$.

Lemma 1.1. *If* $\sigma \in \mathcal{C}^{M+\mu}\mathcal{S}^m_{1,\delta,g}$, *then* $\sigma \in \mathcal{O}^m_{I,g}$ *where* $I = (-(1 - \delta)(M + \mu), M + \mu)$.

Proof. Apply Theorem 1.1.

Although we assume σ is compactly supported in the x variables, g is not necessarily continuous. Various choices of g will be used later.

Proposition 1.3. *Let* $\sigma \in \mathcal{O}^m_{I,g}$, $m \ge 0$, $0 \in I$. *If* g *is integrable on* $(0, 1)$, *then the operator* $\sigma f(x, t) = \int_{t_0}^t \int_{R^n} e^{ix\cdot\xi} \sigma\left(x, t, t', \xi\right) \widetilde{f}(\xi, t') d\xi dt'$ *is bounded from* $H^m_p(\mathbf{R}^n \times (-1, 1))$ *to* $L_p(R^n \times (-1, 1))$. *In particular, this holds for* $\sigma \in \mathcal{C}^\mu \mathcal{S}^m_{1,\delta,g}$, $\mu > 0$.

Proof. We have

$$\begin{aligned}\int_{-1}^1 \int_{R^n} |\sigma f(x, t)|^p\, dxdt &= \int_{-1}^1 \int_{R^n} \left|\int_{t_0}^t \sigma\left(x, t, t', D_x\right) f\left(\cdot, t'\right)(x) dt'\right|^p dxdt \\ &\le \int_{-1}^1 \int_{R^n} \left\{\int_{t_0}^t \left|g\left(|t - t'|\right)\right|^{-\frac{p}{p'}} \left|\sigma\left(x, t, t', D_x\right) f\left(\cdot, t'\right)(x)\right|^p dt'\right\} dx \\ &\quad \times \left(\int_{-1}^1 \left|g\left(|t - t'|\right)\right| dt'\right)^{\frac{p}{p'}} dt\end{aligned}$$

by Hölder's inequality. Since the operators $\sigma(x,t,t',\xi)$ are in $\mathcal{O}^m_{I,g}$ by Lemma 1.1 the above is bounded by

$$\begin{aligned}
&\le C\int_{-1}^{1}\int_{R^n}\int_{t_0}^{t}|g(|t-t'|)|^{-\frac{p}{p'}+p}\,\|f(x,t')\|^p_{H^m_p(R^n_x)}dt'dt\\
&\le C\int_{-1}^{1}\int_{-1}^{1}|g(|t-t'|)|\,\|f(x,t')\|^p_{H^m_p(R^n_x)}dt'dt\\
&\le C\|g\|_1\int_{-1}^{1}\|f(x,t')\|^p_{H^m_p(R^n_x)}dt'\le C\|u\|^p_{H^m_p(R^n_x\times(-1,1))},
\end{aligned}$$

where $\frac{1}{p'}+\frac{1}{p}=1$ and $\|\cdot\|_{H^m_p(R^n_x)}$ denotes the norm in $H^m_p(R^n)$.

Corollary 1.1. *If φ is a zero order ψdo in R^{n+1} with support in $\left\{\xi_{n+1}<\lambda\sum\limits_{i=1}^{n}|\xi_i|^2\right\}$ for some $\lambda>0$, and if σ is a symbol in $\mathcal{C}^{M+\mu}\mathcal{S}^m_{1,\delta,g}$ with g integrable on $(0,1)$ and $\mu>0$, then $\sigma\circ\varphi$ is bounded from $H^{s+m}_p(R^n\times(-1,1))$ to $H^s_p(R^n\times(-1,1))$ for all $s\in[-(1-\delta)M,M]$.*

Proof. This follows from the identities

$$\sigma\circ\varphi=\sum_j\triangle^{-1}\partial_{x_j}\circ\sigma\circ\partial_{x_j}\circ\varphi+\sum_j\triangle^{-1}\partial_{x_j}\circ\sigma_j\circ\varphi,$$

$$\sigma\circ\varphi=\sum_j\partial_{x_j}\circ\sigma\circ\partial_{x_j}\triangle^{-1}\circ\varphi+\sum_j\sigma_j\circ\partial_{x_j}\triangle^{-1}\circ\varphi,$$

where $\sigma_j\in\mathcal{C}^{M+\mu-1}\mathcal{S}^{m+\delta}_{1,\delta,g}$. By Proposition 1.3 and the first identity, we now see that $\sigma\circ\varphi$ is bounded from $H^{m+1}_p(R^n\times(-1,1))$ to $H^1_p(R^n\times(-1,1))$ and so by interpolation from $H^{m+\delta}_p$ to H^δ_p. From the second indentity we see that $\sigma\circ\varphi$ is bounded from $H^{m-1+\delta}_p(R^n\times(-1,1))$ to $H^{-1+\delta}_p(R^n\times(-1,1))$. A simple induction procedure and an interpolation argument imply that $\sigma\circ\varphi$ is bounded from $H^{s+m}_p(R^n\times(-1,1))$ to$H^s_p(R^n\times(-1,1))$ for all $s\in[-(1-\delta)M,M]$.

For all of the special operators we consider, we will obtain the above conclusion for the extended range $s<M+\mu$. See Proposition 1.5 and Lemma 1.7 below.

1.3. Symbol Estimates for *K*, *T* and *R*

Now we turn to estimating some of the special operators appearing in the solution to the oblique derivative problem. We consider the following operators that arise in the section 2 of [6]:

$$\begin{aligned}
Ku(x,t)&=(2\pi)^{-\frac{n}{2}}\int_{t_0}^{t}\int_{|\xi|\ge1}e^{ix\cdot\xi}K(x,t,t',\xi)\,\widetilde{u^*}\,(\xi,t')\,d\xi dt',\\
Tu(x,t)&=(2\pi)^{-\frac{n}{2}}\int_{t_0}^{t}\int_{|\xi|\ge1}e^{ix\cdot\xi}K(x,t,t',\xi)\,a(x,t')\\
&\qquad\cdot Q(x,t',\xi,a(x,t'))\,\widetilde{u^*}\,(\xi,t')\,d\xi dt',\\
\mathcal{K}u(x,t)&=-(2\pi)^{-\frac{n}{2}}\triangle_x^{-1}\int_{|\xi|\ge1}e^{ix\cdot\xi}\tilde\rho(x,t_0)e^{-\widetilde{A}(x,t,t_0,\xi)}\,|\xi|^2\,\widetilde{u^*}\,(\xi,t')\,d\xi,
\end{aligned}\tag{1.9}$$

along with the commutator

$$R=K\circ\tilde aQ-K\tilde aQ,\tag{1.10}$$

where $Q(x,t,\xi,s)$ is smooth and satisfies

$$C^{-1}|\xi|\le ReQ\le C|\xi|,$$

$$\left|\frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\alpha}{\partial \xi^\alpha}\frac{\partial^\ell}{\partial t^\ell}\frac{\partial^m}{\partial s^m}Q\right| \le C_{\beta,\alpha,\ell,m}\left(1+|\xi|\right)^{1+(|\beta|+\ell)\delta-|\alpha|}, \tag{1.11}$$

and

$$\begin{aligned}\widetilde{A}(x,t,t',\xi) &= \int_{t'}^{t} a(x,\theta)Q\left(x,\theta,\xi,a\left(x,\theta\right)\right)d\theta,\\ A(x,t,t') &= \int_{t'}^{t} a(x,\theta)d\theta,\\ K(x,t,t',\xi) &= \tilde{\rho}(x,t)e^{-\widetilde{A}(x,t,t',\xi)}.\end{aligned} \tag{1.12}$$

We use here $\widetilde{a}$ as an abbreviation for $a\left(x,t'\right)$ and a for $a\left(x,t\right)$. We have

$$a \in \mathcal{C}^{\lambda+2}, \quad a\left(x,\pm1\right) \ne 0, \quad \text{and either } a \ge 0 \text{ or } ta \ge 0. \tag{1.13}$$

Also,

$$\begin{aligned} u^* = \rho^* u, \quad \widetilde{u^*}\left(\xi,t'\right) &= \int_{R^n} e^{-ix\cdot\xi}u^*(x,t')dx,\\ u_0^*\left(x\right) = \rho^*\left(x,0\right)u_0\left(x\right), \quad \widetilde{u_0^*}\left(\xi\right) &= \int_{R^n} e^{-ix\cdot\xi}u_0^*(x)dx,\end{aligned}$$

and $t_0 = -1$ if a satisfies case (I), while $t_0 = 0$ if a satisfies case (II) of [6]. The function $\rho^* \in C_c^\infty\left(R^n \times R\right)$ is identically 1 in a neighbourhood of the set where a vanishes. The choice of t_0 is crucial. With the above choice, $a(x,\theta)$ keeps the same sign in (t',t) for $t' \in (t_0,t)$, and $Re\left(\widetilde{A}(x,t,t',\xi)\right) \ge 0$ and $A(x,t,t') \ge 0$ for $t' \in (t_0,t)$.

One of the features in our estimates is evident when we consider the operator T in (1.9)—namely that the presence of a with e^{-A} in the symbol results in greater gain than is first apparent. A simple but effective means for realizing this is the following lemma.

Lemma 1.2. *For a satisfying* (1.13), *we have*

$$\begin{aligned}\left|\nabla_x a\left(x,t\right)\right|^2 &\le C\left|a\left(x,t\right)\right|,\\ \left|\nabla_x A\left(x,t,t'\right)\right|^2 &\le C\left|t-t'\right|A\left(x,t,t'\right), \quad t_0 < t' < t,\\ \left|a\left(x,t\right)\right|^2 &\le CA\left(x,t,t_0\right).\end{aligned} \tag{1.14}$$

Proof. The first two inequalities are proved in ([5], Lemmas 4.3 and 4.4). The last inequality follows from the facts that $A\left(x,t,t_0\right) \ge 0$ for all (x,t) and $a\left(x,t\right) = \frac{\partial}{\partial t}A\left(x,t,t_0\right)$.

The following lemma is crucial for obtaining the gain of $2 - \frac{1}{p(\lambda+3)}$ derivatives from f in Theorem 4 of §1 (see Lemma 5.13 of [5] when k is an integer and see also [7], p. 203 for related results).

Lemma 1.3. *For each real number $k \ge 1$, there is a constant $C_k > 0$, such that*

$$\max_{t\in I}\left|f'(t)\right| \le C_k\left\{\frac{1}{|I|}\max_{t_1,t_2\in I}\left|f(t_1)-f(t_2)\right| + \max_{t_1,t_2\in I}\left|f(t_1)-f(t_2)\right|^{1-\frac{1}{k}}\|f\|_{\mathcal{C}^k(I)}^{\frac{1}{k}}\right\} \tag{1.15}$$

for all intervals I and functions $f \in \mathcal{C}^k(I)$.

Proof. Suppose first that k is a positive integer. Fix $z \in I$, let $P(t) = f'(z)(t-z) + \cdots + \frac{f^{(k-1)}(t)}{(k-1)!}(t-z)^{k-1}$, and let $r(t) = f(t) - f(z) - P(t)$. Let J be an interval such that $z \in J \subset I$, and $|J| = \min\{|I|, \delta\}$, where

$$\delta = \left(\frac{\max_{t_1,t_2\in I}\left|f(t_1)-f(t_2)\right|}{\|f\|_{\mathcal{C}^k(I)}}\right)^{\frac{1}{k}}. \tag{1.16}$$

Since P is a polynomial of degree $k-1$, there is $C_k > 0$, independent of J and P such that $\max_{t\in J}|P'(t)| \le C_k \frac{1}{|J|}\max_{t\in J}|P(t)|$. Then using Taylor's formula to estimate $r(t)$, we have

$$\begin{aligned}|f'(z)| = |P'(z)| &\le \max_{t\in J}|P'(t)| \le C_k \frac{1}{|J|}\max_{t\in J}|P(t)|\\ &\le C_k \frac{1}{|J|}\left[\max_{t\in J}|f(t)-f(z)| + \max_{t\in J}|r(t)|\right]\\ &\le C_k \frac{1}{|J|}\left[\max_{t_1,t_2\in J}|f(t_1)-f(t_2)| + \left(\|f\|_{\mathcal{C}^k(I)}\right)|J|^k\right]\\ &\le C_k \frac{1}{|J|}\max_{t_1,t_2\in I}|f(t_1)-f(t_2)| + \left(\|f\|_{\mathcal{C}^k(I)}\right)\delta^{k-1},\end{aligned}$$

which yields (1.15) using $|J| = \min\{|I|,\delta\}$.

If k is not an integer, then the above argument goes through using $P(t) = f'(z)(t-z)+\cdots+\frac{f^{([k])}(t)}{[k]!}(t-z)^{[k]}$ and $r(t) = f(t)-f(z)-P(t)$, together with the inequality $\max_{t\in J}|r(t)| \le C\left(\|f\|_{\mathcal{C}^k(I)}\right)|J|^k$.

Remark. The inequality $\max\limits_{t\in J}|r(t)| \le C\left(\|f\|_{\mathcal{C}^k(I)}\right)|J|^k$ used above fails with Λ^k in place of $\mathcal{C}^k$ when k is an integer. For example, $f(t) = \sum\limits_{k=o}^{\infty}\frac{e^{i2^k t}}{2^{2k}} \in \Lambda^2$, but

$$|Re\,(f(t)-f(0)-f'(0)t)| \ge c\Big(\log{}_2\frac{1}{t}\Big)t^2.$$

Lemma 1.4. *Suppose $a\in C^{k-1}$ and $t' \in (t_0,t)$. Then*

$$\max_{\theta\in(t',t)}|a(x,\theta)| \le C_k\Big\{\frac{A(x,t,t')}{|t-t'|} + A(x,t,t')^{1-\frac{1}{k}}\,\|a(x,\cdot)\|^{\frac{1}{k}}_{\mathcal{C}^{k-1}((t',t))}\Big\}.$$

In particular, there is C'_k such that

$$|a(x,\theta)| \le C'_k\Big(\frac{A(x,t,t')}{|t-t'|}\Big)^{1-\frac{1}{k}} \ \textit{for all}\ \theta \in (t',t). \tag{1.17}$$

Proof. Since $a(x,\theta)$ keeps the same sign in (t',t) if $t' \in (t_0,t)$, the lemma follows from Lemma 1.3 directly.

The next lemma is the basis for Proposition 1.4 which in turn yields boundedness properties for K and T and their associated operators. We emphasize that in this section, we obtain boundedness results valid for H^s_p for all $1<p<\infty$. Sharp boundedness results for K and T are obtained in section 2.

Lemma 1.5. *Let Q and $\widetilde{A}$ be as in* (1.11) *and* (1.12). *For $t' \in (t_0,t)$, $|\xi|\ge 1$, $s_1+s_2+|\alpha|\le\lambda+2$, and $\mu\in Z_+$, we have with $\delta=\max\left\{\frac{1}{2},\frac{1}{\lambda+1}\right\}$,*

$$\begin{aligned}&|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\widetilde{A}(x,t,t',\xi)|\\ &\le C_{s_1,s_2,\alpha,\beta}(1+|\xi|)^{\delta|\alpha|-|\beta|+s_1+s_2}\left(\widetilde{A}^{\delta|\alpha|+1}(x,t,t',\xi)+1\right),\end{aligned} \tag{1.18}$$

$$\begin{aligned}&|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}(a(x,t')Q^{\mu}(x,t',\xi,a\,(x,t')))|\\ &\le C_{s_1,s_2,\alpha,\beta,k}\,|t-t'|^{-1+\frac{1}{\lambda+3}}\,(1+|\xi|)^{\mu-1+\delta|\alpha|-|\beta|+s_1+s_2+\frac{1}{\lambda+3}}\left(\widetilde{A}^{1-\frac{1}{\lambda+3}}(x,t,t',\xi)+1\right),\end{aligned} \tag{1.19}$$

$$
\begin{aligned}
&|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\widetilde{A}_{x_j}(x,t,t',\xi))| \\
&\le C_{s_1,s_2,\alpha,\beta,q}(1+|\xi|)^{\delta(|\alpha|+1)+s_1+s_2-|\beta|}\left(1+\widetilde{A}(x,t,t',\xi)^{\delta}\right)\quad \text{for } s_1+s_2+|\alpha|\le\lambda+1,
\end{aligned}
\tag{1.20}
$$

$$
\begin{aligned}
&\left|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\left(a(x,t')Q(x,t',\xi)\widetilde{A}_{x_j}(x,t,t',\xi)\right)\right| \\
&\le C_{s_1,s_2,\alpha,\beta}\left|t-t'\right|^{-\frac{1}{2}+\frac{1}{\lambda+3}}(1+|\xi|)^{\delta(|\alpha|+1)+s_1+s_2-|\beta|+\frac{1}{\lambda+3}}\left(1+\widetilde{A}(x,t,t',\xi)^{1-\frac{1}{\lambda+3}+\frac{1}{2}}\right) \\
&\text{for } s_1+s_2+|\alpha|\le\lambda+1.
\end{aligned}
\tag{1.21}
$$

Proof. If $s_1+s_2>0$ or $|\alpha|\ge 2$, (1.18) is trivial since Q satisfies (1.11). Thus we need only check the case $s_1=s_2=0$, $|\alpha|=1$ of (1.18) and this follows from Lemma 1.2 if we note that $\widetilde{A}^{1/2}\le\widetilde{A}^{3/2}+1$. The proofs for (1.19), (1.20) and (1.21) are similar and follow from Lemma 1.2 and Lemma 1.4.

Proposition 1.4. Let $b(x,t,t',\xi)$ be $\mathcal{C}^k$ in the variables x,t,t' for some $k>0$, be $\mathcal{C}^\infty$ in the variable ξ and satisfy for some $N>0$,

$$
\begin{aligned}
&\left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial\xi^\beta}\frac{\partial^{s_1}}{\partial t^{s_1}}\frac{\partial^{s_2}}{\partial t'^{s_2}}b\right| \\
&\le C_{\alpha,\beta,s_1,s_2}\left\{(1+|\xi|)^{\delta(|\alpha|+s_1+s_2)}+\widetilde{A}(x,t,t',\xi)^{|\alpha|+s_1+s_2+N}\right\}(1+|\xi|)^{-|\beta|}, \\
&|\alpha|+s_1+s_2\le k.
\end{aligned}
\tag{1.22}
$$

For $t'\in(t_0,t)$, $|\xi|\ge 1$, $s_1+s_2+|\alpha|\le\min\{\lambda+2,k\}$, and $\mu\in Z_+$, with $\delta=\max\left\{\frac{1}{2},\frac{1}{\lambda+1}\right\}$, we have

$$
|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\left(b(x,t,t',\xi)K(x,t,t',\xi)\right)|\le C_{s_1,s_2\alpha,\beta}(1+|\xi|)^{s_1+s_2+\delta|\alpha|-|\beta|},\tag{1.23}
$$

$$
\begin{aligned}
&\left|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\left(a(x,t')b(x,t,t',\xi)Q^{\mu}(x,t',\xi)K(x,t,t',\xi)\right)\right| \\
&\le C_{s_1,s_2,\alpha,\beta,k}\left|t-t'\right|^{-1+\frac{1}{\lambda+3}}(1+|\xi|)^{\mu-1+s_1+s_2+\delta|\alpha|-|\beta|+\frac{1}{\lambda+3}},
\end{aligned}
\tag{1.24}
$$

and for $|\alpha|+s_1+s_2\le\min\{k,(\lambda+1)\}$, we have

$$
\begin{aligned}
&\left|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\left(b(x,t,t',\xi)\frac{\partial}{\partial x_j}a\left(x,t'\right)K(x,t,t',\xi)\right)\right| \\
&\le C_{s_1,s_2\alpha,\beta}\left|t-t'\right|^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}}(1+|\xi|)^{s_1+s_2+\delta|\alpha|-|\beta|-\frac{1}{2}\frac{\lambda+2}{\lambda+3}},
\end{aligned}
\tag{1.25}
$$

$$
\begin{aligned}
&\left|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\left(b(x,t,t',\xi)\frac{\partial}{\partial t}a\left(x,t'\right)K(x,t,t',\xi)\right)\right| \\
&\le C_{s_1,s_2\alpha,\beta}\left|t-t'\right|^{-1+\frac{1}{2(\lambda+3)}}(1+|\xi|)^{s_1+s_2+\delta|\alpha|-|\beta|-\frac{1}{2}\frac{\lambda+2}{\lambda+3}}
\end{aligned}
\tag{1.26}
$$

$$
\begin{aligned}
&\left|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\left(b(x,t,t',\xi)\widetilde{A}_{x_j}(x,t,t',\xi)K(x,t,t',\xi)\right)\right| \\
&\le C_{s_1,s_2,\alpha,\beta}(1+|\xi|)^{s_1+s_2+\delta(|\alpha|+1)-|\beta|},
\end{aligned}
\tag{1.27}
$$

$$
\begin{aligned}
&\left|\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}\left(a(x,t')Q(x,t',\xi)b(x,t,t',\xi)\widetilde{A}_{x_j}(x,t,t',\xi)K(x,t,t',\xi)\right)\right| \\
&\le C_{s_1,s_2,\alpha,\beta}\left|t-t'\right|^{-\frac{1}{2}+\frac{1}{\lambda+3}}(1+|\xi|)^{s_1+s_2+\delta(|\alpha|+1)-|\beta|+\frac{1}{\lambda+3}}.
\end{aligned}
\tag{1.28}
$$

Proof. Inequalities (1.23), (1.24), (1.25) and (1.28) are consequences of Lemma 1.5. To see this for (1.23), note that the Liebniz formula gives

$$\partial_t^{s_1}\partial_{t'}^{s_2}\partial_x^{\alpha}\partial_\xi^{\beta}e^{-\widetilde{A}(x,t,t',\xi)}=\sum C_{\alpha',\beta',s_1',s_2'}\Big(\prod_{j=1}^{n}\partial_t^{s_{1,j}}\partial_{t'}^{s_{2,j}}\partial_x^{\alpha_j'}\partial_\xi^{\beta_j'}\widetilde{A}(x,t,t',\xi)\Big)e^{-\widetilde{A}(x,t,t',\xi)},$$

where the summation is taken over all

$$\alpha'=(\alpha_1',\cdots,\alpha_n'),\quad \beta'=(\beta_1',\cdots,\beta_n'),\quad s_1'=(s_{1,1},\cdots,s_{1,n}),\quad s_2'=(s_{2,1},\cdots,s_{2,n})$$

with $|\alpha'|=|\alpha|$, $|\beta'|=|\beta|$, $|s_1'|=s_1$, $|s_2'|=s_2$. By (1.18)

$$\begin{aligned}&\Big|\prod_{j=1}^{n}\partial_t^{s_{1,j}}\partial_{t'}^{s_{2,j}}\partial_x^{\alpha_j}\partial_\xi^{\beta_j}\widetilde{A}\Big|e^{-\widetilde{A}}\\
&\le C\prod_{j=1}^{n}\left(\widetilde{A}^{\delta|\alpha_j|+1}+1\right)(1+|\xi|)^{\delta|\alpha_j|-|\beta_j|+s_{1,j}+s_{2,j}}e^{-\widetilde{A}}\\
&\le C(1+|\xi|)^{\delta|\alpha|-|\beta|+s_1+s_2}\left(A^{\delta|\alpha|+|\beta|+s_1+s_2}+1\right)e^{-\widetilde{A}}\le C(1+|\xi|)^{\delta|\alpha|-|\beta|+s_1+s_2}.\end{aligned}$$

as required. The proofs for (1.24), (1.27) and (1.28) are similar.

As for (1.25), we use (1.14) and (1.17) for $k\le\lambda+3$ to obtain

$$\Big|\frac{\partial}{\partial x_j}a\,(x,t')\Big|\le C\,|a\,(x,t')|^{\frac12}\le C\Big(\frac{A\,(x,t,t')}{|t-t'|}\Big)^{\frac12-\frac{1}{2(\lambda+3)}}=C\Big(\frac{A\,(x,t,t')}{|t-t'|}\Big)^{\frac12\frac{\lambda+2}{\lambda+3}},$$

while for (1.18), we use (1.15) with $f=a$ to obtain

$$\begin{aligned}\Big|\frac{\partial}{\partial t}a\,(x,t')\Big|&\le C\Big(\frac{\max\limits_{t'\le\theta'\le\theta\le t}|a\,(x,\theta)-a\,(x,\theta')|}{|t-t'|}\Big)^{1-\frac{1}{\lambda+2}}\le C\Big(\frac{\max\limits_{t'\le\theta\le t}|a\,(x,\theta)|}{|t-t'|}\Big)^{\frac12}\\
&\le C\Big(\frac{A\,(x,t,t')}{|t-t'|}\Big)^{\frac12\left(1-\frac{1}{\lambda+3}\right)}|t-t'|^{-\frac12}\quad\text{by (1.17)},\\
&\le C\,|t-t'|^{-1+\frac{1}{2(\lambda+3)}}A\,(x,t,t')^{\frac12\frac{\lambda+2}{\lambda+3}}\end{aligned}\tag{1.29}$$

and the result now follows easily.

We must also deal with commutators of the above operators with D_x and this introduces factors of $\widetilde{A}_x$ into the symbol. Thus we set for $m\ge0$ an integer and $\delta=\max\left\{\frac12,\frac{1}{\lambda+1}\right\}$,

$$\begin{aligned}G_mu\,(x,t)&=\int_{t_0}^{t}\int_{|\xi|\ge1}e^{ix\cdot\xi}\,|\xi|^{-m\delta}\,b(x,t,t',\xi)\left(\widetilde{A}_x(x,t,t',\xi)\right)^m K(x,t,t',\xi)\widetilde{u^*}\,(\xi,t')d\xi dt',\\
H_mu\,(x,t)&=\int_{t_0}^{t}\int_{|\xi|\ge1}e^{ix\cdot\xi}a(x,t')Q(x,t',\xi)\,|\xi|^{-m\delta}\,b(x,t,t',\xi)\left(\widetilde{A}_x(x,t,t',\xi)\right)^m\\
&\quad\times K(x,t,t',\xi)\widetilde{u^*}\,(\xi,t')d\xi dt',\\
J_mu\,(x,t)&=\int_{t_0}^{t}\int_{|\xi|\ge1}e^{ix\cdot\xi}\,|\xi|^{-m\delta}\,b(x,t,t',\xi)\left(\widetilde{A}_x(x,t,t',\xi)\right)^m Da\,(x,t')\\
&\quad\times K(x,t,t',\xi)\widetilde{u^*}\,(\xi,t')d\xi dt',\end{aligned}\tag{1.30}$$

and in Case II, we define corresponding "Poisson" operators for G_m and H_m,

$$\mathcal{G}_mu\,(x,t)=\int_{|\xi|\ge1}e^{ix\cdot\xi}\,|\xi|^{-m\delta}\,b(x,t,\xi)\left(\widetilde{A}_x(x,t,0,\xi)\right)^m\mathcal{K}(x,t,0,\xi)\widetilde{u^*}\,(\xi,t)d\xi,$$

$$\mathcal{H}_m u\,(x,t) = \int_{|\xi|\geq 1} e^{ix\cdot\xi} a(x,t)Q(x,t,\xi)\,|\xi|^{-m\delta}\, b(x,t,\xi)\left(\widetilde{A}_x(x,t,0,\xi)\right)^m \mathcal{K}(x,t,0,\xi)\widetilde{u^*}\,(\xi,t)d\xi. \tag{1.31}$$

Here we assume that $b(x,t,t',\xi)$ is $\mathcal{C}^k$ in the variables x,t,t' for some $k>0$, and is $\mathcal{C}^\infty$ and homogeneous of degree zero in the variable ξ. Moreover, in order to obtain boundedness up to the cap $\lambda+2$, we assume that $b(x,t,t',\xi)$ satisfies

For any first order constant coefficient differential operator D, there is an integer M such that

$$Db(x,t,t',\xi)=\sum_{j=1}^{M} b'_j\,(x,t,t')\,b_j(x,t,t',\xi), \tag{1.32}$$

where $b'_j\in\mathcal{C}^{k-1}$ and where b_j satisfies the same conditions as b.

Remark. If $\sigma\in\mathcal{C}^k S^0_{1,\delta,g}$ for $k>0$, Corollary 1.1 only yields $\sigma\in O_{(-(1-\delta)[k],[k])}$. As we will see below, property (1.32) permits us to easily obtain the improved range $O_{(-(1-\delta)[k],k)}$.

Lemma 1.6. *Let $b(x,t,t',\xi)$ be $\mathcal{C}^k$ in the variables x,t,t' for some $k\geq 1$, be C^∞ in the variable ξ and satisfy both* (1.22) *and* (1.32). *Let Z_m denote either of the operators G_m, H_m or J_m in* (1.30). *Then there is an integer M such that*

$$D\circ Z_m=\widetilde{Z}\circ|D_x|+\int_{t_0}^{t}\sum_{j=1}^{M}\int_{t_0}^{\theta_1} b_j\,(x,t,\theta)\,Z_j\,(x,t,t',D_x)\,|D_x|^{1-\delta}\,dt'd\theta+Z_0\,(x,t,D_x)\,, \tag{1.33}$$

where $\widetilde{Z}$, Z^i_j have the same form as Z_m and Z_{m+1} respectively, with the same smoothness in x,t,t', where $b_j=0$ if $m=0$ or $Z_m=G_m$ or H_m, and $b^i_j\in\mathcal{C}^{\min\{\lambda,k-1\}}$ otherwise, and where

$$Z_0u=\sum_{i=1}^{k'}\int_{|\xi|\geq 1} e^{ix\cdot\xi}\,|\xi|^{k'-i}\,b^i_0(x,t,\xi)\widetilde{u^*}\,(\xi,t)d\xi, \tag{1.34}$$

where $b^i_0(x,t,\xi)$ is $\mathcal{C}^{\min\{\lambda+2,k\}}$ in the variables x,t, and is C^∞ in the variable ξ and satisfies (1.22). *The corresponding conclusions hold if Z is one of the operators $\mathcal{G}_m$ or $\mathcal{H}_m$ with the corresponding formula as follows:*

$$D\circ Z=\widetilde{Z}\circ|D_x|+\sum_{j=1}^{M}\int_0^t b^i_j\,(x,t,\theta)\,Z_j\,(x,t,D_x)\,|D_x|^{1-\delta}\,d\theta. \tag{1.35}$$

Proof. We need only note that $Q\,|\xi|^{-1}$ satisfies property (1.32) and then compute. The operator Z_0 in (1.33) arises from the action of ∂_t on Z in the case when at least one of the D_i involves ∂_t, while b^i_j arises from the action of ∂_{x_i} on $b\left(\widetilde{A}_x\right)^m$.

The next definition extends the mapping properties of Definition 1.4 to the parameter variable.

Definition 1.5. *An operator W belongs to $\mathcal{O}^m_I$, where I is an interval, if W admits a bounded extension from $H^{s+m}_{p,comp}$ to $H^s_{p,loc}$ and from Λ^s_{comp} to Λ^s_{loc} for all s in the interval I and all $1<p<\infty$. The operator W belongs to $\overline{\mathcal{O}}^m_I$ if in addition, it is bounded from Λ^t_{comp} to Λ^t_{loc} where t is the right endpoint of the interval I.*

Proposition 1.5. *Let φ be a zero order ψdo with constant coefficients and support in the cone $\{|\tau| < C|\xi|\}$. Let $b(x,t,t',\xi)$ be $\mathcal{C}^k$ in the variables x,t,t' for some $k \geq 1$, and be C^∞ in the variable ξ and satisfy both* (1.22) *and* (1.32). *Then*

$$G_0 \circ \varphi, \mathcal{G}_0 \circ \varphi \in \mathcal{O}_I^0,\ H_0 \circ \varphi \in \mathcal{O}_I^{\frac{1}{\lambda+3}},\ \mathcal{H}_0 \in \mathcal{O}_I^{\frac{1}{2}},\ J_0 \circ \varphi \in \mathcal{O}_I^{-\frac{1}{4}}$$

with $I = (-1, \min\{\lambda+2, k\})$ and

$$G_m \circ \varphi, \mathcal{G}_m \in \mathcal{O}_{I'}^0,\ H_m \circ \varphi \in \mathcal{O}_{I'}^{\frac{1}{\lambda+3}},\ \mathcal{H}_m \in \mathcal{O}_{I'}^{\frac{1}{2}},\ J_0 \circ \varphi \in \mathcal{O}_{I'}^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}}$$

with $I' = (-1, \min\{\lambda+1, k\})$ for $m \geq 1$. In particular,

$$K \circ \varphi, \mathcal{K} \in \mathcal{O}_{(-1,\lambda+2)}^0,\ T \circ \varphi \in \mathcal{O}_{(-1,\lambda+2)}^{\frac{1}{\lambda+3}},\ \partial_t \circ \mathcal{K}, D_x \circ \mathcal{K} - \mathcal{K} \circ D_x \in \mathcal{O}_{(-1,\lambda+1)}^{\frac{1}{2}},$$

$$K \circ a \circ \varphi \in \mathcal{O}_{(-1,\lambda+2)}^{\frac{1}{\lambda+3}-1},\ K \circ Da \circ \varphi \in \mathcal{O}_{(-1,\lambda+1)}^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}}.$$

In fact, we have much stronger statements for T and $K \circ a$, namely

$$T \circ \varphi \in \mathcal{O}_{(-1,\lambda+2)}^{\frac{1}{p(\lambda+3)}},\ K \circ a \circ \varphi \in \mathcal{O}_{(-1,\lambda+2)}^{\frac{1}{p(\lambda+3)}-1}.$$

Proof. Since $\min\{\lambda+2, k\} \geq 1$, we have by Lemma 1.6 formulas (1.33) and (1.35). All of the assertions, apart from the last three concerning $K \circ Da$, T and $K \circ a$, follow from Proposition 1.3, Corollary 1.1, Proposition 1.4 (with $\mu = 1$ for H_m), and the fact that $\mathcal{C}^\alpha \subset H_p^\beta$ for $0 \leq \beta < \alpha$. To handle $K \circ Da \circ \varphi$ we note that

$$\begin{aligned} K \circ Da \circ \varphi &= K(Da) \circ \varphi + (K \circ Da - K(Da)) \circ \varphi \\ &= K(Da) \circ \varphi + \left(K \circ Da^\sharp - K\left(Da^\sharp\right)\right) \circ \varphi + \mathcal{O}_{(0,\lambda+1)}^{-1} \\ &= K(Da) \circ \varphi + \mathcal{O}_{(0,\lambda+1)}^{-1}, \end{aligned}$$

by applying (1.3) to $Da^\sharp$. Since $K(Da) \circ \varphi \in \mathcal{O}_{(-1,\lambda+1)}^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}}$, so is $K \circ Da \circ \varphi$.

Finally, we use Theorem 2.2 from the next section to obtain the improvement $T \circ \varphi \in \mathcal{O}_{(-1,\lambda+2)}^{\frac{1}{p(\lambda+3)}}$. Alternatively, we could note that $T : \Lambda^s \to \Lambda^s$ by Corollary 1.2 below, and that $T : \mathcal{B}_{1+\epsilon}^{s+\frac{1}{\lambda+3},1+\epsilon} \to \mathcal{B}_{1+\epsilon}^{s,1+\epsilon}$ for all $\epsilon > 0$ by what we just proved. Interpolation then yields $T \circ \varphi \in \mathcal{O}_{(-1,\lambda+2)}^{\frac{1}{p(\lambda+3)}+\epsilon'}$. Similar arguments apply to show that $K \circ a \circ \varphi \in \mathcal{O}_{(-1,\lambda+2)}^{\frac{1}{p(\lambda+3)}-1}$.

We now dispose of the operator R in (1.10).

Lemma 1.7. *Let $\mu_p = \min\left\{1 - \delta - \frac{1}{p(\lambda+3)}, \frac{\lambda+2}{2(\lambda+3)}\right\}$ where $\delta = \max\left\{\frac{1}{2}, \frac{1}{\lambda+1}\right\}$. Let φ be supported in the nonelliptic cone as in Proposition* 1.5. *Then $R \circ \varphi \in \mathcal{O}_{[0,\lambda+2)}^{-\mu_p}$.*

Proof. We begin by decomposing $R \circ \varphi$ as

$$\begin{aligned} R \circ \varphi &= \left(K \circ \widetilde{a}\widetilde{Q} - K\widetilde{a}\widetilde{Q}\right) \circ \varphi \\ &= (K \circ \widetilde{a} - K\widetilde{a}) \circ \widetilde{Q} \circ \varphi + \left(K\widetilde{a} \circ \widetilde{Q} - K\widetilde{a}\widetilde{Q}\right) \circ \varphi \\ &= I + II. \end{aligned}$$

Write $\widetilde{a} = \widetilde{a}^\sharp + \widetilde{a}^\flat$ where $\widetilde{a}^\flat \in \mathcal{C}^{\lambda+2} S_{1,\delta}^{-(\lambda+2)\delta}$. Now let φ_1 be homogeneous of degree zero and satisfy $\varphi_1 = 1$ on the support of φ. Then there exists $\widetilde{\varphi}$ of the same type such that

$$\begin{aligned} \widetilde{Q} \circ \varphi = \widetilde{Q} \circ \varphi_1 \circ \varphi &= \varphi_1 \circ \widetilde{Q} \circ \varphi - \left\{ \sum_{1 \leq |\ell| \leq N} \left(\nabla_\xi^\ell \varphi_1\right) \cdot \left(\nabla_x^\ell \widetilde{Q}\right) \right\} \circ \varphi + \mathcal{O}_{[0,\lambda+2)}^{-(1-\delta)N} \\ &= \widetilde{\varphi} \circ \mathcal{O}_{[0,\lambda+2)}^1 + \mathcal{O}_{[0,\lambda+2)}^{-(1-\delta)N}. \end{aligned}$$

Then we have

$$I = \left(K \circ \widetilde{a}^{\sharp} - K\widetilde{a}^{\sharp}\right) \circ \widetilde{\varphi} \circ \mathcal{O}^{1}_{[0,\lambda+2)} + \mathcal{O}^{1-(\lambda+2)\delta}_{[0,\lambda+2)}.$$

Now we have

$$\left(K \circ \widetilde{a}^{\sharp} - K\widetilde{a}^{\sharp}\right) \circ \widetilde{\varphi} = \nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\sharp} \circ \widetilde{\varphi} + \mathcal{O}^{-2}_{[0,\lambda+2)}$$

by (1.3). Also,

$$\nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\sharp} \circ \widetilde{\varphi} = \nabla_{\xi} K \cdot \nabla_x \widetilde{a} \circ \widetilde{\varphi} - \nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\flat} \circ \widetilde{\varphi} = \nabla_{\xi} K \cdot \nabla_x \widetilde{a} \circ \widetilde{\varphi} + \mathcal{O}^{-2-\frac{1}{\lambda+1}}_{[0,\lambda+1)},$$
$$\nabla_{\xi} K \cdot \nabla_x \widetilde{a} \in \mathcal{C}^{\lambda+1} S^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}-1}_{1,\delta},$$

by (1.25). Thus $\nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\sharp} \circ \widetilde{\varphi} \in \mathcal{O}^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}-1}_{[0,\lambda+1)}$, and to obtain the improvement in the cap up to $\lambda+2$, i.e. $\nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\sharp} \circ \widetilde{\varphi} \in \mathcal{O}^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}-1}_{[0,\lambda+2)}$, we compose with D :

$$D_x \circ \nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\sharp} = \nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\sharp} \circ D_x + \left(\nabla_x \nabla_{\xi} K\right) \widetilde{a}^{\sharp} + \nabla_{\xi} K \cdot \nabla_x^2 \widetilde{a}^{\sharp},$$
$$D_t \circ \nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\sharp} = KaQ\nabla_{\xi}\widetilde{A}\widetilde{a}^{\sharp} - Ka\nabla_{\xi} Q\widetilde{a}^{\sharp}.$$

Thus $\nabla_{\xi} K \cdot \nabla_x \widetilde{a}^{\sharp} \circ \widetilde{\varphi} \in \mathcal{O}^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}-1}_{[0,\lambda+2)}$ and we get $I \in \mathcal{O}^{-\frac{1}{2}\frac{\lambda+2}{\lambda+3}}_{[0,\lambda+2)}$.

For term II, we write $\widetilde{Q} = \widetilde{Q}^{\sharp} + \widetilde{Q}^{\flat}$ where $\widetilde{Q}^{\sharp} = Q\left(x, t', \xi, \widetilde{a}^{\sharp}\right)$ and $\widetilde{Q}^{\flat} = \widetilde{Q} - \widetilde{Q}^{\sharp} = \widetilde{a}^{\flat}\widetilde{Q}' \in \mathcal{C}^{\lambda+2} S^{1-(\lambda+2)\delta}_{1,\delta}$ upon applying Taylor's formula to the final variable in $\widetilde{Q}$. Then

$$\begin{aligned} II &= \left(K\widetilde{a} \circ \widetilde{Q}^{\sharp} - K\widetilde{a}\widetilde{Q}^{\sharp}\right) \circ \varphi + \left(K\widetilde{a} \circ \widetilde{Q}^{\flat} - K\widetilde{a}\widetilde{Q}^{\flat}\right) \circ \varphi \\ &= \left\{\nabla_{\xi}\left(K\widetilde{a}\right) \cdot \nabla_x \widetilde{Q}^{\sharp} + \mathcal{O}^{-1+\frac{1}{p(\lambda+3)}+1+2(\delta-1)}_{[0,\lambda+2)}\right\} \circ \varphi + \mathcal{O}^{\frac{1}{\lambda+3}-(\lambda+2)\delta}_{[0,\lambda+2)} \\ &= \nabla_{\xi}\left(K\widetilde{a}\right) \cdot \nabla_x \widetilde{Q}^{\sharp} \circ \varphi + \mathcal{O}^{\frac{1}{p(\lambda+3)}+2(\delta-1)}_{[0,\lambda+2)}. \end{aligned}$$

Finally, $\nabla_{\xi}\left(K\widetilde{a}\right) \cdot \nabla_x \widetilde{Q}^{\sharp} \circ \varphi \in \mathcal{O}^{-1+\frac{1}{p(\lambda+3)}-1+(1+\delta)}_{[0,\lambda+2)} = \mathcal{O}^{\frac{1}{p(\lambda+3)}+\delta-1}_{[0,\lambda+2)}$ and so $R \circ \varphi \in \mathcal{O}^{-\mu_p}_{[0,\lambda+2)}$ as required.

Remark. If $\overrightarrow{\ell}$ satisfies $\mathcal{A}^{\mp}_{p,\alpha}$ on Γ, etc. as in part (A) of Theorem 1.3 of [6], then $R \circ \varphi \in \mathcal{O}^{-(1-\delta-\alpha)}_{[0,\lambda+2)}$ since Theorem 2.2 in section 2 below (or more precisely its proof) applies to show that $\nabla_{\xi}\left(K\widetilde{a}\right) \cdot \nabla_x \widetilde{Q}^{\sharp} \circ \varphi \in \mathcal{O}^{-(1-\delta-\alpha)}_{[0,\lambda+2)}$.

1.4. Boundedness of Operators on Λ^s

Here we discuss the behaviour of the operators $\mathcal{K}$, K and T on the Hölder spaces Λ^s, $s > 0$. We begin by considering the the following types of operator with $\delta = \max\left\{\frac{1}{2}, \frac{1}{\lambda+1}\right\}$:

$$\begin{aligned} G_{m,\sigma} u(x,t) &= \int_{t_0}^{t} \int_{|\xi| \geq 1} e^{ix\cdot\xi} |\xi|^{-\sigma-m\delta} b(x,t,t',\xi) \left(\widetilde{A}_x(x,t,t',\xi)\right)^m \\ &\quad \times K(x,t,t',\xi)\widetilde{u^*}(\xi,t') d\xi dt', \\ H_{m,\sigma} u(x,t) &= \int_{t_0}^{t} \int_{|\xi| \geq 1} e^{ix\cdot\xi} a(x,t') Q(x,t',\xi) |\xi|^{-\sigma-m\delta} b(x,t,t',\xi) \\ &\quad \times \left(\widetilde{A}_x(x,t,t',\xi)\right)^m K(x,t,t',\xi)\widetilde{u^*}(\xi,t') d\xi dt', \\ \mathcal{G}_{m,\sigma} u(x,t) &= \int_{|\xi| \geq 1} e^{ix\cdot\xi} |\xi|^{-\sigma-m\delta} b(x,t,\xi) \left(\widetilde{A}_x(x,t,t_0,\xi)\right)^m K(x,t,t_0,\xi)\widetilde{u^*}(\xi) d\xi, \end{aligned}$$

where $\sigma \in R$, $m \in Z^+$ and $b \in \mathcal{C}^\ell S^0_{1,\delta}$. Let

$$G_{m,\sigma}(x,x',t,t'),\quad H_{m,\sigma}(x,x',t,t')\ \text{ and } \mathcal{G}_{m,\sigma}u\,(x,x',t)$$

be the distribution kernels of $G_{m,\sigma}$, $H_{m,\sigma}$ and $\mathcal{G}_{m,\sigma}$ respectively. It is then easy to show (see also [4]) the following

Lemma 1.8 *If $c\,(x,t,t',\xi)$ satisfies*

$$\left|\partial_\xi^\alpha c\,(x,t,t',\xi)\right| \le C_\alpha \left(1+\widetilde{A}\,(x,t,t',\xi)\right)^N (1+|\xi|)^{-|\alpha|+\sigma}$$

for some $N \ge 0$ and $0 \le |\alpha| \le [\sigma]+n+1$, if Z is an operator of the form

$$Zu\,(x,t,t') = \int_{|\xi|\ge 1} e^{ix\cdot\xi} c(x,t,t'\xi)K(x,t,t',\xi)\widetilde{u^*}\,(\xi)d\xi,$$

and if $Z\,(x,x',t,t')$ is the distribution kernel of Z, then for $|x-x'|+A\,(x,t,t')>0$,

$$|Z\,(x,x',t,t')| \le C\Big(\max_{\substack{|\alpha|\le[\gamma]+n+1\\ |\xi|=1}} \left|\partial_\xi^\alpha c(x,t,t',\xi)\right|\Big)\frac{1}{(|x-x'|+A(x,t,t'))^{n+\sigma}}. \tag{1.36}$$

Proof. We consider two cases: (i) $A\,(x,t,t') \ge |x-x'|$ and (ii) $A\,(x,t,t') \le |x-x'|$. In case (i), we use $K=e^{-\widetilde{A}}$ where $\frac{1}{c}A\,|\xi| \le Re\left(\widetilde{A}\right) \le cA\,|\xi|$ to obtain

$$\Big|\int_{|\xi|\ge 1} e^{i\left(x-x'\right)\cdot\xi} c\,(x,t,t'\xi)\,e^{-\widetilde{A}\left(x,t,t'\xi\right)}d\xi\Big| \le \int_{|\xi|\ge 1}|\xi|^\gamma\, e^{-A|\xi|}d\xi \le CA^{-n-\sigma}.$$

In case (ii), choose $\rho \in C_c^\infty\,(R_+)$ so that $\rho=0$ on $\left(0,\frac{1}{2}\right)$ and $\rho=1$ on $[1,\infty)$. Now

$$\begin{aligned}
Z\,(x,x',t,t') &= \int_{|\xi|\ge 1} e^{i\left(x-x'\right)\cdot\xi} c\,(x,t,t'\xi)\,\rho\,(|x-x'|\,|\xi|)\,e^{-\widetilde{A}\left(x,t,t'\xi\right)}d\xi\\
&\quad+\int_{|\xi|\ge 1} e^{i\left(x-x'\right)\cdot\xi} c\,(x,t,t'\xi)\,\{1-\rho\,(|x-x'|\,|\xi|)\}\,e^{-\widetilde{A}\left(x,t,t'\xi\right)}d\xi\\
&= I+II,
\end{aligned}$$

and we have

$$|II| \le \int_{|\xi|\ge 1}|\xi|^\sigma\,\{1-\rho\,(|x-x'|\,|\xi|)\}\,d\xi \le \int_{|x-x'|^{-1}\ge|\xi|\ge 1}|\xi|^\sigma\,d\xi \le |x-x'|^{-\sigma-n}.$$

As for I, we may assume $(x-x')=(|x-x'|\,,0,\cdots,0)$, and we then have

$$I=\int_{|\xi|\ge 1}\Big(\frac{1}{i\,|x-x'|}\Big)^k\left\{\partial_{\xi_1}^k e^{i\left(x-x'\right)\cdot\xi}\right\} c\,(x,t,t'\xi)\,\rho\,(|x-x'|\,|\xi|)\,e^{-\widetilde{A}\left(x,t,t'\xi\right)}d\xi.$$

Now perform integration by parts to get

$$\begin{aligned}
I &= \int_{|\xi|\ge 1} e^{i\left(x-x'\right)\cdot\xi}\sum_{j=1}^k\frac{c_{j,k}}{|x-x'|^j}\partial_{\xi_1}^j\rho\,(|x-x'|\,|\xi|)\,\frac{1}{|x-x'|^{k-j}}\partial_{\xi_1}^{k-j}\\
&\quad\times\left\{c\,(x,t,t'\xi)\,e^{-\widetilde{A}\left(x,t,t'\xi\right)}\right\}d\xi\\
&\quad+\int_{|\xi|\ge 1} e^{i\left(x-x'\right)\cdot\xi}\frac{c_k}{|x-x'|^k}\rho\,(|x-x'|\,|\xi|)\,\partial_{\xi_1}^k\left\{c\,(x,t,t'\xi)\,e^{-\widetilde{A}\left(x,t,t'\xi\right)}\right\}d\xi.
\end{aligned}$$

Since $\left|\partial_{\xi_1}^j\rho\,(|x-x'|\,|\xi|)\right| \le C_j\,|x-x'|^j$ for $\frac{1}{2|x-x'|} \le |\xi| \le \frac{1}{|x-x'|}$, and 0 otherwise, we conclude that

$$|I| \le C \int_{\frac{1}{2|x-x'|} \le |\xi| \le \frac{1}{|x-x'|}} \sum_{j=1}^{k} \frac{1}{|x-x'|^{k-j}} \Big(\max_{\substack{1\le|\alpha|\le k \\ |\xi|=1}} \left|\partial_\xi^\alpha c(x,t,t',\xi)\right| \Big) |\xi|^{\sigma-k+j}\, d\xi$$

$$+ C \int_{\frac{1}{2|x-x'|} \le |\xi|} \sum_{j=1}^{k} \frac{1}{|x-x'|^{k}} \Big(\max_{\substack{1\le|\alpha|\le k \\ |\xi|=1}} \left|\partial_\xi^\alpha c(x,t,t',\xi)\right| \Big) |\xi|^{\sigma-k}\, d\xi.$$

If we choose $k = [\sigma]+n+1$, then the integral in the second term on the right is convergent, and we obtain the desired estimate for term I.

Lemma 1.9. *There is a constant C independent of σ, x, x', t and t' such that for $x \ne x'$,*

$$|G_{m,\sigma}(x,x',t,t')| \le C\Big(\max_{\substack{|\alpha|\le[\sigma]+n+1 \\ |\xi|=1}} \left|\partial_\xi^\alpha b(x,t,t',\xi)\right| \Big) \frac{|A_x(x,t,t')|^m}{(|x-x'|+A(x,t,t'))^{n+\frac{m}{2}-\sigma}},$$

$$|H_{m,\sigma}(x,x',t,t')| \le C\Big(\max_{\substack{|\alpha|\le[\sigma]+n+1 \\ |\xi|=1}} \left|\partial_\xi^\alpha b(x,t,t',\xi)\right| \Big) \frac{|A_x(x,t,t')|^m\, |a(x,t')|}{(|x-x'|+A(x,t,t'))^{n+1+\frac{m}{2}-\sigma}},$$

$$|\mathcal{G}_{m,\sigma}(x,x',t)| \le C\Big(\max_{\substack{|\alpha|\le[\sigma]+n+1 \\ |\xi|=1}} \left|\partial_\xi^\alpha b(x,t,\xi)\right| \Big) \frac{|A_x(x,t,t_0)|^m}{(|x-x'|+A(x,t,t_0))^{n+\frac{m}{2}-\sigma}},$$

Proof. This is a direct consequence of Lemma 1.8 and the inequality

$$\max_{|\xi|=1} \left|\partial_\xi^\alpha \widetilde{A}_x(x,t,t',\xi)\right| \le C_\alpha \left|A_x\,(x,t,t')\right|.$$

We now have

Lemma 1.10. *If b satisfies (1.22) with $k=1$ and $0<\sigma<1$, then there is a positive constant C_σ such that for every $\epsilon \in R$, $\epsilon \ne 0$ and $|t|<1$, the following estimates hold where Z_σ is either $G_{m,\sigma}$ or $H_{m,\sigma}$:*

$$\int_0^t \int_{|x-x'|<|\epsilon|} |Z_\sigma(x,x',t,t')|\, dx'dt' \le C_\sigma\, |\epsilon|^\sigma, \tag{1.37}$$

$$\int_0^t \int_B |Z_\sigma(x+h,x',t,t') - Z_\sigma(x,x',t,t')|\, dx'dt' \le C_\sigma\, |h|^\sigma, \tag{1.38}$$

and if in addition $a(x,t)a(x,t+\epsilon) \ge 0$, then

$$\int_t^{t+\epsilon} \int_B |Z_\sigma(x,x',t,t')|\, dx'dt' \le C_\sigma\, |\epsilon|^\sigma, \tag{1.39}$$

$$\int_0^t \int_B |Z_\sigma(x,x',t+\epsilon,t') - Z_\sigma(x,x',t,t')|\, dx'dt' \le C_\sigma\, |\epsilon|^\sigma. \tag{1.40}$$

If Z_σ is $\mathcal{G}_{m,\sigma}$, we have

$$\int_{|x-x'|<|\xi|} |Z_\sigma(x,x',t)|\, dx' \le C_\sigma\, |\epsilon|^\sigma,$$

$$\int_B |Z_\sigma(x+h,x',t) - Z_\sigma(x,x',t)|\, dx' \le C_\sigma\, |h|^\sigma,$$

and

$$\int_B |Z_\sigma(x,x',t+\epsilon) - Z_\sigma(x,x',t)|\, dx' \le C_\sigma\, |\epsilon|^\sigma.$$

Proof. We prove only the estimates for $H_{0,\sigma}$, the cases for $G_{0,\sigma}$ and $\mathcal{G}_{0,\sigma}$ being easier, and the cases for $G_{m,\sigma}$, $H_{m,\sigma}$ and $\mathcal{G}_{m,\sigma}$ being similar, but using $|A_x| \le C\sqrt{A}$. We have

$$\begin{aligned}\int_0^t \int_{|x-x'|<|\epsilon|} |H_{0,\sigma}(x,x',t,t')|\, dx'dt' &\le C\int_0^t \int_{|x-x'|<|\epsilon|} \frac{|a(x,t')|}{(|x-x'|+A(x,t,t'))^{n+1-\sigma}} dx'dt' \\ &\le C\int_0^t \int_0^{|\epsilon|} \frac{|a(x,t')|}{(r+A(t,t',x))^{2-\sigma}} drdt' \\ &\le C\int_0^{|\epsilon|} \left| \int_0^t \frac{\frac{d}{dt'}A(x,t,t')}{(r+A(x,t,t'))^{2-\sigma}} dt' \right| dr\end{aligned}$$

since $a(x,t')$ keeps the same sign in $(0,t)$, and thus we have

$$\begin{aligned}&\int_0^t \int_{|x-x'|<|\epsilon|} |H_{0,\sigma}(x,x',t,t')|\, dx'dt' \\ &\le C\int_0^{|\epsilon|} \Big\{ (r+A(x,t,t))^{-1+\sigma} + (r+A(x,t,0))^{-1+\sigma} \Big\} dr \le C_\sigma |\epsilon|^\sigma ,\end{aligned}$$

which proves (1.37).

To obtain (1.38) we write

$$\begin{aligned}&\int_0^t \int_B |H_{0,\sigma}(x+h,x',t,t') - H_{0,\sigma}(x,x',t,t')|\, dx'dt' \\ &= \int_0^t \Big\{ \int_{|x-x'|\le 4|h|} + \int_{\{x'\in B:|x-x'|\ge 4|h|\}} \Big\} |H_{0,\sigma}(x+h,x',t,t') - H_{0,\sigma}(x,x',t,t')|\, dx'dt' \\ &= I + II.\end{aligned}$$

From (1.37) we have $I \le C_\sigma |h|^\sigma$. To estimate II, fix h and set

$$F(x,x',t,t',s) = H_{0,\sigma}(x+sh,x',t,t').$$

Then by Lemma 1.9 we have

$$\begin{aligned}II &= \int_0^t \int_{\{x'\in B:|x-x'|\ge 4|h|\}} \left| \int_0^1 \frac{d}{ds} F(x,x',t,t',s) ds \right| dx'dt' \\ &\le \int_0^t \int_{\{x'\in B:|x-x'|\ge 4|h|\}} \int_0^1 Ch \Big\{ \frac{|a(x+sh,t')|}{(|x-x'|+A(x+sh,t,t'))^{n+2-\sigma}} \\ &\quad + \frac{1}{(|x-x'|+A(x+sh,t,t'))^{n+1-\sigma}} \Big\} dsdx'dt' \\ &= III + IV.\end{aligned}$$

Now we have

$$\begin{aligned}III &= Ch\int_{|x-x'|\ge 4|h|} \int_0^1 \left| \int_0^t \frac{\frac{d}{dt'}A(x+sh,t,t')}{(|x-x'|+A(x+sh,t,t'))^{n+2-\sigma}} dt' \right| dsdx' \\ &\le Ch \int_{\{x'\in B:|x-x'|\ge 4|h|\}} \int_0^1 \frac{1}{|x-x'|^{n+1-\sigma}} dsdx' \le C_\sigma |h|\, |h|^{-1+\sigma} = C_\sigma |h|^\sigma,\end{aligned}$$

and similarly,

$$IV \le Ch \int_0^t \int_{\{x'\in B:|x-x'|\ge 4|h|\}} \int_0^1 \frac{1}{|x-x'|^{n+1-\sigma}} dsdx'dt' \le C_\sigma |h|^\sigma.$$

The proofs for (1.39) and (1.40) are similar. We need only notice that $a(x,t)$ keeps the same sign in the interval $[t,t+\epsilon]$ (or $[t+\epsilon,t]$) if $a(x,t)a(x,t+\epsilon) \ge 0$.

Combining the estimates in Lemma 1.10 with a standard argument as in [4], we obtain the following proposition.

Proposition 1.6. *Let $0<\sigma\le\lambda$. If $b\in\mathcal{C}^{2+\lambda}$, then $G_{m,\sigma}\circ\varphi$, $H_{m,\sigma}\circ\varphi$ and $\mathcal{G}_{m,\sigma}$ are bounded from Λ^s_{comp} to $\Lambda^{s+\sigma}_{loc}$ provided $0<s+\sigma\le\lambda+1$.*

Corollary 1.2. *If $\lambda>0$, then $\mathcal{K}$, $K\circ\varphi$, $T\circ\varphi$, $K\widetilde{a}\,|D_x|\circ\varphi$ and $KaQ\circ\varphi$ are bounded on Λ^s for $0\le s\le\lambda+2$.*

Proof. By Proposition 1.6 $\mathcal{K}$, $K\circ\varphi$, $T\circ\varphi$, $K\widetilde{a}\,|D_x|\circ\varphi$ and $KaQ\circ\varphi$ are bounded on Λ^s for $0\le s\le\lambda+1$. Then if Z is one of $\mathcal{K}$, K, or T, we have $\partial_t\circ Z=Z'\circ|D_x|+\widetilde{Z}$ where $\widetilde{Z}=b\,(x,t,D_x)\,Q\,(x,t,D_x)$ with $b\in\mathcal{C}^{\lambda+2}S^0_{1,0}$ and $\partial_x\circ Z=Z\circ\partial_x+\widetilde{Z}'$, where Z' has the form $G_{0,0}$, $H_{0,0}$ or $\mathcal{G}_{0,0}$, and where $\widetilde{Z}'$ has the form $G_{0,1}$, $H_{0,1}$ or $\mathcal{G}_{0,1}$. The Corollary now follows from Proposition 1.6.

We can also now improve the conclusions of Proposition 1.5 to include boundedness on Λ^s up to the cap.

Proposition 1.7. *For $1<p\le\infty$, we have*

$\mathcal{K}$,$K\circ\varphi\in\overline{\mathcal{O}}^0_{(-1,\lambda+2)}$; $\quad T\circ\varphi:\Lambda^s\to\Lambda^s,\quad 0<s\le\lambda+2$;

$T\circ\varphi:\mathcal{B}_p^{s+\frac{1}{p(\lambda+3)}}\to\mathcal{B}^s_p,\ 1<s\le\lambda+2$; $\quad K\circ a\circ\varphi:\Lambda^s\to\Lambda^{s+1},\quad 0<s\le\lambda+1$;

$K\circ a\circ\varphi:\mathcal{B}_p^{s+\frac{1}{p(\lambda+3)}}\to\mathcal{B}^{s+1}_p,\quad -1<s\le\lambda+1$;

$K\circ Da\circ\varphi\in\overline{\mathcal{O}}^{-\frac14}_{(-1,\lambda+1)}$; $\quad R\circ\varphi\in\overline{\mathcal{O}}^{-\mu_p}_{(-1,\lambda+2)}$,

where $\mu_p=\min\left\{1-\delta+\frac{1}{p(\lambda+3)},\frac{\lambda+2}{2(\lambda+3)}\right\}$ *for* $1<p<\infty$ *and* $0<\mu_\infty<\min\left\{1-\delta,\frac{\lambda+2}{2(\lambda+3)}\right\}$, $\delta=\max\left\{\frac12,\frac{1}{\lambda+1}\right\}$.

Proof. For K, $\mathcal{K}$ and T, the assertions follow from Proposition 1.5 and Corollary 1.2. As for $K\circ a$, we write $a=a^\sharp+a^\flat$ with $a^\flat\in\mathcal{C}^{\lambda+2}S^{-1-\frac{1}{\lambda+1}}_{1,\frac{1}{\lambda+1}}$. Then using the improved estimates (1.3) on $a^\sharp$, we have

$$\begin{aligned}K\circ a&=K\circ a^\sharp+K\circ a^\flat=K\widetilde{a}^\sharp+\sum_{1\le|\alpha|\le N}c_\alpha\partial^\alpha_\xi K\partial^\alpha_x\widetilde{a}^\sharp+\overline{\mathcal{O}}^{-1-\frac{1}{\lambda+1}}_{(-1,\lambda+2)}\\&=K\widetilde{a}-K\widetilde{a}^\flat+\overline{\mathcal{O}}^{-1}_{(-1,\lambda+2)}=K\widetilde{a}+\overline{\mathcal{O}}^{-1}_{(-1,\lambda+2)}\\&=(K\widetilde{a}\,|D_x|)\circ|D_x|^{-1}+\overline{\mathcal{O}}^{-1}_{(-1,\lambda+2)}.\end{aligned}$$

Now $K\widetilde{a}\,|D_x|$ has the same mapping properties as T, and so the assertions regarding $K\circ a\circ\varphi$ follow.

Now we turn our attention to $K\circ Da\circ\varphi$. We have

$$\begin{aligned}K\circ Da&=K\circ Da^\sharp+K\circ Da^\flat=K\left(D\widetilde{a}^\sharp\right)+\overline{\mathcal{O}}^{-1}_{(-1,\lambda+1)}\\&=K\,(D\widetilde{a})-K\left(D\widetilde{a}^\flat\right)+\overline{\mathcal{O}}^{-1}_{(-1,\lambda+1)}=K\,(D\widetilde{a})+\overline{\mathcal{O}}^{-1}_{(-1,\lambda+1)}.\end{aligned}$$

So by Proposition 1.5, $K\circ Da\in\overline{\mathcal{O}}^{-\frac14-\epsilon(\lambda)}_{(0,\lambda+1)}$. Since $\Lambda^s\subset H_p^{s-\frac{\epsilon}{2}}$ for all $1<p<\infty$, we have $K\circ Da:\Lambda^s\to\Lambda^{s+\frac14}$ for $0<s<\lambda+\frac34$, and it remains only to show that $K\circ Da:\Lambda^{\lambda+\frac34}\to\Lambda^{\lambda+1}$. For this it suffices to show $\partial_t\circ K\circ Da:\Lambda^{\lambda+\frac34}\to\Lambda^\lambda$, $\partial_x\circ K\circ Da:\Lambda^{\lambda+\frac34}\to\Lambda^\lambda$. Now $\partial_t\circ K\circ Da=-KaQ\circ Da+Da$ and by Corollary 1.2,

$$\begin{aligned}K\circ a\circ\varphi&:\Lambda^s\to\Lambda^s,\quad 0<s\le\lambda+1,\\Da&:\Lambda^{s+1}\to\Lambda^s,\quad 0<s\le\lambda+1,\end{aligned}$$

which shows that $\partial_t \circ K \circ Da : \Lambda^{\lambda+\frac{3}{4}} \to \Lambda^{\lambda}$. As for $\partial_x \circ K \circ Da$, we have

$$\begin{aligned}\partial_x \circ K \circ Da &= K \circ \partial_x \circ Da + K\widetilde{A}_x \circ Da \\ &= (K \circ Da) \circ \partial_x + K \circ \partial_x (Da) + K\widetilde{A}_x \circ Da.\end{aligned}$$

We have $\partial_x (Da) : \Lambda^s \to \Lambda^s, 0 < s \le \lambda$ and $K\widetilde{A}_x \circ \varphi : \Lambda^{s+\frac{1}{2}} \to \Lambda^s, 0 < s \le \lambda + 1$ and so $\partial_x \circ K \circ Da \circ \varphi : \Lambda^{\lambda+\frac{3}{4}} \to \Lambda^{\lambda}$, i.e., $K \circ Da \circ \varphi \in \overline{\mathcal{O}}^{-\frac{1}{4}}_{(0,\lambda+2)}$.

For R, we have by (1.19) that $R \circ \varphi \in \overline{\mathcal{O}}^{-\mu_\infty}_{(0,\lambda+2)}$, since as before, $\Lambda^s \subset H_p^{s-\frac{\epsilon}{2}}$ for all $1 < p < \infty$ and we can take p very large. Thus $R \circ \varphi : \Lambda^s \to \Lambda^{s+\mu_\infty}, \quad 0 < s < \lambda + 2 - \mu_\infty$. To show that $R \circ \varphi : \Lambda^{\lambda+2-\mu_\infty} \to \Lambda^{\lambda+2}$, it suffices to show $\partial_t \circ R, \partial_x \circ R : \Lambda^{\lambda+2-\mu_\infty} \to \Lambda^{\lambda+1}$. For this we compute

$$\begin{aligned}\partial_t \circ R &= KaQ \circ \widetilde{a}\widetilde{Q} - KaQ\widetilde{a}\widetilde{Q} \\ &= (KaQ) \circ \widetilde{a} \circ \widetilde{Q} - KaQ\widetilde{a}\widetilde{Q} \\ &= \left(KaQ \circ \widetilde{a}^\sharp\right) \circ \widetilde{Q} - KaQ\widetilde{a}^\sharp\widetilde{Q} + \mathcal{O}^{\frac{1}{\lambda+3}-1-\frac{1}{\lambda+1}+1}_{(0,\lambda+2)} \\ &= \sum_{|\alpha|\le N} c_\alpha \partial_\xi^\alpha (KaQ)\, \partial_x^\alpha \left(\widetilde{a}^\sharp\right) \circ \widetilde{Q} - KaQ\widetilde{a}^\sharp\widetilde{Q} + \mathcal{O}^{\frac{1}{\lambda+3}-\frac{1}{\lambda+1}}_{(0,\lambda+2)} \\ &= \left(KaQ \circ \widetilde{a}^\sharp\right) \circ \widetilde{Q} - KaQ\widetilde{a}^\sharp\widetilde{Q} + \mathcal{O}^{\frac{1}{\lambda+3}}_{(0,\lambda+2)}.\end{aligned}$$

Now $\widetilde{Q} = Q(x, t', \xi, \widetilde{a})$. Letting $Q^\sharp = Q\left(x, t', \xi, \widetilde{a}^\sharp\right)$, we obtain

$$\begin{aligned}\partial_t \circ R &= \left(KaQ\widetilde{a}^\sharp\right) \circ Q^\sharp - KaQ\widetilde{a}^\sharp Q^\sharp + \mathcal{O}^{\frac{1}{\lambda+3}}_{(0,\lambda+2)}. \\ &= (KaQ\widetilde{a}) \circ Q^\sharp - KaQ\widetilde{a}Q^\sharp + \mathcal{O}^{\frac{1}{\lambda+3}}_{(0,\lambda+2)} = \mathcal{O}^{\frac{1}{\lambda+3}}_{(0,\lambda+2)},\end{aligned}$$

by the sharp estimates (1.3) for $\widetilde{a}^\sharp$. Also, using Proposition 1.6,

$$\begin{aligned}\partial_x \circ R &= \partial_x \circ K \circ \widetilde{a}\widetilde{Q} - \partial_x \circ K\widetilde{a}\widetilde{Q} \\ &= K \circ \partial_x \circ \widetilde{a}\widetilde{Q} - K\widetilde{a}\widetilde{Q} \circ \partial_x + K\widetilde{A}_x\widetilde{a}\widetilde{Q} - K\left(\widetilde{a}\widetilde{Q}\right)_x - K\widetilde{A}_x \circ \widetilde{a}\widetilde{Q} \\ &= K \circ \widetilde{a}\widetilde{Q} \circ \partial_x - K\widetilde{a}\widetilde{Q} \circ \partial_x + K \circ \left(\widetilde{a}\widetilde{Q}\right)_x - K\left(\widetilde{a}\widetilde{Q}\right)_x + K\widetilde{A}_x\widetilde{a}\widetilde{Q}_x - K\widetilde{A}_x \circ \widetilde{a}\widetilde{Q} \\ &= R \circ \partial_x + K \circ \left(\widetilde{a}\widetilde{Q}\right)_x - K\left(\widetilde{a}\widetilde{Q}\right)_x + K\widetilde{A}_x\widetilde{a}\widetilde{Q} - K\widetilde{A}_x \circ \widetilde{a}^\sharp\widetilde{Q} - K\widetilde{A}_x \circ \widetilde{a}^\flat \circ \widetilde{Q} \\ &= R \circ \partial_x + K \circ \left(\widetilde{a}_x\widetilde{Q}\right) - K\widetilde{a}_x\widetilde{Q} + K \circ \widetilde{a} \circ \widetilde{Q}_x \\ &\quad - K\widetilde{a}\widetilde{Q}_x + K\widetilde{A}_x\widetilde{a}\widetilde{Q} - K\widetilde{A}_x\widetilde{a}^\sharp \circ \widetilde{Q} + \overline{\mathcal{O}}^{\frac{1}{2}}_{(0,\lambda+1)} \\ &= R \circ \partial_x + K \circ \left(\widetilde{a}_x\widetilde{Q}\right) - K\widetilde{a}_x\widetilde{Q} + K \circ \widetilde{a} \circ \widetilde{Q}_x \\ &\quad - K\widetilde{a}\widetilde{Q}_x + K\widetilde{A}_x\widetilde{a}\widetilde{Q} - K\widetilde{A}_x\widetilde{a} \circ \widetilde{Q} + \overline{\mathcal{O}}^{\frac{1}{2}}_{(0,\lambda+1)}.\end{aligned}$$

Now

$$K \circ \widetilde{a}_x = K \circ \left(\widetilde{a}^\sharp\right)_x + K \circ \left(\widetilde{a}^\flat\right)_x = K\widetilde{a}^\sharp_x + \overline{\mathcal{O}}^{-\delta(\lambda+2)}_{(0,\lambda+1)} = K\widetilde{a}_x + \overline{\mathcal{O}}^{-\delta(\lambda+2)}_{(0,\lambda+1)},$$

and $K\widetilde{a}_x \in \overline{\mathcal{O}}^{-\frac{1}{4}}_{(0,\lambda+1)}$ and $K\widetilde{A}_x\widetilde{a} : \Lambda^s \to \Lambda^{s+\frac{1}{2}}, 0 < s \le \lambda + \frac{1}{2}$ and $K\widetilde{A}_x\widetilde{a}\widetilde{Q} : \Lambda^s \to \Lambda^{s-\frac{1}{2}}, 0 < s \le \lambda + \frac{3}{2}$ by Proposition 1.6. Also $K\widetilde{a}, K \circ \widetilde{a} : \Lambda^{s-1} \to \Lambda^s, 0 < s \le \lambda + 2$ by the first part of the proof. Also

$$\widetilde{Q}_x = \frac{\partial}{\partial x} Q\left(x, t', \xi, a\left(x, t'\right)\right) = \widetilde{a}_x \cdot Q_z + \mathcal{C}^{\lambda+2} s^{1+\delta}_{1,\delta},$$

and so we conclude that $\partial_x \circ R \circ \varphi : \Lambda^{\lambda+2-\mu_\infty} \to \Lambda^{\lambda+1}$, and this completes the proof that $R \circ \varphi \in \overline{\mathcal{O}}^{-\mu_p}_{(-1,\lambda+2)}$.

1.5. Reduction of the Operators $\boldsymbol{K}$, $\boldsymbol{\mathcal{K}}$, and $\boldsymbol{T}$

In the second subsection above, we discussed the behaviour of the operators K, $\mathcal{K}$, and T in the nonelliptic cone $\{|\tau| < C\,|\xi|\}$. Now in general, T is not bounded on H^s_p or $\mathcal{B}^{s,p}_p$ and so Proposition 1.5 need not hold with $\frac{1}{\lambda+3}$ replaced by 0, and neither K nor $\mathcal{K}$ need have any gain. In order to establish sharp mapping properties for these operators in the next section, we will show here that the boundedness of T, K, or $\mathcal{K}$ into H^s_p (or $\mathcal{B}^{s,p}_p$) for some p, s satisfying $1 < p < \infty$ and $-1 < s < \lambda + 2$, is equivalent to the boundedness of the corresponding operators $\varphi \circ T \circ \varphi$, $\varphi \circ K \circ \varphi$, or $\varphi \circ \mathcal{K}$ into L^p (or $\mathcal{B}^{0,p}_p$). We continue to denote by φ and ψ zero order ψdo's with support in the nonelliptic cone $\{|\tau| < C\,|\xi|\}$ and the elliptic cone $\{|\xi| < C\,|\tau|\}$ respectively.

Define $K_\nu = K \circ |D_x|^\nu$ and $\mathcal{K}_\nu = \mathcal{K} \circ |D_x|^\nu$ for $0 \le \nu \le \frac{1}{2}$.

Lemma 1.11. *We have*

$$\begin{gathered} T \circ \nabla_x \circ \varphi = \nabla_x \circ T \circ \varphi + B_T, \quad K_\nu \circ \nabla_x \circ \varphi = \nabla_x \circ K_\nu \circ \varphi + B_\nu, \\ \mathcal{K}_\nu \circ \nabla_x = \nabla_x \circ \mathcal{K}_\nu + \mathcal{B}_\nu, \quad K \circ a \circ \nabla_x \circ \varphi = \nabla_x \circ K \circ a \circ \varphi + B^*, \end{gathered} \tag{1.41}$$

where $B_T \in \mathcal{O}^{\frac{1}{\lambda+3}+\delta}_{(-1,\lambda+1)}$, $B^* \in \mathcal{O}^{\frac{1}{\lambda+3}-(1-\delta)}_{(-1,\lambda+1)}$ *and* $B_\nu, \mathcal{B}_\nu \in \mathcal{O}^{\nu+\delta}_{(-1,\lambda+1)}$.

Proof. This follows from $T \in \mathcal{C}^{\lambda+2} S^{\frac{1}{\lambda+3}}_{1,\delta}$, $K_\nu \in \mathcal{C}^{\lambda+2} S^{\nu}_{1,\delta}$, $\mathcal{K}_\nu \in \mathcal{C}^{\lambda+2} S^{\nu}_{1,\delta}$, and $K \circ a \in \mathcal{C}^{\lambda+2} S^{\frac{1}{\lambda+3}-1}_{1,\delta}$.

Lemma 1.12. *We have* $\psi \circ T \circ \varphi, \psi \circ K \circ a \circ \varphi, \psi \circ K_\nu \circ \varphi, \psi \circ \mathcal{K}_\nu \in \overline{\mathcal{O}}^{0}_{(-1,\lambda+2)}$. *If Z denotes either $T \circ \varphi$, $K_\nu \circ \varphi$ or $\mathcal{K}_\nu$, then Z is bounded from* $H^{s+\nu-\frac{1}{p}}_p \to H^s_p$ *or from* $\mathcal{B}^{s+\nu-\frac{1}{p},p}_p \to \mathcal{B}^{s,p}_p$ *for some $\gamma \ge 0$, $-1 < s < \lambda + 2$ and $1 < p < \infty$ if and only if $\varphi \circ Z \circ \varphi$ is bounded from* $H^\gamma_p \to L^p$ *or from* $\mathcal{B}^{\gamma,p}_p \to \mathcal{B}^{0,p}_p$.

Proof. We have

$$\begin{gathered} \psi \circ K_\nu \circ \varphi = \partial_t^{-1} \circ \psi \circ \partial_t \circ K_\nu \circ \varphi = \partial_t^{-1} \circ \psi \circ \{|D_x|^\nu - KaQ \circ |D_x|^\nu\} \in \overline{\mathcal{O}}^{-1+\frac{1}{\lambda+3}+\nu}_{(-1,\lambda+2)}, \\ \psi \circ \mathcal{K}_\nu = \partial_t^{-1} \circ \psi \circ \partial_t \circ \mathcal{K}_\nu \circ \varphi = -\partial_t^{-1} \circ \psi \circ \mathcal{K}aQ \circ |D_x|^\nu \circ \varphi \in \overline{\mathcal{O}}^{-1+\frac{1}{2}+\nu}_{(-1,\lambda+2)}. \end{gathered}$$

Since $\frac{1}{\lambda+3} + \nu < \frac{1}{2} + \nu \le 1$, we have $\psi \circ K_\nu \circ \varphi$ and $\psi \circ \mathcal{K}_\nu \in \overline{\mathcal{O}}^{0}_{(-1,\lambda+2)}$. Now

$$\begin{aligned} \psi \circ K \circ a \circ \varphi &= \partial_t^{-1} \circ \psi \circ \partial_t \circ K \circ a \circ \varphi = \partial_t^{-1} \circ \psi \circ \{a - Ka \circ a\} \circ \varphi \\ &= \partial_t^{-1} \circ \psi \circ \left\{a - Ka\widetilde{a}\widetilde{Q}\right\} \circ \varphi + \alpha O^{-1}_{(0,\lambda)}, \end{aligned}$$

$$\psi \circ T \circ \varphi = \partial_t^{-1} \circ \psi \circ \partial_t \circ T \circ \varphi = \partial_t^{-1} \circ \psi \circ \left\{aQ - KaQ\widetilde{a}\widetilde{Q}\right\} \circ \varphi,$$

and $KaQ\widetilde{a}\widetilde{Q}$ is bounded from H^1_p to L^p. Indeed, the last assertion follows from the argument at the top of page 62 in [5] as follows. It suffices to show that $Ka\widetilde{a}Q$ is bounded on L^p, but this is a consequence of the argument used to prove Theorem 2.1 in the next section below, since after the change of variables $s = A_x\left(t\right) = \int_{t_0}^{t} a\left(x, \theta\right) d\theta$ as in (2.24), matters reduce to the validity of the Hardy inequality (2.25) with weights $w = \frac{a\left(A_x^{-1}(s)\right)^{p-1}}{s^p}$ and $v = \frac{1}{a\left(A_x^{-1}(s)\right)}$. But $w \leq s^{-p}$ and $v \geq 1$, and so (2.25) holds trivially by (2.27). Thus $\psi \circ T \circ \varphi : H_p^1 \to H_p^1$. The first two identities in Lemma 1.1 together with Proposition 1.7 show that $\psi \circ T$ is bounded on H_p^s for $-1 < s < \lambda + 1$ and $1 < p < \infty$. By interpolation, $\psi \circ T \circ \varphi$ is bounded on $\mathcal{B}_p^{s,p}$ for $-1 < s < \lambda$ and $1 < p < \infty$. Finally, the rest of the lemma follows from Proposition 1.7 and Lemma 1.11. This completes the proof of Lemma 1.12.

§2. Boundedness Properties of $\mathcal{K}$, K and T.

In the previous paper [6], we reduced matters regarding the gain from f in the oblique derivative problem to the boundedness of the operator

$$Tf(x,t) = \int_{R^n} e^{ix\cdot\xi} \int_0^t a(x,t')Q(x,t',\xi)e^{-\int_{t'}^t a(x,\theta)Q(x,\theta,\xi)d\theta} f^{\sim}(\xi,t')dt'd\xi$$

from a Sobolev space into L^p (or $\mathcal{B}_p^{0,p}$). Earlier, in the previous section, we observed that $aK \in \mathcal{C}^\lambda S_{1,\frac{1}{2}}^{-\frac{\lambda}{\lambda+3}}$, and it follows that T is in $\mathcal{C}^\lambda S_{1,\frac{1}{2}}^{\frac{1}{\lambda+3}}$, and so of order $\frac{1}{\lambda+3}$ on L^p Sobolev spaces (of the appropriate index). This is best possible for **all** $1 < p < \infty$, but if we fix attention on a particular p, then we can do better, namely $T \in \mathcal{O}^{\frac{1}{p(\lambda+3)}}$. To see this, we apply some techniques from harmonic analysis to characterize the boundedness on L^p of the operators $T_\alpha = TQ^{-\alpha}$:

$$T_\alpha f(x,t) = \int_{R^n} e^{ix\cdot\xi} \int_0^t a(x,t')\left(Q(x,t',\xi)\right)^{1-\alpha} e^{-\int_{t'}^t a(x,\theta)Q(x,\theta,\xi)d\theta} f^{\sim}(\xi,t')dt'd\xi. \tag{2.1}$$

In [5], we showed that for the case $\alpha = 0$, $T_0 = T$ is bounded on $L^p(R^n \times (0,1))$ if and only if a satisfies the $\mathcal{A}_p^-$ condition,

$$\Big[\frac{1}{\int_\sigma^\beta a(x,t)dt} \int_\sigma^\beta a(x,t)^{p'}dt\Big]^{p-1} \leq C\frac{1}{\gamma-\beta}\int_\beta^\gamma a(x,t)dt, \tag{2.2}$$

for all x in R^n and $0 < \sigma < \beta < \gamma < 1$ such that $\int_\sigma^\beta a(x,t)dt = \int_\beta^\gamma a(x,t)dt$. This was accomplished by using the change of variables $s = A_x(t) = \int_0^t a(x,\theta)d\theta$ (for each fixed x) and the Calderón reproducing formula to reduce the boundedness of T on $L^p(R^n \times (0,1))$ to the family of vector-valued weighted norm inequalities,

$$\int_0^{A_x(1)} \Big(\sum_{k=0}^\infty |M^- h_k(s)|^q\Big)^{p/q} w_x(s)ds \leq C\int_0^{A_x(1)} \Big(\sum_{k=0}^\infty |h_k(s)|^q\Big)^{p/q} w_x(s)ds, \tag{2.3}$$

for all sequences $\{h_k\}_{k=0}^\infty$ of nonnegative functions on $(0, A_x(1))$, where M^- denotes the one-sided Hardy-Littlewood maximal operator,

$$M^- f(s) = \sup_{0<\delta<s} \frac{1}{\delta}\int_{s-\delta}^s |f(t)|\,dt,$$

and $w_x(s) = \frac{d}{ds}A_x^{-1}(s) = \frac{1}{a(A_x^{-1}(s))}$. Using [10], these inequalities were in turn shown to be equivalent to the family of A_p^- conditions,

$$\Big[\frac{1}{\delta}\int_b^{b+\delta} w_x(s)ds\Big]\Big[\frac{1}{\delta}\int_{b-\delta}^b w_x(s)^{1-p'}ds\Big]^{p-1} \leq C, \quad 0 < \delta < b < A_x(1) - \delta,$$

which become (2.2) upon reversing the change of variable $s = A_x(t)$.

However, a simpler approach is to exploit the fact that for functions f which have been microlocalized to have *support* $\hat{f} \subset \{|\xi| > |\tau|\}$, the Besov space norm $\|f\|_{\mathcal{B}_p^{0,p}}$ is comparable to the mixed norm

$$\|f\|_{L^p\left(\mathcal{B}_p^{0,p}\right)} = \left\{\int \|f(\cdot,t)\|^p_{\mathcal{B}_p^{0,p}(dx)}\, dt\right\}^{\frac{1}{p}}. \tag{2.4}$$

From this, it is not hard to reduce matters to the boundedness of T_α on the mixed Lebesgue-Besov space $L^p\left(\mathcal{B}_p^{0,p}\right)$ normed by (2.4).

In the next subsection, we characterize the boundedness of T_α on $L^p\left(\mathcal{B}_p^{0,p}\right)$ in terms of a simple modification of the the $\mathcal{A}_p^-$ condition (2.2). The proofs are modelled after the arguments in section 6 of [5].

2.1 Boundedness of T_α

Our main theorem here is

Theorem 2.1. *Suppose $1 < p < \infty$ and $a(x,t)$ is a bounded nonnegative function on $R^n \times (0,1)$. Then T_α is bounded on $L^p_{(0,1)}\left(\mathcal{B}_p^{0,p}(R^n)\right)$, i.e.*

$$\Big(\int_0^1 \|T_\alpha f(\cdot,t)\|^p_{\mathcal{B}_p^{0,p}}\, dt\Big)^{\frac{1}{p}} \le C\Big(\int_0^1 \|f(\cdot,t)\|^p_{\mathcal{B}_p^{0,p}}\, dt\Big)^{\frac{1}{p}},$$

if and only if a satisfies the $\mathcal{A}^-_{p,\alpha}$ condition:

$$\Big[\frac{1}{\int_\sigma^\beta a(x,t)dt}\int_\sigma^\beta a(x,t)^{p'}dt\Big]^{p-1} \le C\frac{1}{\gamma-\beta}\Big[\int_\beta^\gamma a(x,t)dt\Big]^{1-p\alpha}, \tag{2.5}$$

for all x in R^n and $0<\sigma<\beta<\gamma<1$ with $\int_\sigma^\beta a(x,t)dt = \int_\beta^\gamma a(x,t)dt > 0$.

Proof. To show the necessity of the $\mathcal{A}^-_{p,\alpha}$ condition, we first prove the apparently weaker condition

$$\Big[\frac{1}{\int_\sigma^\beta a(x,t)dt}\int_\sigma^\beta a(x,t)^{p'}dt\Big]^{p-1} \le C\frac{1}{\delta-\gamma}\Big[\int_\gamma^\delta a(x,t)dt\Big]^{1-p\alpha}, \tag{2.6}$$

for all x in R^n and $0<\sigma<\beta<\gamma<\delta<1$ such that $\int_\sigma^\beta a(x,t)dt = \int_\beta^\gamma a(x,t)dt = \int_\gamma^\delta a(x,t)dt > 0$. For this, fix w in R^n and let $r = \int_\sigma^\beta a(w,\theta)d\theta = \int_\beta^\gamma a(w,\theta)d\theta = \int_\gamma^\delta a(w,\theta)d\theta > 0$. As in [5], let $\hat{\phi} \in C^\infty(R^n)$ have support contained in $\{\xi : \frac{1}{2} \le |\xi| \le 4\}$ and satisfy $\hat{\phi}(\xi) = 1$ for $1 \le |\xi| \le 2$. Set $\phi_r(x) = r^{-n}\phi(\frac{x}{r})$ so that $\hat{\phi}_r(\xi) = \hat{\phi}(r\xi)$. Letting ϕ_r denote the operation of convolution in the x-variable (in $R^n \times (0,1) = \{(x,t) : x \in R^n, 0<t<1\}$), then just as in [5], the real part of the kernel of $T_\alpha\phi_r$ satisfies the following estimate for $|x-w|, |x'-w| < cr$, $t' \in (\sigma,\beta)$ and $t \in (\gamma,\delta)$:

$$Re(T_\alpha\phi_r)(x,t,x',t') \ge ca(x,t')r^{\alpha-n-1}. \tag{2.7}$$

As in [5], define f by

$$f(x',t') = \chi_{B(w,cr)}(x')\chi_{(\sigma,\beta)}(t')a(w,t')^{p'-1}. \tag{2.8}$$

Combining (2.7) and (2.8) we obtain for $|x-w| < cr$, $t \in (\gamma,\delta)$,

$$Re(T_\alpha\phi_r f)(x,t) \ge cr^{\alpha-1}\int_\sigma^\beta a(x,t')a(w,t')^{p'-1}dt'. \tag{2.9}$$

where the last integral in (2.9) satisfies

$$\int_\sigma^\beta a(w,\theta)^{p'}d\theta \le 2\int_\sigma^\beta a(x,\theta)a(w,\theta)^{p'-1}d\theta \tag{2.10}$$

for $|x-w|<cr$.

Now choose $\eta \in \mathcal{C}_c^\infty(R^n)$ such that $\eta \geq 0$, support $\eta \subset B(0,\frac{1}{2})$ and $\int_{R^n}\eta(x)dx=1$. With $\eta_r(x)=r^{-n}\eta\left(\frac{x}{r}\right)$, it follows easily from (2.9) and (2.10) that

$$Re(\eta_r * T_\alpha \phi_r f)(x,t) \geq cr^{\alpha-1}\int_\sigma^\beta a(w,t')^{p'}dt' \tag{2.11}$$

for $|x-w|<cr$, $t \in (\gamma,\delta)$, with a perhaps smaller constant c—note that $r\leq(\delta-\gamma)/3$ if, as we may assume, $a(x,t)\leq 1/3$. We will also need the following inequality, valid for any $g\in\mathcal{B}_p^{0,p}(R^n)$, $1<p<\infty$:

$$\begin{aligned}\|\eta_r * g\|_{L^p} &= \Big\|\sum_{k=0}^\infty \varphi_k*\varphi_k*\eta_r*g\Big\|_{L^p} \leq \sum_{k=0}^\infty \|\varphi_k*\eta_r\|_{L^1}\|\varphi_k*g\|_{L^p}\\ &\leq \Big(\sum_{k=0}^\infty \|\varphi_k*\eta_r\|_{L^1}^{p'}\Big)^{\frac{1}{p'}}\Big(\sum_{k=0}^\infty\|\varphi_k*g\|_{L^p}^p\Big)^{\frac{1}{p}}\\ &\leq \|\eta_r\|_{\dot{\mathcal{B}}_1^{0,p'}}\|g\|_{\mathcal{B}_p^{0,p}} = \|\eta\|_{\dot{\mathcal{B}}_1^{0,p'}}\|g\|_{\mathcal{B}_p^{0,p}} \leq C\|g\|_{\mathcal{B}_p^{0,p}}.\end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12) with the boundedness of T_α on $L^p\left(\mathcal{B}_p^{0,p}\right)$, we obtain

$$\begin{aligned}&\Big(cr^{\alpha-1}\int_\sigma^\beta a(w,t')^{p'}dt'\Big)^p cr^n(\delta-\gamma)\\ &\leq \int_\gamma^\delta\int_{B(w,cr)}|\eta_r*T_\alpha\phi_r f(x,t)|^p dxdt \leq \int_0^1 \|\eta_r*T_\alpha\phi_r f\|_{L^p(dx)}^p dt\\ &\leq C\int_0^1\|T_\alpha\phi_r f\|^p_{\mathcal{B}_p^{0,p}(dx)}dt \leq C\int_0^1\|\phi_r f\|^p_{\mathcal{B}_p^{0,p}(dx)}dt\\ &= C\int_0^1\left(\sum_{k=0}^\infty\|\varphi_k*\phi_r f\|^p_{L^p(dx)}\right)dt \leq C\int_0^1\|f\|^p_{L^p(dx)}dt\\ &= C\int_0^1\int_{R^n}|f(x',t')|^p dx'dt' = cr^n\int_\sigma^\beta a(w,t')^{p'}dt'.\end{aligned} \tag{2.13}$$

If we rewrite (2.13) as

$$\Big(r^{\alpha-1}\int_\sigma^\beta a(w,t')^{p'}dt'\Big)^{p-1} \leq C\frac{r^{1-\alpha}}{\delta-\gamma},$$

and use $r=\int_\sigma^\beta a(w,t')dt'=\int_\gamma^\delta a(w,t')dt'$, we obtain (2.6). It remains to show that (2.6) implies the $\mathcal{A}_{p,\alpha}^-$ condition (2.5), and this follows using the argument beginning on page 43 of [5]. This completes the proof that $\mathcal{A}_{p,\alpha}^-$ is necessary for the boundedness of T_α on $L^p\left(\mathcal{B}_p^{0,p}\right)$.

Conversely, to show that the $\mathcal{A}_{p,\alpha}^-$ condition (2.5) implies the boundedness of T_α on $L^p(\mathcal{B}_p^{0,p})$, we begin by using the Calderón reproducing formula to reduce matters to a Littlewood-Paley decomposition of T_α. Choose $\hat{\phi}_0$ and $\hat{\phi}_1$ nonnegative and infinitely differentiable on R^n with supports in $\{\xi:|\xi|\leq 1\}$ and $\{\xi:\frac{1}{2}\leq|\xi|\leq 2\}$ respectively so that $\sum_{k=0}^\infty \hat{\phi}_k(\xi)^2=1$, for all $\xi \in R^n$, where $\hat{\phi}_k(\xi)=\hat{\phi}_1(2^{-k}\xi)$ for $k\geq 2$. Also choose $\hat{\psi}_1\geq 0$, C^∞ with support in $\{\xi:\frac{1}{4}\leq|\xi|\leq 4\}$ such that $\hat{\psi}_1=1$ on the support of $\hat{\phi}_1$. Set $\hat{\psi}_k(\xi)=\hat{\psi}_1(2^{-k}\xi)$. Letting ϕ_k and ψ_k denote the operation of convolution in the x-variable (in $R^n\times(0,1)=\{(x,t):x \in R^n, 0<t<1\}$) with $(\hat{\phi}_k)^\vee$ and $(\hat{\psi}_k)^\vee$, the inverse Fourier transforms of $\hat{\phi}_k$ and $\hat{\psi}_k$ respectively, we have for any $f,\ g \in C_c^\infty(R^n\times(0,1))$,

$$\begin{aligned}\int_0^1\int_{R^n} T_\alpha f(x,t)g(x,t)dxdt &= \sum_{k=0}^\infty \int_0^1\int_{R^n} T_\alpha\phi_k^2(x,t)g(x,t)dxdt\\ &= \int_0^1\int_{R^n}(T_\alpha\phi_0^2)f(x,t)g(x,t)dxdt\\ &\quad+\sum_{k=1}^\infty\int_0^1\int_{R^n}[(1-\psi_k)T_\alpha\phi_k](\phi_k f)(x,t)g(x,t)dxdt\\ &\quad+\sum_{k=1}^\infty\int_0^1\int_{R^n}(T_\alpha\phi_k)(\phi_k f)(x,t)(\psi_k g)(x,t)dxdt\\ &= I+II+III. \end{aligned}\tag{2.14}$$

Term I in (2.14) is estimated as in [5]. Note first that by the argument on page 45 of [5], the kernel of $T_\alpha\phi_0^2$ satisfies

$$\left|T_\alpha\phi_0^2(x,t,x',t')\right| \le C(1+|x-x'|)^{-(n+\frac12)}$$

uniformly in t' and t. Thus $T_\alpha\phi_0^2$ is bounded on $L^p\left(\mathcal{B}_p^{0,p}\right)$ and

$$|I| \le \|T_\alpha\phi_0^2 f\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)} \le C\|f\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)} \tag{2.15}$$

for $f,\ g \in C_c^\infty(R^n\times(0,1))$.

To estimate term II in (2.14), we apply Propositions 1.1 and 1.2 to the composition $(I-\psi_k)\circ T_\alpha\phi_k$, with $M=1$, to obtain that $(I-\psi_k)\,T_\alpha\phi_k$ maps $\mathcal{B}_p^{-\mu,p}(dx)$ into $\mathcal{B}_p^{0,p}(dx)$ with norm independent of k,t and t'. Thus

$$\begin{aligned}|II| &\le \sum_{k=1}^\infty \|(I-\psi_k)T_\alpha\phi_k(\phi_k f)\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\ &\le \sum_{k=1}^\infty C\|\phi_k f\|_{L^p\left(\mathcal{B}_p^{-\mu,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\ &\le \sum_{k=1}^\infty C2^{-k\mu}\|f\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\ &= C\|f\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\end{aligned}\tag{2.16}$$

for $f,\ g \in C_c^\infty(R^n\times(0,1))$.

To estimate the main term III in (2.14), we need the following inequality, uniformly in k:

$$\int_0^1\int_{R^n}|T_\alpha\phi_k h(x,t)|^p dxdt \le C\int_0^1\int_{R^n}|h(x,t)|^p dxdt,\quad k\ge 1, \tag{2.17}$$

for all $h\in\mathcal{S}(R^n\times(0,1))$, the subspace of $\mathcal{S}(R^{n+1})$ whose elements are supported in $R^n\times(0,1)$. Assuming (2.17), we have for $f,\ g \in C_c^\infty(R^n\times(0,1))$,

$$\begin{aligned}|III| &\le \int_0^1\int_{R^n}\Big(\sum_{k=1}^\infty|T_\alpha\phi_k(\phi_k f)(x,t)|^p\Big)^{\frac1p}\Big(\sum_{k=1}^\infty|\psi_k g(x,t)|^{p'}\Big)^{\frac1{p'}}dxdt\\ &\le \Big\{\int_0^1\int_{R^n}\Big(\sum_{k=1}^\infty|T_\alpha\phi_k(\phi_k f)(x,t)|^p\Big)dxdt\Big\}^{\frac1p}\Big\{\int_0^1\int_{R^n}\Big(\sum_{k=1}^\infty|\psi_k g(x,t)|^{p'}\Big)dxdt\Big\}^{\frac1{p'}}\end{aligned}$$

$$\leq C\Big\{\int_0^1\int_{R^n}\Big(\sum_{k=1}^{\infty}|\phi_k f(x,t)|^p\Big)dxdt\Big\}^{\frac{1}{p}}\Big\{\int_0^1\int_{R^n}\Big(\sum_{k=1}^{\infty}|\psi_k g(x,t)|^{p'}\Big)dxdt\Big\}^{\frac{1}{p'}}$$

by (2.17) applied with $h=\phi_k f\in\mathcal{S}(R^n\times(0,1))$,

$$\leq C\Big(\int_0^1\|f(\cdot,t)\|^p_{\mathcal{B}^{0,p}_p}\,dt\Big)^{\frac{1}{p}}\Big(\int_0^1\|g(\cdot,t)\|^{p'}_{\mathcal{B}^{0,p'}_{p'}}\,dt\Big)^{\frac{1}{p'}}$$

$$=C\|f\|_{L^p\left(\mathcal{B}^{0,p}_p\right)}\|g\|_{L^{p'}\left(\mathcal{B}^{0,p'}_{p'}\right)}. \tag{2.18}$$

Combining (2.14), (2.15), (2.16) and (2.18) shows that T_α is bounded on $L^p\left(\mathcal{B}^{0,p}_p\right)$, and thus it remains only to establish (2.17).

To estimate the kernel $T_\alpha\phi_k(x,t,x',t')$ of $T_\alpha\phi_k$, we write, using (2.1),

$$\begin{aligned}&|T_\alpha\phi_k(x,t,x',t')|\\
&=\chi_{(0,t)}(t')\Big|\int_{R^n}e^{i(x-x')\cdot\xi}a(x,t')\left(Q(x,t',\xi)\right)^{1-\alpha}e^{-\int_{t'}^{t}a(x,\theta)Q(x,\theta,\xi)d\theta}\hat{\phi}_k(\xi)d\xi\Big|\\
&=\chi_{(0,t)}(t')\Big|\int_{R^n}\Big(\Big(\frac{2^{-2k}-\triangle_\xi}{2^{-2k}+|x-x'|^2}\Big)^N e^{i(x-x')\cdot\xi}\Big)\\
&\quad\times a(x,t')\left(Q(x,t',\xi)\right)^{1-\alpha}e^{-\int_{t'}^{t}a(x,\theta)Q(x,\theta,\xi)d\theta}\hat{\phi}_k(\xi)d\xi\Big|\\
&\leq\chi_{(0,t)}(t')\,(2^{-2k}+|x-x'|^2)^{-N}2^{nk}2^{-2kN}a(x,t')2^{k(1-\alpha)}e^{-2^{k-m}\int_{t'}^{t}a(x,\theta)d\theta}\end{aligned} \tag{2.19}$$

since $|\left(-\triangle_\xi\right)^\ell\hat{\phi}_k(\xi)|\leq C_\ell 2^{-2k\ell}$, $2^{1-m}|\xi|\leq Q(x,\theta,\xi)\leq 2^{m-1}|\xi|$ for some fixed m, and $\hat{\phi}_k$ is supported in $\{\xi: 2^{k-1}\leq|\xi|\leq 2^{k+1}\}$. Denote by M the Hardy-Littlewood maximal operator in the x-variable, $Mh(x,t)=\sup\limits_{r>0}r^{-n}\int_{|x-x'|\leq r}|h(x',t)|dx'$. Then ([11], p. 63)

$$\int_{R^n}2^{kn}[1+(2^k|x-x'|^2)]^{-N}\,|h(x',t)|\,dx'\leq C_N Mh(x,t) \tag{2.20}$$

for $N>\frac{n}{2}$. For $k\in Z_+$ and $a(t)\geq 0$ for $0<t<1$, define $T^\alpha_{a,k}$ by

$$T^\alpha_{a,k}g(t)=\int_0^t a(t')2^{k(1-\alpha)}e^{-2^k\int_{t'}^{t}a(\theta)d\theta}g(t')dt',\quad 0<t<1 \tag{2.21}$$

for any g integrable on $(0,1)$. Using (2.19), (2.20) and (2.21) we then have

$$\begin{aligned}&|T_\alpha\phi_k h(x,t)|\\
&\leq C_N\int_0^t a(x,t')2^{k(1-\alpha)}e^{-2^{k-m}\int_{t'}^{t}a(x,\theta)d\theta}\int_{R^n}\frac{2^{kn}}{[1+(2^k|x-x'|)^2]^N}|h(x',t')|dx'dt'\\
&\leq C_N\int_0^t a(x,t')2^{k(1-\alpha)}e^{-2^{k-m}\int_{t'}^{t}a(x,\theta)d\theta}Mh(x,t')dt'\\
&\leq C_N T^\alpha_{a_x,k-m}(Mh)(x,t),\end{aligned} \tag{2.22}$$

where $a_x(t)=a(x,t)$.

We now claim that

$$\int_0^1|T^\alpha_{a_x,k}g(t)|^p dt\leq C\int_0^1|g(t)|^p dt \tag{2.23}$$

for all sequences of functions $\{f_k\}$. To see this, let $A_x(t)=\int_0^t a(x,\theta)d\theta$ and make the change of variable $s=A_x(t)=\int_0^t a(x,\theta)d\theta$ and $s'=A_x(t')$ in (2.23) and (2.21). Then with

$\tilde{g}(s) = g(A_x^{-1}(s))$ and $(T^\alpha_{a_x,k}g)^\sim(s) = T^\alpha_{a_x,k}g(A_x^{-1}(s))$, we have

$$\begin{aligned}(T^\alpha_{a_x,k}g)^\sim(s) &= \int_0^{A_x^{-1}(s)} a(x,t')2^{k(1-\alpha)}e^{-2^k(A_x(t)-A_x(t'))}g(t')dt' \\ &= \int_0^s 2^{k(1-\alpha)}e^{-2^k(s-s')}\tilde{g}(s')ds',\end{aligned}$$

and so with $w_x(s) = \frac{d}{ds}A_x^{-1}(s) = \frac{1}{a(A_x^{-1}(s))}$ so that $dt = w_x(s)ds$, (2.22) is equivalent to

$$\int_0^{A_x(1)} \left|\int_0^s 2^{k(1-\alpha)}e^{-2^k(s-s')}g(s')ds'\right|^p w_x(s)ds \le C\int_0^{A_x(1)} |g(s)|^p w_x(s)ds \tag{2.24}$$

for all nonnegative functions g on $(0, A_x(1))$. We now claim that (2.24) follows from the $\mathcal{A}^-_{p,\alpha}$ condition (2.5) and the weighted norm inequality for the Hardy operator (see [8]), namely

$$\int_0^A \left|\int_0^s h(s')ds'\right|^p w(s)ds \le C\int_0^A |h(s)|^p v(s)ds, \tag{2.25}$$

for all $h \ge 0$ if and only if

$$\sup_{0<b<A}\Big(\int_b^A w(s)ds\Big)\Big(\int_0^b v(s)^{1-p'}ds\Big)^{p-1} < \infty. \tag{2.26}$$

To see that (2.24) follows, replace $g(s')$ by $e^{-2^k s'}g(s')$ in (2.24) to obtain (2.25) with $A = A_x(1)$, $w(s) = 2^{pk(1-\alpha)}e^{-p2^k s}w_x(s)$ and $v = e^{-p2^k s}w_x(s)$. Thus we must show that (2.26) holds for these weights, i.e.,

$$\sup_{0<b<A_x(1)}\Big(\int_b^{A_x(1)} 2^{pk(1-\alpha)}e^{-p2^k s}w_x(s)ds\Big)\Big(\int_0^b e^{p'2^k s}w_x(s)^{1-p'}ds\Big)^{p-1} < \infty. \tag{2.27}$$

We now rewrite the $\mathcal{A}^-_{p,\alpha}$ condition (2.5) using the change of variable $s = A_x(t) = \int_0^t a(x,\theta)d\theta$ to get

$$\delta^{p\alpha}\Big[\frac{1}{\delta}\int_b^{b+\delta} w_x(s)ds\Big]\Big[\frac{1}{\delta}\int_{b-\delta}^b w_x(s)^{1-p'}ds\Big]^{p-1} \le C \tag{2.28}$$

for $0 < \delta < b < A_x(1) - \delta$. Set $r = 2^{-k}$. If $1 < p \le 2$, then we estimate the product in (2.27) by

$$\begin{aligned}& r^{p(\alpha-1)}\Big(\sum_{i=0}^\infty \int_{b+ir}^{b+(i+1)r} e^{-ps/r}w_x(s)ds\Big)\Big(\sum_{j=0}^\infty \int_{b-(j+1)r}^{b-jr} e^{p's/r}w_x(s)^{1-p'}ds\Big)^{p-1} \\ &\le r^{p(\alpha-1)}\sum_{i=0}^\infty\Big(\int_{b+ir}^{b+(i+1)r} e^{-ps/r}w_x(s)ds\Big)\sum_{j=0}^\infty\Big(\int_{b-(j+1)r}^{b-jr} e^{p's/r}w_x(s)^{1-p'}ds\Big)^{p-1} \\ &\le Cr^{p(\alpha-1)}\sum_{i,j\ge 0} e^{-p(i+j)}\Big(\int_{b+ir}^{b+(i+1)r} w_x(s)ds\Big)\Big(\int_{b-(j+1)r}^{b-jr} w_x(s)^{1-p'}ds\Big)^{p-1} \\ &\le Cr^{p(\alpha-1)}\sum_{i,j\ge 0} e^{-p(i+j)}\left[(i+j)\,r\right]^{p(1-\alpha)} \quad \text{by (2.28) with } \delta = (i+j)\,r \\ &\le C\sum_{i,j\ge 0} e^{-p(i+j)}\,(i+j)^{p(1-\alpha)} \le C.\end{aligned}$$

On the other hand, if $2 \le p < \infty$, then we raise the product in (2.27) to the power $p' - 1$

and estimate

$$r^{p'(\alpha-1)}\Big(\sum_{i=0}^{\infty}\int_{b+ir}^{b+(i+1)r}e^{-ps/r}w_x(s)ds\Big)^{p'-1}\Big(\sum_{j=0}^{\infty}\int_{b-(j+1)r}^{b-jr}e^{p's/r}w_x(s)^{1-p'}ds\Big)$$

by a constant C as above. This establishes (2.23) with C independent of k.

We thus have

$$\begin{aligned}\int_0^1\int_{R^n}|T_\alpha\phi_k h(x,t)|^p dxdt &\le C\int_0^1\int_{R^n}|T_{a_x,k-m}(Mh)(x,t)|^p dtdx \quad \text{by (2.23)},\\ &\le C\int_0^1\int_{R^n}|Mh(x,t)|^p dtdx \quad \text{by (2.23)}\\ &= C\int_0^1\int_{R^n}|Mh(x,t)|^p dxdt\\ &\le C\int_0^1\int_{R^n}|h(x,t)|^p dxdt\end{aligned}$$

since M is bounded on L^p. This establishes (2.17) and completes the proof of Theorem 2.1.

We now turn to the local estimates for T_α on $\mathcal{B}_p^{s,p}(R^{n+1})$ that we need in [6]. In this setting, both Γ and $\overrightarrow{\mathbf{T}}$ have been straightened out and the flow for $\overrightarrow{\mathbf{T}}=\frac{\partial}{\partial t}$ through $(x,0)$ in Γ is given by $\overrightarrow{\gamma}\left((x,0),t\right)=(x,t)$. Thus the $\mathcal{A}_{p,\alpha}^{\mp}$ condition becomes

Definition 2.1. *The function a satisfies the $\mathcal{A}_{p,\alpha}^{\mp}$ condition at the fibre $F_{(y,0)}$, $(y,0)\in\Gamma$, if there are constants $r>0$, $R^-<0<R^+$, such that $a\left(x,R^-\right)\neq 0$ and $a\left(x,R^+\right)\neq 0$ for $x\in R^n$, $|x-y|<r$ and both of the following conditions hold:*

$$\Big[\frac{1}{\int_\sigma^\beta a\left(x,t\right)dt}\int_\sigma^\beta a\left(x,t\right)^{p'}dt\Big]^{p-1}\le C\frac{1}{\gamma-\beta}\Big[\int_\beta^\gamma a\left(x,t\right)dt\Big]^{1-p\alpha} \tag{2.29}$$

for all $x\in\Gamma$, $|x-y|<r$ and all $0<\sigma<\beta<\gamma<R^+$ with $\int_\sigma^\beta a\left(x,t\right)dt=\int_\beta^\gamma a\left(x,t\right)dt>0$, and also

$$\Big[\frac{1}{\int_\beta^\gamma |a\left(x,t\right)|dt}\int_\beta^\gamma |a\left(x,t\right)|^{p'}dt\Big]^{p-1}\le C\frac{1}{\beta-\sigma}\Big[\int_\sigma^\beta |a\left(x,t\right)|dt\Big]^{1-p\alpha} \tag{2.30}$$

for all $x\in\Gamma$, $|x-y|<r$ and all $R^-<\sigma<\beta<\gamma<0$ with $\int_\sigma^\beta|a\left(x,t\right)|dt=\int_\beta^\gamma|a\left(x,t\right)|dt>0$.

We shall need to know that for functions $f\left(x,t\right)$ with frequencies in the cone $\{|\xi|>|\tau|\}$, the mixed norm $\|\cdot\|_{L^p(\mathcal{B}_p^{0,p})}$ is equivalent to the Besov norm $\|\cdot\|_{\mathcal{B}_p^{0,p}}$.

Lemma 2.1. *Suppose supp $\hat{f}\left(\xi,\tau\right)\subset\{(\xi,\tau)\in R^n\times R:|\xi|>|\tau|\}$. Then*

$$\|f\|_{B_p^{0,p}}\cong\|f\|_{L^p(\mathcal{B}_p^{0,p})}.$$

Proof. Let $\{\Phi_k\}_{k=0}^\infty$ (respectively $\{\phi_k\}_{k=0}^\infty$) satisfy the usual conditions for the Calderon reproducing formula in R^{n+1} (respectively R^n). Then by the condition on the support of $\hat{f}$, we have $\Phi_k f\cong\phi_k f$, and so

$$\begin{aligned}\|f\|_{\mathcal{B}_p^{0,p}}^p&=\sum_{k=0}^\infty\|\Phi_k f\|_{L^p(R^{n+1})}^p\cong\sum_{k=0}^\infty\|\phi_k f\|_{L^p(R^{n+1})}^p\\ &=\sum_{k=0}^\infty\int_R\|\phi_k f\left(\cdot,t\right)\|_{L^p(R^n)}^p dt=\int_R\sum_{k=0}^\infty\|\phi_k f\left(\cdot,t\right)\|_{L^p(R^n)}^p dt=\|f\|_{L^p(\mathcal{B}_p^{0,p})}^p.\end{aligned}$$

Now we can state our characterization of the local boundedness of T_α on $B_p^{s,p}$ in Case II.

Theorem 2.1. *Let $a(x,t)$ be a $\mathcal{C}^{\lambda+2}$ function with $a(x,t)\geq 0$ when $t\geq 0$ and $a(x,t)\leq 0$ when $t\leq 0$, let $y\in R^n$ and suppose that $a\in\mathcal{A}_{p,\alpha}^{\mp}$ at the fibre $F_{(y,0)}$. Let $U=\{(x,y):|x-y|<r, R^-<t<R^+\}$ where r, R^- and R^+ are as in (2.29) and (2.30). Let φ, ψ be $\mathcal{C}^\infty$ functions supported in U, and suppose that P denotes a multiplier with symbol supported in the cone $\{|\xi|>|\tau|\}$. Then if M_φ, M_ψ are the operators of multiplication by φ and ψ respectively, the operator $PM_\varphi T_\alpha M_\psi P$ is bounded on $\mathcal{B}_p^{s,p}$ for $-\frac{1}{2}<s<\lambda+2$. Conversely, if $\varphi=\psi=1$ on U, and $M_\varphi T_\alpha M_\psi$ is bounded on $B_p^{s,p}$ for some $s\in\left(-\frac{1}{2},\lambda+2\right)$, then $a\in\mathcal{A}_{p,\alpha}^{\mp}$ at the fibre $F_{(y,0)}$. In particular, $PM_\varphi T_{\frac{1}{p(\lambda+3)}}M_\psi P$ is bounded on $\mathcal{B}_p^{s,p}$ for $-\frac{1}{2}<s<\lambda+2$.*

Proof. Choose $\rho\in\mathcal{C}_c^\infty(R^n)$ with *supp* $\rho\subset\{x:|x-y|<r\}$ and $\rho(x)=1$ on the support of ψ. Then $\rho(x)a(x,t)$ satisfies $\mathcal{A}_{p,\alpha}^{\mp}$ since for any fixed x, the inequality in $\mathcal{A}_{p,\alpha}^{\mp}$ is unaffected by multiplying $a(x,t)$ by a positive constant. Thus the operator T_α^*, obtained from T_α by replacing a with ρa, is bounded on $L^p(\mathcal{B}_p^{0,p})$ by Theorem 2.1 together with the same result scaled to $R^n\times(-R,0)$. Now if *supp* φ, *supp* $\psi\subset U$, then $PM_\varphi T_\alpha M_\psi P=PM_\varphi T_\alpha^* M_\psi P$ is bounded on $L^p(\mathcal{B}_p^{0,p})$, and so also on $\mathcal{B}_p^{0,p}$ by Lemma 2.1. By Lemma 1.12, we now conclude that $PM_\varphi T_\alpha M_\psi P$ is bounded on $\mathcal{B}_p^{s,p}$ for $-\frac{1}{2}<s<\lambda+2$.

Conversely, if $\varphi=\psi=1$ on U and $M_\varphi T_\alpha M_\psi$ is bounded on $\mathcal{B}_p^{s,p}$ for some $s\in\left(-\frac{1}{2},\lambda+2\right)$, then by Lemma 1.12, it is bounded on $\mathcal{B}_p^{0,p}$. So suppose $\widetilde{T}_\alpha=M_\varphi T_\alpha M_\psi$ is bounded on $\mathcal{B}_p^{0,p}$. The proof of necessity in Theorem 2.1 carries over here with just a few changes, as follows. With notation as in the proof of Theorem 2.1, we have from (2.9) and (2.10) that

$$Re(\widetilde{T}_\alpha\phi_r f)(x,t)=Re(M_\varphi T_\alpha\phi_r f)(x,t)\geq cr^{\alpha-1}\int_\sigma^\beta a(w,t')^{p'}dt' \tag{2.31}$$

for $|x-w|<cr$, $t\in(\gamma,\delta)$. Now choose $\eta\in\mathcal{C}_c^\infty(R^{n+1})$ such that $\eta\geq 0$, *support* $\eta\subset B(0,\frac{1}{2})$ and $\int_{R^{n+1}}\eta(x)dx=1$ (note the use of R^{n+1} in place of R^n here). With $\eta_r(x)=r^{-n}\eta\left(\frac{x}{r}\right)$, it follows easily from (2.31) that

$$Re(\eta_r*\widetilde{T}_\alpha\phi_r f)(x,t)\geq cr^{\alpha-1}\int_\sigma^\beta a(w,t')^{p'}dt' \tag{2.32}$$

for $|x-w|<cr$, $t\in(\gamma,\delta)$, with a perhaps smaller constant c—note that $r\leq(\delta-\gamma)/3$ if, as we may assume, $a(x,t)\leq 1/3$. If we use the inequality $\|\eta_r*g\|_{L^p}\leq C\|g\|_{\mathcal{B}_p^{0,p}}$, valid for any $g\in\mathcal{B}_p^{0,p}(R^{n+1})$, $1<p<\infty$, (this is (2.12) with R^{n+1} in place of R^n), the $\mathcal{A}_{p,\alpha}^{\mp}$ condition in the open set $\overrightarrow{\gamma}(U_R)$ can be derived as in the proof of Theorem 2.1.

Finally, $a\in\mathcal{C}^{\lambda+2}$ implies $a\in\mathcal{A}^{\mp}_{p,\frac{1}{p(\lambda+3)}}$ by the following elementary computation using Lemma 1.4:

$$\begin{aligned}\Big[\frac{1}{\int_\sigma^\beta a(x,t)dt}\int_\sigma^\beta a(x,t)^{p'}dt\Big]^{p-1}&\leq\sup_{\sigma\leq t\leq\gamma}a\,(x,t)\\&\leq C\Big(\frac{1}{\gamma-\sigma}\int_\sigma^\gamma a\,(x,t)\,dt\Big)^{1-\frac{1}{\lambda+3}}\quad\text{by (1.11)}\\&\leq C\frac{1}{\gamma-\beta}\Big[\int_\beta^\gamma a(x,t)dt\Big]^{1-p\alpha}\end{aligned}$$

for $0<\sigma<\beta<\gamma$ since $\int_\sigma^\gamma a\,(x,t)\,dt=2\int_\beta^\gamma a(x,t)dt$, $\frac{1}{\lambda+3}=p\alpha$ and $\gamma-\beta\leq\gamma-\sigma\leq 1$.

Similarly for $\mathcal{A}^{=}_{p,\alpha}$ and $\mathcal{A}^{\ddagger}_{p,\alpha}$. Simple examples show this is sharp. This of course shows that $PM_\varphi T_{\frac{1}{p(\lambda+3)}} M_\psi P$ is bounded on $\mathcal{B}^{s,p}_p$ for $-\frac{1}{2} < s < \lambda + 2$ as required.

2.2. Boundedness of K_ν

In [6] we reduced matters regarding the gain from g in the oblique derivative problem to the boundedness of the operator

$$Kf(x,t) = \int_{R^n} e^{ix\cdot\xi} \int_0^t e^{-\int_{t'}^t a(x,\theta)Q(x,\theta,\xi)d\theta} f^\sim(\xi,t')dt'd\xi$$

from a Besov space into L^p. It is known that if a is C^∞ and has type k, then K gains $\frac{1}{k+1}$ derivatives, i.e. $K \in \mathcal{O}^{\frac{1}{k+1}}$. In this subsection, we extend this result to rough a by replacing the type condition on a with the condition

$$(\mathcal{T}_\nu) \quad \beta - \alpha \le C \left(\int_\alpha^\beta |a(x,t)|\, dt \right)^\nu \quad \text{for } \alpha < \beta, \text{ and } x \in R^n.$$

We begin by proving the analogue of Theorem 2.1 for the operators $K_\nu = KQ^\nu$ given by

$$K_\nu f(x,t) = \int_{R^n} e^{ix\cdot\xi} \int_0^t Q(x,t',\xi)^\nu e^{-\int_{t'}^t a(x,\theta)Q(x,\theta,\xi)d\theta} f^\sim(\xi,t')dt'd\xi. \tag{2.34}$$

Theorem 2.3. *Suppose* $1 < p < \infty$ *and* $a(x,t)$ *is a bounded nonnegative function on* $R^n \times (0,1)$. *Then* K_ν *is bounded on* $L^p_{(0,1)}\left(\mathcal{B}^{0,p}_p(R^n)\right)$ *if and only if* a *satisfies the* $(\mathcal{T}_\nu)$ *condition.*

Proof. As in the proof of Theorem 2.1, fix w in R^n and let $r = \int_\alpha^\beta a(w,\theta)d\theta$. Then with ϕ_r as before, the real part of the kernel of $K_\nu\phi_r$ satisfies the following estimate for $|x-w|$, $|x'-w| < cr$, and $t,\ t' \in (\alpha,\beta)$:

$$\begin{aligned}
&Re(K_\nu\phi_r)(x,t,x't') \\
&= \int_{R^n} \cos((x-x')\cdot\xi)\left(Q(x,t',\xi)\right)^\nu e^{-\int_{t'}^t a(x,\theta)Q(x,\theta,\xi)d\theta} \hat{\phi}(r\xi)d\xi \\
&\ge \int_{\frac{1}{2r}\le|\xi|\le\frac{4}{r}} cr^{-\nu} e^{-\int_{t'}^t a(x,\theta)Cr^{-1}d\theta} d\xi \ge cr^{-\nu-n}.
\end{aligned} \tag{2.35}$$

Now define f by

$$f(x',t') = \chi_{B(w,cr)}(x')\chi_{(\alpha,\beta)}(t'). \tag{2.36}$$

Combining (2.35) and (2.36) we obtain for $|x-w| < cr,\ t \in (\alpha,\beta)$,

$$Re(K_\nu\phi_r f)(x,t) \ge c\int_{B(w,cr)} r^{-\nu-n}(\beta-\alpha) \ge cr^{-\nu}(\beta-\alpha). \tag{2.37}$$

As in the previous subsection, choose $\eta \in \mathcal{C}^\infty_c(R^n)$ such that $\eta \ge 0$, support $\eta \subset B(0,\frac{1}{2})$ and $\int_{R^n}\eta(x)dx = 1$. With $\eta_r(x) = r^{-n}\eta\left(\frac{x}{r}\right)$, it follows immediately from (2.37) that

$$Re(\eta_r * K_\nu\phi_r f)(x,t) \ge cr^{-\nu}(\beta-\alpha). \tag{2.38}$$

or $|x-w| < cr,\ t \in (\alpha,\beta)$, with a perhaps smaller constant c—note that $r \le (\beta-\alpha)/3$ if, as we may assume, $a(x,t) \le 1/3$. From (2.36), (2.38), (2.12) and the boundedness of $K_\nu\phi_r$

on $L^p\left(\mathcal{B}_p^{0,p}\right)$, we now obtain

$$\begin{aligned}
&\left(cr^{-\nu}(\beta-\alpha)\right)^p cr^n(\beta-\alpha)\\
&\le \int_{B(w,cr)}\int_\alpha^\beta |\eta_r * K_\nu\phi_r f(x,t)|^p dxdt \le \int_0^1 \|\eta_r * K_\nu\phi_r f\|^p_{L^p(dx)}\,dt\\
&\le C\int_0^1 \|K_\nu\phi_r f\|^p_{\mathcal{B}_p^{0,p}(dx)}\,dt \le C\int_0^1 \|\phi_r f\|^p_{\mathcal{B}_p^{0,p}(dx)}\,dt\\
&= C\int_0^1 \Big(\sum_{k=0}^\infty \|\varphi_k * \phi_r f\|^p_{L^p(dx)}\Big)dt \le C\int_0^1 \|f\|^p_{L^p(dx)}\,dt\\
&= C\int_0^1\int_{R^n} |f(x',t')|^p dx'dt' = Cr^n(\beta-\alpha),
\end{aligned}$$

which yields the $\mathcal{T}_\nu$ condition since $r=\int_\alpha^\beta a(w,t')dt'$.

To show that the $\mathcal{T}_\nu$ condition implies the boundedness of K_ν on $L^p\left(\mathcal{B}_p^{0,p}\right)$, we proceed as before by using the Calderón reproducing formula on a Littlewood- Paley decomposition of K_ν. Let ϕ_k and ψ_k be as in subsection 2.1. We have for any $f,\ g\ \in\ C_c^\infty(R^n\times(0,1))$,

$$\begin{aligned}
\int_0^1\int_{R^n} K_\nu f(x,t)g(x,t)dxdt &= \sum_{k=0}^\infty \int_0^1\int_{R^n} K_\nu\phi_k^2(x,t)g(x,t)dxdt,\\
&= \int_0^1\int_{R^n} (K_\nu\phi_0^2)f(x,t)g(x,t)dxdt\\
&+\sum_{k=1}^\infty \int_0^1\int_{R^n} [(1-\psi_k)K_\nu\phi_k](\phi_k f)(x,t)g(x,t)dxdt\\
&+\sum_{k=1}^\infty \int_0^1\int_{R^n} (K_\nu\phi_k)(\phi_k f)(x,t)(\psi_k g)(x,t)dxdt\\
&= I+II+III. \qquad (2.39)
\end{aligned}$$

Term I in (2.39) is handled just as in the previous subsection, obtaining

$$|I|\le C\|f\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)} \qquad (2.40)$$

for $f,\ g\ \in\ C_c^\infty(R^n\times(0,1))$.

To estimate term II in (2.39), we apply Propositions 1.1 and 1.2 to the composition $(I-\psi_k)\circ K_\nu\phi_k$, with $M=1$, to obtain that $(I-\psi_k)\,K_\nu\phi_k$ maps $\mathcal{B}_p^{-\mu,p}(dx)$ into $\mathcal{B}_p^{0,p}(dx)$ with norm independent of k, t and t'. Thus

$$\begin{aligned}
|II| &\le \sum_{k=1}^\infty \|\,(I-\psi_k)\,K_\nu\phi_k(\phi_k f)\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\
&\le \sum_{k=1}^\infty C\|\phi_k f\|_{L^p\left(\mathcal{B}_p^{-\mu,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\
&\le \sum_{k=1}^\infty C2^{-k\mu}\|f\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\
&= C\|f\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)} \qquad (2.41)
\end{aligned}$$

for $f,\ g\ \epsilon\ C_c^\infty(R^n\times(0,1))$.

To estimate the main term III in (2.39), we need the following inequality, uniformly in k:

$$\int_0^1 \int_{R^n} |K_\nu \phi_k h(x,t)|^p dxdt \le C \int_0^1 \int_{R^n} |h(x,t)|^p dxdt, \quad k \ge 1, \tag{2.42}$$

for all $h \in \mathcal{S}(R^n \times (0,1))$, the subspace of $\mathcal{S}(R^{n+1})$ whose elements are supported in $R^n \times (0,1)$. Assuming (2.42), we have for $f,\ g \in C_c^\infty(R^n \times (0,1))$,

$$\begin{aligned}
|III| &\le \int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |K_\nu \phi_k(\phi_k f)(x,t)|^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^\infty |\psi_k g(x,t)|^{p'}\Big)^{\frac{1}{p'}} dxdt \\
&\le \Big\{\int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |K_\nu \phi_k(\phi_k f)(x,t)|^p\Big) dxdt\Big\}^{\frac{1}{p}} \Big\{\int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |\psi_k g(x,t)|^{p'}\Big) dxdt\Big\}^{\frac{1}{p'}} \\
&\le C\Big\{\int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |\phi_k f(x,t)|^p\Big) dxdt\Big\}^{\frac{1}{p}} \Big\{\int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |\psi_k g(x,t)|^{p'}\Big) dxdt\Big\}^{\frac{1}{p'}} \\
&\quad \text{by (2.42) applied with } h = \phi_k f \in \mathcal{S}(R^n \times (0,1)), \\
&\le C\Big(\int_0^1 \|f(\cdot,t)\|^p_{\mathcal{B}_p^{0,p}}\, dt\Big)^{\frac{1}{p}} \Big(\int_0^1 \|g(\cdot,t)\|^{p'}_{\mathcal{B}_{p'}^{0,p'}}\, dt\Big)^{\frac{1}{p'}} \\
&= C\|f\|_{L^p(\mathcal{B}_p^{0,p})} \|g\|_{L^{p'}(\mathcal{B}_{p'}^{0,p'})}.
\end{aligned} \tag{2.43}$$

Combining (2.39), (2.40), (2.41) and (2.43) shows that K_ν is bounded on $L^p\left(\mathcal{B}_p^{0,p}\right)$, and thus it remains only to establish (2.42).

To estimate the kernel $K_\nu\phi_k(x,t,x',t')$ of $K_\nu\phi_k$, we write, using (2.34),

$$\begin{aligned}
|K_\nu \phi_k(x,t,x',t')| &= \chi_{(0,t)}(t') \Big| \int_{R^n} e^{i(x-x')\cdot\xi} \left(Q(x,t',\xi)\right)^\nu e^{-\int_{t'}^t a(x,\theta) Q(x,\theta,\xi) d\theta} \hat{\phi}_k(\xi) d\xi \Big| \\
&= \chi_{(0,t)}(t') \Big| \int_{R^n} \left(\left(\frac{2^{-2k} - \triangle_\xi}{2^{-2k} + |x-x'|^2} \right)^N e^{i(x-x')\cdot\xi} \right) \left(Q(x,t',\xi)\right)^\nu \\
&\qquad \times e^{-\int_{t'}^t a(x,\theta) Q(x,\theta,\xi) d\theta} \hat{\phi}_k(\xi) d\xi \Big| \\
&\le \chi_{(0,t)}(t')\ (2^{-2k} + |x-x'|^2)^{-N} 2^{nk} 2^{-2kN} 2^{k\nu} e^{-2^{k-m}\int_{t'}^t a(x,\theta)d\theta}
\end{aligned} \tag{2.44}$$

since $|\left(-\triangle_\xi\right)^\ell \hat{\phi}_k(\xi)| \le C_\ell 2^{-2k\ell}$, $2^{1-m}|\xi| \le Q(x,\theta,\xi) \le 2^{m-1}|\xi|$ for some fixed m, and $\hat{\phi}_k$ is supported in $\{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$. Denote by M the Hardy-Littlewood maximal operator in the x-variable, so that

$$\int_{R^n} 2^{kn}[1 + (2^k|x-x'|^2)]^{-N}\, |h(x',t')|\, dx' \le C_N Mh(x,t'), \tag{2.45}$$

for $N > \frac{n}{2}$. Since a satisfies the $\mathcal{T}_\nu$ condition (2.33), we have

$$2^{k\nu} e^{-2^{k-m}\int_{t'}^t a(x,\theta)d\theta} \le 2^{k\nu} e^{-2^{k-m} C^{-1/\nu} \left|t-t'\right|^{1/\nu}}. \tag{2.46}$$

Now the right side of (2.46) is an even integrable function of $t - t'$ satisfying

$$\int_0^1 2^{k\nu} e^{-2^{k-m} C^{-1/\nu} \left|t-t'\right|^{1/\nu}} dt \text{ (or } dt') \le C_{m,\nu}$$

uniformly in k, and so we conclude that

$$\int_0^1 2^{k\nu} e^{-2^{k-m}\int_{t'}^t a(x,\theta)d\theta} Mh(x,t')dt' \le C_{m,\nu} \widetilde{M}\left(Mh\right)(x,t), \tag{2.47}$$

where $\widetilde{M}$ denotes the Hardy-Littlewood maximal operator in the t-variable. Combining (2.44), (2.45) and (2.47) yields

$$\begin{aligned}|K_\nu \phi_k h(x,t)| &\le C_N \int_0^t 2^{k\nu} e^{-2^{k-m}\int_{t'}^t a(x,\theta)d\theta} \int_{R^n} \frac{2^{kn}}{[1+(2^k|x-x'|)^2]^N} |h(x',t')|dx'dt' \\ &\le C_N \int_0^t 2^{k\nu} e^{-2^{k-m}\int_{t'}^t a(x,\theta)d\theta} Mh(x,t')dt' \\ &\le C_N \widetilde{M}(Mh)(x,t). \qquad (2.48)\end{aligned}$$

We thus have

$$\begin{aligned}\int_0^1 \int_{R^n} |K_\nu \phi_k h(x,t)|^p dxdt &\le C \int_0^1 \int_{R^n} |\widetilde{M}(Mh)(x,t)|^p dtdx, \quad \text{by (2.48)} \\ &\le C \int_0^1 \int_{R^n} |Mh(x,t)|^p dtdx \\ &\le C \int_0^1 \int_{R^n} |h(x,t)|^p dxdt\end{aligned}$$

since both $\widetilde{M}$ and M are bounded on L^p. This establishes (2.42) and completes the proof of Theorem 2.3

In order to state a local version of this result, we recast the definition of the $\mathcal{T}_\nu$ condition in terms of open sets. Let

$$U_R = \{(x,t)\,\epsilon R^n \times R : |x|\,, |t| < R\}\,,$$

and note that $\overrightarrow{\gamma}\,((x,s)\,,t) = (x, s+t)$ is the flow for $\overrightarrow{\mathbf{T}} = \frac{\partial}{\partial t}$ through (x,s).

Definition 2.2. *The function a satisfies the $\mathcal{T}_\nu$ condition in the open set U_R if*

$$\beta - \alpha \le C\left(\int_\alpha^\beta |a\,(x,t)|\,dt\right)^\nu \quad \textit{for all } -R<\alpha<\beta<R, \textit{ and } |x|<R.$$

Now we can state our characterization of the local boundedness of K_ν on $\mathcal{B}_p^{s,p}$.

Theorem 2.4. *Let $a(x,t)$ be a $\mathcal{C}^{\lambda+2}$ function, and suppose that $a\in\mathcal{T}_\nu$ in the open set U_R. Let φ, ψ be $\mathcal{C}^\infty$ functions supported in U_R, and suppose that P denotes a multiplier with symbol supported in the cone $\{|\xi|>|\tau|\}$. Then if M_φ, M_ψ are the operators of multiplication by φ and ψ respectively, the operator $PM_\varphi K_\nu M_\psi P$ is bounded on $\mathcal{B}_p^{s,p}$ for $-\frac{1}{2}<s<\lambda+2$. Conversely, if $\varphi=\psi=1$ on U_R, and $M_\varphi K_\nu M_\psi$ is bounded on $\mathcal{B}_p^{s,p}$ for some $s\in\left(-\frac{1}{2},\lambda+2\right)$, then $a\in\mathcal{T}_\nu$ in the open set $U_{R'}$ for some $R'>0$.*

Proof. The operator K_ν is bounded on $L^p(\mathcal{B}_p^{0,p})$ by Theorem 2.3 together with the same result scaled to $R^n\times(-R,0)$. Now if *supp* φ, *supp* $\psi\subset U_R$, then $PM_\varphi K_\nu M_\psi P$ is bounded on $L^p(\mathcal{B}_p^{0,p})$, and so also on $\mathcal{B}_p^{0,p}$ by Lemma 2.1. By Lemma 1.12, we now conclude that $PM_\varphi K_\nu M_\psi P$ is bounded on $\mathcal{B}_p^{s,p}$ for $-\frac{1}{2}<s<\lambda+2$.

Conversely, if $\varphi=\psi=1$ on U_1 and $M_\varphi K_\nu M_\psi$ is bounded on $\mathcal{B}_p^{s,p}$ for some $s\in\left(-\frac{1}{2},\lambda+2\right)$, then by Lemma 1.12, it is bounded on $B_p^{0,p}$. So suppose $\widetilde{K}_\nu = M_\varphi K_\nu M_\psi$ is bounded on $\mathcal{B}_p^{0,p}$. The proof of necessity in Theorem 2.3 carries over here with just a few changes, as follows. With notation as in the proof of Theorem 2.3, we have from (2.37) that

$$Re(\widetilde{K}_\nu \phi_r f)(x,t) = Re(M_\varphi K_\nu \phi_r f)(x,t) \ge cr^{-\nu}(\beta-\alpha) \qquad (2.49)$$

for $|x-w|<cr$, $t\in(\alpha,\beta)$. Now choose $\eta\in\mathcal{C}_c^\infty(R^{n+1})$ such that $\eta\ge0$, *support* $\eta\subset B(0,\frac{1}{2})$ and $\int_{R^{n+1}}\eta(x)dx=1$ (note the use of R^{n+1} in place of R^n here). With $\eta_r(x)=r^{-n}\eta\left(\frac{x}{r}\right)$,

it follows easily from (2.49) that

$$Re(\eta_r * \widetilde{K}_\nu \phi_r f)(x,t) \geq cr^{-\nu} \int_\alpha^\beta a(w,t')^{p'} dt'$$

for $|x-w| < cr$, $t \in (\alpha, \beta)$, with a perhaps smaller constant c—note that $r \leq (\beta - \alpha)/3$ if, as we may assume, $a(x,t) \leq 1/3$. If we use the inequality $\|\eta_r * g\|_{L^p} \leq C\, \|g\|_{\mathcal{B}_p^{0,p}}$, valid for any $g \in \mathcal{B}_p^{0,p}(R^{n+1})$, $1 < p < \infty$ (this is (2.12) with R^{n+1} in place of R^n), the $\mathcal{T}_\nu$ condition in the open set $U_{R'}$ can be derived as in the proof of Theorem 2.3.

2.3. Boundedness of $\mathcal{K}$

In [6], we reduced matters regarding the gain from h in the oblique derivative problem to the boundedness of the operator

$$\mathcal{K}f(x,t) = \int_{R^n} e^{ix\cdot\xi} e^{-\int_0^t a(x,\theta)Q(x,\theta,\xi)d\theta} \hat{f}\,(\xi)\, d\xi$$

from a Besov space into $\mathcal{B}_p^{0,p}$. In this subsection, we show that if a has type k, then $\mathcal{K}$ gains $\frac{1}{p(k+1)}$ derivatives from h in the L^p scale of smoothness spaces (note that since a vanishes at the origin, $k \geq 1$; for $k = 0$, this would agree with the classical result for the Poisson integral in the half space). More generally, we extend this result to fractional γ replacing the type condition on a with a type conditon at the origin,

$$(\mathcal{P}_\gamma)\; |\beta| \leq C\Big|\int_0^\beta a(x,t)dt\Big|^\gamma \quad \text{for all } \beta \in R,\ x \in R^n. \tag{2.50}$$

We begin by proving the analogue of Theorems 2.1 and 2.3 for the operators $\mathcal{K}_\gamma = KQ^\gamma$ given by

$$\mathcal{K}_\gamma f(x,t) = \int_{R^n} e^{ix\cdot\xi} Q(x,t',\xi)^\gamma e^{-\int_0^t a(x,\theta)Q(x,\theta,\xi)d\theta} \hat{f}\,(\xi)\, d\xi. \tag{2.51}$$

Theorem 2.5. *Suppose $1 < p < \infty$ and $a(x,t)$ is a bounded nonnegative function on $R^n \times (0,1)$. Then $\mathcal{K}_\gamma$ is bounded from $\mathcal{B}_p^{0,p}$ to $L^p_{(0,1)}\left(\mathcal{B}_p^{0,p}\left(R^n\right)\right)$ if and only if a satisfies the $\mathcal{P}_{p\gamma}$ condition.*

Proof. As in the proof of Theorem 2.3, fix w in R^n and let $r = \int_\alpha^\beta a(w,\theta)d\theta$. Then with ϕ_r as before, the real part of the kernel of $\mathcal{K}_\gamma \phi_r$ satisfies the following estimate for $|x-w|$, $|x'-w| < cr$, and $t \in (0,\beta)$:

$$\begin{aligned} Re(\mathcal{K}_\gamma \phi_r)(x,t,x') &= \int_{R^n} \cos((x-x')\cdot\xi)\,(Q(x,t',\xi))^\gamma\, e^{-\int_0^t a(x,\theta)Q(x,\theta,\xi)d\theta} \hat{\phi}(r\xi) d\xi \\ &\geq \int_{\frac{1}{2r} \leq |\xi| \leq \frac{4}{r}} cr^{-\gamma} e^{-\int_0^t a(x,\theta)Cr^{-1}d\theta} d\xi \geq cr^{-\gamma-n}. \end{aligned} \tag{2.52}$$

Now define f by

$$f(x') = \chi_{B(w,cr)}(x'). \tag{2.53}$$

Combining (2.51) and (2.52) we obtain for $|x-w| < cr$, $t \in (0,\beta)$,

$$Re(\mathcal{K}_\gamma \phi_r f)(x,t) \geq c \int_{B(w,cr)} r^{-\gamma-n} \geq cr^{-\gamma}. \tag{2.54}$$

As in the previous subsection, choose $\eta \in \mathcal{C}_c^\infty(R^n)$ such that $\eta \geq 0$, *support* $\eta \subset B(0, \frac{1}{2})$ and $\int_{R^n} \eta(x)dx = 1$. With $\eta_r(x) = r^{-n}\eta\left(\frac{x}{r}\right)$, it follows immediately from (2.54) that

$$Re(\eta_r * \mathcal{K}_\gamma \phi_r f)(x,t) \geq cr^{-\gamma} \tag{2.55}$$

for $|x-w|<cr$, $t \in (0,\beta)$, with a perhaps smaller constant c—note that $r \le \beta/3$ if, as we may assume, $a(x,t) \le 1/3$. From (2.53), (2.55), (2.12) and the boundedness of $\mathcal{K}_\gamma \phi_r$ from $\mathcal{B}_p^{0,p}$ to $L^p\left(\mathcal{B}_p^{0,p}\right)$, we now obtain

$$\begin{aligned}
\left(cr^{-\gamma}\right)^p cr^n\beta &\le \int_{B(w,cr)}\int_0^\beta |\eta_r * \mathcal{K}_\gamma\phi_r f(x,t)|^p dxdt \le \int_0^1 \|\eta_r * \mathcal{K}_\gamma\phi_r f\|^p_{L^p(dx)}\,dt\\
&\le C\int_0^1 \|\mathcal{K}_\gamma\phi_r f\|^p_{\mathcal{B}_p^{0,p}(dx)}\,dt \le C\,\|\phi_r f\|^p_{\mathcal{B}_p^{0,p}(dx)}\\
&= C\sum_{k=0}^\infty \|\varphi_k * \phi_r f\|^p_{L^p(dx)} \le C\,\|f\|^p_{L^p(dx)}\\
&= C\int_{R^n} |f(x',t')|^p dx' = Cr^n,
\end{aligned}$$

which yields the $\mathcal{P}_{p\gamma}$ condition since $r = \int_0^\beta a(w,t')dt'$.

To show that the $\mathcal{P}_{p\gamma}$ condition implies the boundedness of $\mathcal{K}_\gamma$ from $\mathcal{B}_p^{0,p}$ to $L^p\left(\mathcal{B}_p^{0,p}\right)$, we proceed as before by using the Calderón reproducing formula on a Littlewood- Paley decomposition of $\mathcal{K}_\gamma$. Let ϕ_k and ψ_k be as in subsection 5.1. We have for any $f \in C_c^\infty(R^n)$ and $g \in C_c^\infty(R^n \times (0,1))$,

$$\begin{aligned}
\int_0^1\int_{R^n} \mathcal{K}_\gamma f(x,t)g(x,t)dxdt &= \sum_{k=0}^\infty \int_0^1\int_{R^n} \mathcal{K}_\gamma\phi_k^2 f(x,t)g(x,t)dxdt,\\
&= \int_0^1\int_{R^n} (\mathcal{K}_\gamma\phi_0^2)f(x,t)g(x,t)dxdt\\
&\quad + \sum_{k=1}^\infty \int_0^1\int_{R^n} [(1-\psi_k)\mathcal{K}_\gamma\phi_k](\phi_k f)(x,t)g(x,t)dxdt\\
&\quad + \sum_{k=1}^\infty \int_0^1\int_{R^n} (\mathcal{K}_\gamma\phi_k)(\phi_k f)(x,t)(\psi_k g)(x,t)dxdt\\
&= I + II + III. \qquad (2.56)
\end{aligned}$$

To estimate term I in (2.56), we proceed as in the previous section to obtain

$$|I| \le C\|f\|_{\mathcal{B}_p^{0,p}}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)} \qquad (2.57)$$

for $f \in C_c^\infty(R^n)$ and $g \in C_c^\infty(R^n \times (0,1))$.

To estimate term II in (2.56), we apply Propositions 1.1 and 1.2 to the composition $(I-\psi_k)\circ\mathcal{K}_\gamma\phi_k$, with $M=1$, to obtain that $(I-\psi_k)\,\mathcal{K}_\gamma\phi_k$ maps $\mathcal{B}_p^{-\mu,p}(dx)$ into $\mathcal{B}_p^{0,p}(dx)$ with norm independent of k and t. Thus

$$\begin{aligned}
|II| &\le \sum_{k=1}^\infty \|\,(I-\psi_k)\,\mathcal{K}_\gamma\phi_k(\phi_k f)\|_{L^p\left(\mathcal{B}_p^{0,p}\right)}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\
&\le \sum_{k=1}^\infty C\|\phi_k f\|_{\mathcal{B}_p^{-\mu,p}}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\
&\le \sum_{k=1}^\infty C2^{-k\mu}\|f\|_{\mathcal{B}_p^{0,p}}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}\\
&= C\|f\|_{\mathcal{B}_p^{0,p}}\|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)} \qquad (2.58)
\end{aligned}$$

for $f \in C_c^\infty(R^n)$ and $g \in C_c^\infty(R^n \times (0,1))$.

To estimate the main term III in (2.56), we need the following inequality, uniformly in k:

$$\int_0^1 \int_{R^n} |\mathcal{K}_\gamma \phi_k h(x,t)|^p dxdt \le C \int_{R^n} |h(x)|^p dx, \quad k \ge 1, \tag{2.59}$$

for all $h \in \mathcal{S}(R^n)$. Assuming (2.59), we have for $f \in C_c^\infty(R^n)$ and $g \in C_c^\infty(R^n \times (0,1))$,

$$\begin{aligned}
|III| &\le \int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |\mathcal{K}_\gamma \phi_k(\phi_k f)(x,t)|^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^\infty |\psi_k g(x,t)|^{p'}\Big)^{\frac{1}{p'}} dxdt \\
&\le \Big\{\int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |\mathcal{K}_\gamma \phi_k(\phi_k f)(x,t)|^p\Big) dxdt\Big\}^{\frac{1}{p}} \\
&\quad \times \Big\{\int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |\psi_k g(x,t)|^{p'}\Big) dxdt\Big\}^{\frac{1}{p'}} \\
&\le C \Big\{\int_{R^n} \Big(\sum_{k=1}^\infty |\phi_k f(x)|^p\Big) dx\Big\}^{\frac{1}{p}} \Big\{\int_0^1 \int_{R^n} \Big(\sum_{k=1}^\infty |\psi_k g(x,t)|^{p'}\Big) dxdt\Big\}^{\frac{1}{p'}} \\
&\quad \text{by (2.59) applied with } h = \phi_k f \in \mathcal{S}(R^n), \\
&\le C \, \|f\|_{\mathcal{B}_p^{0,p}} \left(\int_0^1 \|g(\cdot,t)\|_{\mathcal{B}_{p'}^{0,p'}}^{p'} dt\right)^{\frac{1}{p'}} = C \|f\|_{\mathcal{B}_p^{0,p}} \|g\|_{L^{p'}\left(\mathcal{B}_{p'}^{0,p'}\right)}.
\end{aligned} \tag{2.60}$$

Combining (2.56), (2.57), (2.58) and (2.60) shows that $\mathcal{K}_\gamma$ is bounded from $\mathcal{B}_p^{0,p}$ to $L^p\left(\mathcal{B}_p^{0,p}\right)$, and thus it remains only to establish (2.59).

To estimate the kernel $\mathcal{K}_\gamma \phi_k(x,t,x')$ of $\mathcal{K}_\gamma \phi_k$, we write, using (2.51),

$$\begin{aligned}
&|\mathcal{K}_\gamma \phi_k(x,t,x')| \\
&= \Big|\int_{R^n} e^{i(x-x')\cdot\xi} \left(Q(x,t,\xi)\right)^\gamma e^{-\int_0^t a(x,\theta) Q(x,\theta,\xi) d\theta} \hat{\phi}_k(\xi) d\xi\Big| \\
&= \Big|\int_{R^n} \Big(\Big(\frac{2^{-2k} - \triangle_\xi}{2^{-2k} + |x-x'|^2}\Big)^N e^{i(x-x')\cdot\xi}\Big) \left(Q(x,t,\xi)\right)^\gamma e^{-\int_0^t a(x,\theta) Q(x,\theta,\xi) d\theta} \hat{\phi}_k(\xi) d\xi\Big| \\
&\le (2^{-2k} + |x-x'|^2)^{-N} 2^{nk} 2^{-2kN} 2^{k\gamma} e^{-2^{k-m} \int_0^t a(x,\theta) d\theta}
\end{aligned} \tag{2.61}$$

since $|\left(-\triangle_\xi\right)^\ell \hat{\phi}_k(\xi)| \le C_\ell 2^{-2k\ell}$, $2^{1-m}|\xi| \le Q(x,t,\xi) \le 2^{m-1}|\xi|$ for some fixed m, and $\hat{\phi}_k$ is supported in $\{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$. As in subsection 5.1, denote by M the Hardy-Littlewood maximal operator in the x-variable, so that

$$\int_{R^n} 2^{kn} [1 + (2^k |x-x'|^2)]^{-N} \, |h(x')| \, dx' \le C_N M h(x), \tag{2.62}$$

for $N > \frac{n}{2}$. Since a satisfies the $\mathcal{P}_{p\gamma}$ condition (see (2.50) with γ replaced by $p\gamma$), we have

$$2^{k\gamma} e^{-2^{k-m} \int_0^t a(x,\theta) d\theta} \le 2^{k\gamma} e^{-2^{k-m} c t^{\frac{1}{p\gamma}}}. \tag{2.63}$$

Combining (2.61), (2.62) and (2.63) yields

$$\begin{aligned}
|\mathcal{K}_\gamma \phi_k h(x,t)| &\le C_N 2^{k\gamma} e^{-2^{k-m} c t^{\frac{1}{p\gamma}}} \int_{R^n} \frac{2^{kn}}{[1 + (2^k |x-x'|)^2]^N} |h(x')| dx' \\
&\le C_N 2^{k\gamma} e^{-2^{k-m} c t^{\frac{1}{p\gamma}}} M h(x).
\end{aligned} \tag{2.64}$$

We thus have

$$\int_0^1 \int_{R^n} |\mathcal{K}_\gamma \phi_k h(x,t)|^p dxdt \le C\Big(\int_0^1 \Big|2^{k\gamma} e^{-2^{k-m} ct^{\frac{1}{p\gamma}}}\Big|^p dt\Big) \int_{R^n} |Mh(x)|^p\, dx,$$
$$\le C \int_{R^n} |Mh(x)|^p\, dx \le C \int_{R^n} |h(x)|^p dx,$$

since

$$\int_0^1 \Big|2^{k\gamma} e^{-2^{k-m} ct^{\frac{1}{p\gamma}}}\Big|^p dt = 2^{kp\gamma} \int_0^{2^k} e^{-2^{-m} cs} d\left(\frac{s}{2^k}\right)^{p\gamma} \le C_{p,\gamma},$$

and M is bounded on L^p. This establishes (2.59) and completes the proof of Theorem 2.5.

Now, as at the end of the previous subsection, we recast the definition of the $\mathcal{P}_\gamma$ condition in terms of open sets. Recall that

$$U_R = \{(x,t) \in R^n \times R : |x|\,, |t| < R\}\,,$$

and that $\overrightarrow{\gamma}\,((x,s)\,,t) = (x, s+t)$ is the flow for $\vec{T} = \frac{\partial}{\partial t}$ through (x,s).

Definition 2.3. *The function a satisfies the $\mathcal{P}_\gamma$ condition in the open set U_R if*

$$|\beta| \le C\Big|\int_0^\beta a\,(x,t)\, dt\Big|^\gamma \quad \text{for all } -R < \beta < R, \textit{ and } |x| < R.$$

Now we can state our characterization of the local boundedness of $\mathcal{K}_\gamma$ on $\mathcal{B}_p^{s,p}$.

Theorem 2.6. *Let $a(x,t)$ be a $\mathcal{C}^{\lambda+2}$ function, and suppose that $a \in \mathcal{P}_{p\gamma}$ in the open set U_R. Let φ , ψ be $\mathcal{C}^\infty$ functions supported in U_R, and suppose that P denotes a multiplier with symbol supported in the cone $\{|\xi| > |\tau|\}$. Then if M_φ, M_ψ are the operators of multiplication by φ and ψ respectively, the operator $PM_\varphi \mathcal{K}_\gamma M_\psi$ is bounded $\mathcal{B}_p^{s,p}$ on for $-\frac{1}{2} < s < \lambda + 2$. Conversely, if $\varphi = \psi = 1$ on U_R, and $M_\varphi \mathcal{K}_\gamma M_\psi$ is bounded on $\mathcal{B}_p^{s,p}$ for some $s \in \left(-\frac{1}{2}, \lambda+2\right)$, then $a \in \mathcal{P}_{p\gamma}$ in the open set $U_{R'}$ for some $R' > 0$.*

Proof. The operator $\mathcal{K}_\gamma$ is bounded on $L^p(\mathcal{B}_p^{0,p})$ by Theorem 2.5 together with the same result scaled to $R^n \times (-R, 0)$. Now if *supp* φ, *supp* $\psi \subset U_R$, then $PM_\varphi \mathcal{K}_\gamma M_\psi$ is bounded on $L^p(\mathcal{B}_p^{0,p})$, and so also on $\mathcal{B}_p^{0,p}$ by Lemma 2.1. By Lemma 1.12, we now conclude that $PM_\varphi \mathcal{K}_\gamma M_\psi$ is bounded on $\mathcal{B}_p^{s,p}$ for $-\frac{1}{2} < s < \lambda + 2$.

Conversely, if $\varphi = \psi = 1$ on U_R and $M_\varphi \mathcal{K}_\gamma M_\psi$ is bounded on $\mathcal{B}_p^{s,p}$ for some $s \in \left(-\frac{1}{2}, \lambda+2\right)$, then by Lemma 1.12, it is bounded on $\mathcal{B}_p^{0,p}$. So suppose $\mathcal{K}'_\gamma = M_\varphi \mathcal{K}_\gamma M_\psi$ is bounded on $\mathcal{B}_p^{0,p}$. The proof of necessity in Theorem 2.5 carries over here with just a few changes, as follows. With notation as in the proof of Theorem 2.5, we have from (2.54) that

$$Re(\mathcal{K}'_\gamma \phi_r f)(x,t) = Re(M_\varphi \mathcal{K}_\gamma \phi_r f)(x,t) \ge cr^{-\gamma}\beta \tag{2.65}$$

for $|x-w| < cr$, $t \in (0,\beta)$. Now choose $\eta \in \mathcal{C}_c^\infty(R^{n+1})$ such that $\eta \ge 0$, support $\eta \subset B(0, \frac{1}{2})$ and $\int_{R^{n+1}} \eta(x)dx = 1$ (note the use of R^{n+1} in place of R^n here). With $\eta_r(x) = r^{-n}\eta\left(\frac{x}{r}\right)$, it follows easily from (2.65) that

$$Re(\eta_r * \mathcal{K}'_\gamma \phi_r f)(x,t) \ge cr^{-\gamma} \int_\alpha^\beta a(w,t')^{p'} dt'$$

for $|x-w| < cr$, $t \in (0,\beta)$, with a perhaps smaller constant c— note that $r \le \beta/3$ if, as we may assume, $a(x,t) \le 1/3$. If we use the inequality $\|\eta_r * g\|_{L^p} \le C\,\|g\|_{\mathcal{B}_p^{0,p}}$, valid for any $g \in \mathcal{B}_p^{0,p}(R^{n+1})$, $1 < p < \infty$, (this is (2.12) with R^{n+1} in place of R^n), the $\mathcal{P}_{p\gamma}$ condition in the open set $U_{R'}$ can be derived as in the proof of Theorem 2.5.

2.4. Boundedness of K in Hölder Spaces

We study K when a satisfies the $\mathcal{T}_\nu$ condition:

$$|t-t'| \leq C\left|\int_{t'}^{t} a(x,\theta)d\theta\right|^{\nu} \tag{2.66}$$

for all $x \in R^n$ and all $t, t' \in R$ with $tt' \geq 0$. Denote the unit ball in R^n by B. Using the argument in [4] we have the following lemma.

Lemma 2.2. *Suppose $\nu > 0$ and a satisfies the $\mathcal{T}_\nu$ condition* (2.66). *Then there is a constant C_ν such that for every $\epsilon > 0$,*

$$\int_0^t \int_{|x-x'|<\epsilon} |K(x,x',t,t')|\, dx'dt' \leq C_\nu \epsilon^\nu, \tag{2.67}$$

$$\int_0^t \int_B |K(x+h,x',t,t') - K(x,x',t,t')|\, dx'dt' \leq C_\nu \left|h\right|^\nu, \tag{2.68}$$

and if in addition $a(x,t)a(x,t+\eta) \geq 0$, then

$$\int_t^{t+\eta} \int_B |K(x,x',t,t')|\, dx'dt' \leq C_\nu \left|\eta\right| \ln \left|\eta\right|^{-1}, \tag{2.69}$$

$$\int_0^t \int_B |K(x,x',t+\eta,t') - K(x,x',t,t')|\, dx'dt' \leq C_\nu \left|\eta\right| \ln \left|\eta\right|^{-1}. \tag{2.70}$$

If we apply the argument used in the proof of Corollary 1.2, we now obtain

Theorem 2.7. *If $\lambda > 0$, and in addition a satisfies the $\mathcal{T}_\nu$ condition* (2.66), *then K maps Λ^s to $\Lambda^{s+\nu}$ for all $s > 0$ with $s + \nu \leq \lambda + 2$.*

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