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ON ABELIAN AUTOMORPHISM GROUPS OF SURFACES OF GENERAL TYPE

Cai Jinxing*

Abstract

It is proved that for a complex minimal smooth projective surface S of general type, its abelian automorphism group is of order $\leq 36K_S^2+24$, provided $\chi(\mathcal{O}_S)\geq 8$, where K_S is the canonical divisor of S, and $\chi(\mathcal{O}_S)$ the Euler characteristic of the structure sheaf of S.

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§1. Introduction

It is well-known that for a complex curve of genus $g\geq 2$, its total automorphism group (resp. abelian automorphism group) is of order $\leq 84(g-1)=42\deg K_C$ (resp. $\leq 4g+4$), where K_C is the canonical divisor of C (cf. [3, 4]). It is an intriguing problem to generalise these bounds to higher dimensions. Several authors have studied this problem (see [5, 6] for details). Recently, Xiao has generalised these results to surfaces of general type, in good analogy with the case of curves. He has proved that for a complex minimal smooth projective surface S of general type, its total automorphism group (resp. abelian automorphism group) is of order $\leq 42^2K_S^2$ (resp. $\leq 52K_S^2+32$, provided $K_S^2\geq 140$), where K_S is the canonical divisor of S (cf. [5, 6, 7]). Clearly, the coefficient 42^2 is the best for the total automorphism group; but the coefficient 52 does not seem to be best possible as Xiao pointed out in [5]. In this paper, we improve on this result. Our main result is the following

Theorem 1.1. *Let S be a minimal smooth surface of general type over the complex number field, K the canonical divisor of S, and $\chi(\mathcal{O}_S)$ the Euler characteristic of the structure sheaf of S. Let G be an abelian group of automorphisms of S (i.e., $G\subset \mathrm{Aut}(S)$). Then $\#G\leq 36K^2+24$, provided $\chi(\mathcal{O}_S)\geq 8$.*

The arguments here are inspired by the work of Xiao[5]. We consider the natural action of the abelian group G on the space $H_n=H^0(S,\ nK_S)$, for a fixed positive integer n. Because G is finite abelian, such an action is diagonalisable, in other words H_n has a basis consisting of semi-invariant vectors. Consider two such semi-invariants $v_1,\ \ v_2$ in H_n, with

$$\sigma(v_i)=\alpha_i(\sigma)v_i \ \text{ for } \ \sigma\in G,$$

where α_i are the corresponding characters of G. Suppose that the two semi-invariants v_1,v_2 correspond to the same character of G (i.e., $\alpha_1=\alpha_2$), and let D_1 and D_2 be the

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*Department of Mathematics, East China Normal University, Shanghai 200062, China

corresponding divisors in $|nK_S|$. Then D_1 and D_2 generate a pencil Λ whose general fibre F is fixed by G. Therefore $\#G$ is limited by the order of the group of automorphisms of the normalijation $\widetilde{F}$ of F as a smooth curve. But $\#\,\mathrm{Aut}(\tilde{F})$ increases proportionally with K_S^2, as $g(\widetilde{F})$ so does.

Instead of considering the natural map

$$H_n \otimes H_n \to H_{2n},$$

we may consider the natural map

$$H_{n-t} \otimes H_{n+t} \oplus H_n \otimes H_n \to H_{2n},$$

which is compatible with the above actions of G, i.e., if $v_1 \in H_{n-t}$, $v_2 \in H_{n+t}$ (resp. $w_i \in H_n$, $i = 1,\ 2$) are two semi-invariants, then $v_1 \otimes v_2$ (resp. $w_1 \otimes w_2$) is semi-invariant in H_{2n}. If there are more than dim H_i semi-invariants in H_i for some $i \leq n+t$, then there are semi-invariants in H_i (therefore in H_{2n}) with the same character, and we are done. So we may assume that there are exactly $\delta_i = \dim H_i$ $(i \leq n+t)$ semi-invariants v_j^i $(j = 1, \cdots, \delta_i)$ in H_i, corresponding to mutually different characters. Each vector v_j^i corresponds to a unique divisor D_j^i in $|iK|$. The relation $v_j^i \otimes v_k^l = cv_m^r \otimes v_n^s$ (where c is a constant, and $i + l = r + s = 2n$) in H_{2n} translates into a relation

$$D_j^i + D_k^l = D_m^r + D_n^s \qquad (*)$$

between these divisors.

Fix a semi-invariant $u \in H_t$ when $H_t \neq 0$. Then u corresponds to a unique divisor U in $|tK|$ which is fixed by G. We can consider the finite set Σ_i of points corresponding to D_j^i in a certain divisorial space P_i defined in [5, §1], and there are natural embeddings:

$$\Sigma_{n-t} \to \Sigma_n \to \Sigma_{n+t}, \qquad P_{n-t} \overset{l_2}{\to} P_n \overset{l_3}{\to} P_{n+t}$$

defined by U. In such a setting, a semi-invariant in H_{2n} of the form $v_j^i \otimes v_k^l$ corresponds naturally to the mid-point of two points in Σ_{n+t} corresponding to D_j^i and D_k^l, and a relation of the form $(*)$ means that the corresponding mid-points coincide.

Denote by $\mathcal{S}_1$ the set of mid-points of two points p, q in Σ_{n+t} such that either p is in Σ_{n-t} and q is in Σ_{n+t} or p and q are in Σ_n. Now the problem has been reduced to that of comparing the number of points in $\mathcal{S}_1$ and the dimension of H_{2n}. In this way we show that for $n = 3$ and $t = 1$ the number of points in $\mathcal{S}_1$ is larger than the dimension of H_6, provided that $\chi(\mathcal{O}_S) \geq 8$ and S has no pencils of curves of genus 2. The case that S has pencils of curves of genus 2 has been studied completely by using the properties of pencils of curves of genus 2 (cf. [2, 5]).

The estimation in Theorem 1.1 appears to be crude (cf. [5, Example 1], [2, Example 5.7]). An interesting question is, roughly speaking: what is asymptotically the best upper bound for G? I hope to return to this subject in a subsequent paper.

§2. Proof of Theorem 1.1

We fix a smooth complex projective minimal surface of general type S in the future, and let K be the canonical divisor of S, $H_n = H^0(S,\ nK)$, and $\chi = \chi(\mathcal{O}_S)$. We also fix an abelian group G of automorphisms of S.

For the reader's convenience, we recall some notation defined in [5].

Definition 2.1.[5] *Let $v_1, \cdots, v_{\delta_n}$ be a basis of H_n consisting of semi-invariants for the natural action of G on H_n, and $D_1, \cdots, D_{\delta_n}$ the divisors in $|nK|$ corresponding to these vectors, where $\delta_n = \dim H_n$. We say that H_n is uniquely decomposable (under the action of G) if the set $\{D_i\}$ is uniquely determined, or equivalently if there are exactly δ_n different characters for the natural action of G on H_n.*

Fix the divisor D_1. Denote by P'_n the set

$$\{\mathbb{Q}\text{-divisors } D \text{ on } S \mid \text{there is an } m \in \mathbb{Z}^+ \text{ such that } mD \text{ is linearly equivalent to } mD_1\}.$$

Denote by $[D]$ the element of P'_n corresponding to the $\mathbb{Q}$-divisor D. We define addition and scalar multiplication as follows:

$$[D] + [D'] = [D + D' - D_1],$$
$$c[D] = [cD + (1-c)D_1], \quad c \in \mathbb{Q}.$$

Then P'_n is a generally infinite dimensional linear space, with $[D_1]$ as the origin.

The subset I in P'_n of points corresponding to the integral divisors linearly equivalent to D_1 is an additive subgroup, and there is a set of generators of I which form a basis of P'_n. Under such a basis, I is a subset of points with integral coordinates.

Denote by P_n the finite dimensional subspace generated by the set

$$\{[D_1],\ [D_2], \cdots, [D_{\delta_n}]\}.$$

Let Σ_n be the finite set in P_n consisting of the points corresponding to the effective divisors in $|nK|$ fixed by G. Then P_n, therefore Σ_n, is uniquely determined up to the choices of D_1 only if H_n is uniquely decomposable. We will call the set Σ_n a basic set in P_n.

Clearly, P_n depends on the choice of D_1; but if we replace D_1 by another divisor, say D_i, P_n differs only by an integral translation. Because $\#\Sigma_n$ and the number of middle points of Σ_n, which are all the properties about Σ_n we use, are integral translation invariants, it dose not matter which D_i is selected. Also, Σ_n is determined up to integral translations as above iff H_n has exactly δ_n semi-invariants, and thus iff there are δ_n different characters for the action of G on H_n.

Assume $H_1 \neq 0$. Fix a semi-invariant $u \in H_1$ for the natural action of G on H_1. Let U be the divisor in $|K|$ corresponding to u. We have natural maps:

$$H_2 \overset{\otimes u}{\to} H_3 \overset{\otimes u}{\to} H_4; \quad |2K| \overset{+U}{\to} |3K| \overset{+U}{\to} |4K|; \quad P'_2 \overset{l'_2}{\to} P'_3 \overset{l'_3}{\to} P'_4.$$

If we take $[nU]$ to be the origin of P'_n, then $l'_i (i = 2, 3)$ are embeddings of linear spaces. We will identify P'_2 and P'_3 as subspaces of P'_4 in this way in the future. Since U is fixed by G, l'_i $(i = 2,\ 3)$ induce natural embeddings:

$$\Sigma_2 \to \Sigma_3 \to \Sigma_4, \qquad P_2 \overset{l_2}{\to} P_3 \overset{l_3}{\to} P_4.$$

Let v (resp. w) be a semi-invariant in H_2 (resp. H_4), and p (resp. q) the corresponding point in Σ_2 (resp. Σ_4). Let D be the divisor in $|6K|$ corresponding to the vector $v \otimes w$. Then $l_3([\frac{1}{2}D])$ corresponds to a point in P_4, which is just the mid-point $\frac{1}{2}(l_3 l_2(p) + q)$.

Similarly, let $v_1,\ v_2$ be two semi-invariants in H_3, and p, q the corresponding points in Σ_3.

Let D be the divisor in $|6K|$ corresponding to the vector $v_1 \otimes v_2$. Then $l_3([\frac{1}{2}D])$ corresponds to a point in P_4, which is just the mid-point $\frac{1}{2}(l_3(p)+l_3(q))$.

Definition 2.2.[5] *Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets of points in a linear space P. We define $\mathcal{A}.\ \mathcal{B}$ to be the set of mid-points $\frac{1}{2}(p+q)$ of two points p in $\mathcal{A}$ and q in $\mathcal{B}$ (p and q may be the same point if $\mathcal{A}\cap\mathcal{B}\neq\emptyset$; so $\mathcal{A}\cap\mathcal{B}\subset\mathcal{A}.\mathcal{B}$).*

We define the dimension of $\mathcal{A}$ to be the dimension of the (affine) space generated by $\mathcal{A}$. Let $\mathcal{A}$ be a finite set of points in P, and $\mathcal{B}$ a subset of $\mathcal{A}$. The set $\mathcal{B}$ is said to be relatively convex in $\mathcal{A}$, if no point of $\mathcal{A}-\mathcal{B}$ is contained in the convex hull of $\mathcal{B}$. The set $\mathcal{B}$ is called integrally convex if it is relatively convex in some lattice $\mathcal{A}$ generating P. With such a lattice $\mathcal{A}$ fixed, we will call the points in $\mathcal{A}$ integral points.

A chain in a set $\mathcal{B}$ is by definition a series of points $p_1,\cdots,p_n$ in $\mathcal{B}$ such that the vectors p_i-p_{i-1} $(i=2,\cdots,n)$ are equal. In this case, n is called the length of the chain. If $\mathcal{B}$ is integrally convex (in a fixed lattice $\mathcal{A}$) and p, q are two points in $\mathcal{B}$, then the integral points on the line segment joining p and q form a chain in $\mathcal{B}$ in an obvious way.

Remark. The set Σ_n is integrally convex with respect to the lattice $\mathcal{L}$ consisting of the points corresponding to divisors linearly equivalent to nK (cf. [5, Lemma 3]). Clearly it is easy to verify that Σ_i is relatively convex in Σ_{i+1} for $i=2$, 3, as we consider Σ_i as a subset of Σ_{i+1} in the above way.

Lemma 2.1. *Suppose that $\chi\geq 5$, and that P_6 is uniquely decomposable. Then the dimension of Σ_3 is* 3.

Proof. Let d be the dimension of Σ_3. Then

$$\#(\Sigma_3.\Sigma_3)\geq (d+1)\Big(\#\Sigma_3-\frac{d}{2}\Big)\quad \text{(cf. [5, Lemma 1]).}$$

Now assume $d\geq 4$. Then taking into account the inequality $\#\Sigma_3\geq d+1$, we have

$$\#(\Sigma_3.\Sigma_3)\geq 5\#\Sigma_3-10.$$

On the other hand, we have

$$\dim H_6=15K^2+\chi,\quad \dim H_3=3K^2+\chi.$$

So we get $\#(\Sigma_3.\Sigma_3)\geq 5\dim H_3-10>\dim H_6$ when $\chi\geq 3$, a contradiction. Hence we have $d\leq 3$. Now $d\leq 2$ is impossible, otherwise the image of the 3-canonical map of S is either a rational curve or a rational surface (cf. [5, Lemma 3]).

We may copy the proof of [5, Lemma 4] to get the following

Lemma 2.2. *Suppose $K^2\geq 10$, and let Σ_i be a basic set in H_i for $i=3$, 4. Assume that either there is a chain in Σ_4 of length $\geq\frac{1}{6}\#\Sigma_4$ or there is a chain in Σ_3 of length $\geq\frac{1}{4}\#\Sigma_3$. Then S has a pencil of curves of genus* 2.

Lemma 2.3. *Let $\mathcal{A}_i$ $(i=1,\ 2,\ 3)$ be finite integral sets in a $\mathbb{Q}$-linear space P with $\mathcal{A}_1\subset\mathcal{A}_2\subset\mathcal{A}_3$. Let P_2 be the enveloping space of $\mathcal{A}_2$. Assume* $\dim P_2<\dim P$. *Then there exists an integral linear map φ (i.e., φ maps integral points to integral points) from P to P_2 such that*

(i) *$\varphi|_{\mathcal{A}_3}$ is injective, and $\varphi|_{P_2}$ is identity;*

(ii) $\#(\mathcal{A}_1.\mathcal{A}_3\cup\mathcal{A}_2.\mathcal{A}_2)\geq\#(\mathcal{A}_1.\varphi(\mathcal{A}_3)\cup\mathcal{A}_2.\mathcal{A}_2)$.

Proof. Let $x_1,\ \ldots,\ x_n$ be a (integral) basis of P. We can suppose that $\dim P_2=\dim P-1$, and P_2 is the hyperplane $x_n=0$. Since $\mathcal{A}_3$ is a finite set, we can choose an integer t such that $|a_i|+|b_i|<t$ $(i=1,\cdots,n)$ for any two points $(a_1,\cdots,a_n)$, $(b_1,\cdots,b_n)$ in $\mathcal{A}_3$. We define

$$\varphi:\quad P\to P_2$$
$$(x_1,\ x_2,\cdots,x_n)\mapsto(x_1+tx_n,\ x_2+tx_n,\cdots,\ x_{n-1}+tx_n,\ 0).$$

Clearly, φ satisfies (i) by the choice of t. Now the mid-point of two points p in $\mathcal{A}_1$ and q in $\varphi(\mathcal{A}_3)$ is the image of the mid-point of two points p in $\mathcal{A}_1$ and $\varphi^{-1}(q)$ in $\mathcal{A}_3$, and if $\frac{1}{2}(p_1+q_1)=\frac{1}{2}(p_2+q_2)$ for p_i in $\mathcal{A}_1$ and q_i in $\mathcal{A}_3$, then

$$\frac{1}{2}(p_1+\varphi(q_1))=\frac{1}{2}(p_2+\varphi(q_2)).$$

Hence φ satisfies (ii).

Definition 2.3.[5] *Let $\mathcal{A}$ be a finite integral set in a space P with a basis $x_1,\cdots,x_n$. The arrangement of $\mathcal{A}$ with respect to x_i is a new integral set $\mathcal{A}'$ such that a point $(a_1,\cdots,\ a_n)$ is contained in $\mathcal{A}'$ iff*

(i) *$a_i\ge 0$, and*

(ii) *there are at least a_i+1 points $(b_1,\cdots,b_n)$ in $\mathcal{A}$ such that $a_j=b_j$ for all $j\ne i$.*

It is immediate that $\mathcal{A}'$ is integral, and has the same cardinality and dimension as that of $\mathcal{A}$. But if $\mathcal{A}$ is convex, $\mathcal{A}'$ need not be convex. Clearly, if $\mathcal{A}_i$ $(i=1,\ 2,\ 3)$ are finite integral sets in P with $\mathcal{A}_1\subset\mathcal{A}_2\subset\mathcal{A}_3$, then by the definition we have

$$\mathcal{A}'_1\subset\mathcal{A}'_2\subset\mathcal{A}'_3.$$

Lemma 2.4. *Let $\mathcal{A}_i$ $(i=1,2,3)$ be finite integral sets of dimension 3 with $\mathcal{A}_1\subset\mathcal{A}_2\subset\mathcal{A}_3$, and let $\mathcal{A}'_i$ be the arrangement of $\mathcal{A}_i$ with respect to a coordinate axis. Then*

$$\#(\mathcal{A}_1.\mathcal{A}_3\cup\mathcal{A}_2.\mathcal{A}_2)\ge\#(\mathcal{A}'_1.\mathcal{A}'_3\cup\mathcal{A}'_2.\mathcal{A}'_2).$$

Proof. For the sake of simplicity, let $i=1$, and use x, y, z instead of x_1, x_2, x_3. Take any two integers y_m, z_m, and assume that there are k points in $\mathcal{A}'_1.\mathcal{A}'_3\cup\mathcal{A}'_2.\mathcal{A}'_2$ with $y=y_m$, $z=z_m$. Then there exist two points $p=(x_1,\ y_1,\ z_1)$ and $q=(x_2,\ y_2,\ z_2)$ in $\mathcal{A}'_3$ whose mid-point is $(\frac{1}{2}(k-1),\ y_m,\ z_m)$. Now $(\frac{1}{2}(k-1),\ y_m,\ z_m)$ is either in $\mathcal{A}'_1.\mathcal{A}'_3$ or in $\mathcal{A}'_2.\mathcal{A}'_2$. Hence we have either $p\in\mathcal{A}'_1$, $q\in\mathcal{A}'_3$ or $p,\ q\in\mathcal{A}'_2$. Now we suppose $p\in\mathcal{A}'_1$, $q\in\mathcal{A}'_3$. (For the latter case, the proof is similar). By the definition of arrangement, this means that there are at least x_1+1 points with $y=y_1$, $z=z_1$ in $\mathcal{A}_1$ and at least x_2+1 points with $y=y_2$, $z=z_2$ in $\mathcal{A}_3$. Because $x_1+x_2+1=k$, we see that the points in $\mathcal{A}_1$ with $y=y_1$, $z=z_1$ and the points in $\mathcal{A}_3$ with $y=y_2$, $z=z_2$ produce at least k mid-points with $y=y_m$, $z=z_m$ in $\mathcal{A}_1.\mathcal{A}_3$. In fact, let the x_1+1 (resp. x_2+1) points in the first (resp. second) row of $\mathcal{A}_1$ (resp. $\mathcal{A}_3$) be $p_1,\cdots,p_{x_1+1}$ (resp. $q_1,\cdots,q_{x_2+1}$) such that if $i<j$ then the x coordinate of p_i (resp. q_i) is less than p_j (resp. q_j). Then the mid-points

$$\frac{1}{2}(p_1+q_1),\ \frac{1}{2}(p_1+q_2),\cdots,\frac{1}{2}(p_1+q_{x_2+1}),\ \frac{1}{2}(p_2+q_{x_2+1}),\cdots,\frac{1}{2}(p_{x_1+1}+q_{x_2+1})$$

form the desired subset of $\mathcal{A}_1.\mathcal{A}_3$.

Lemma 2.5. *Let $\mathcal{A}_i$ $(i=1,2,3)$ be finite integrally convex sets with $\mathcal{A}_1\subset\mathcal{A}_2\subset\mathcal{A}_3$, and the dimension of $\mathcal{A}_1$ is 3. Suppose that the length of the longest chain in $\mathcal{A}_3$ (resp. $\mathcal{A}_2$) is less then $\frac{1}{6}\#\mathcal{A}_3$ (resp. $\frac{1}{4}\#\mathcal{A}_2$). And suppose*

$$\#\mathcal{A}_2\ge 21,\quad \#\mathcal{A}_3\le 2\#\mathcal{A}_2.$$

Then

$$\#(\mathcal{A}_1.\mathcal{A}_3 \cup \mathcal{A}_2.\mathcal{A}_2) \geq \min \begin{cases} \#\mathcal{A}_3 + 3\#\mathcal{A}_2 - 23, \\ \frac{5}{6}\#\mathcal{A}_3 + \frac{10}{3}\#\mathcal{A}_2 - 10, \\ \frac{5}{6}\#\mathcal{A}_3 + \frac{13}{4}\#\mathcal{A}_2 - 2, \\ \frac{7}{12}\#\mathcal{A}_3 + \frac{15}{4}\#\mathcal{A}_2 - 6, \\ \frac{1}{2}\#\mathcal{A}_3 + 4\#\mathcal{A}_2 - 4, \\ 5\#\mathcal{A}_2 - 31. \end{cases}$$

Proof. By Lemma 2.3, we can suppose that the dimension of the enveloping space of $\mathcal{A}_3$ is 3. Let $p_1, \cdots, p_l$ be a longest chain in $\mathcal{A}_2$. Assume that p_1 is the origin of the enveloping space, and that $p_2 = (1,\ 0,\ 0)$. Let x, y, z be this basis. Then

$$\sqrt[3]{\#\mathcal{A}_2} \leq l \leq \frac{1}{4}\#\mathcal{A}_2.$$

Arrange $\mathcal{A}_n$ $(i = 1, \cdots, 3)$ with respect to x-axis, then with respect to y, and then to z. By Lemma 2.4, we need only to count the number of points in the set of mid-points of the new set $\#(\mathcal{A}'_1.\mathcal{A}'_3 \cup \mathcal{A}'_2.\mathcal{A}'_2)$ thus produced.

Let m_{xi} (resp. m_{yi}, m_{zi}) be the number of points of $\mathcal{A}'_i$ on the x (resp. y, z) axis, $i = 1,\ 2,\ 3$. We have $m_{x2} = l \leq \frac{1}{4}\#\mathcal{A}_2$ (cf. [5, p. 625]) and $m_{x3} \leq \frac{1}{6}\#\mathcal{A}_3$, $m_{x1} \leq m_{x2} \leq m_{x3}$, etc. If the basis elements y and z are chosen carefully, $\mathcal{A}'_1$ will also be of dimension 3. We may assume that the points of $\mathcal{A}'_3$ have either $z = 0$, or $z = 1$, $y = 0$ (cf. [5, p. 626]).

Now for simplicity of the notation we replace $\mathcal{A}'_i$ by $\mathcal{A}_i$. Remark that $(0,\ 0,\ 1)$ is in $\mathcal{A}_1$, and although $\mathcal{A}_i$ is not convex now, it has the property that if a point $(a,\ b,\ c)$ is in $\mathcal{A}_i$, then all integral points $(a',\ b',\ c')$ such that

$$0 \leq a' \leq a, \quad 0 \leq b' \leq b, \quad 0 \leq c' \leq c$$

are in $\mathcal{A}_i$ too.

Let $\mathcal{A}_{i0}$ be just the subset of points of $\mathcal{A}_i$ in the plane $z = 0$. Let t_i $(i = 1,\ 2,\ 3)$ be the number of points in $\mathcal{A}_i$ with $z = 1$. Clearly,

$$1 \leq t_1 \leq t_2 \leq t_3 \leq m_{x3} \leq \frac{1}{6}\#\mathcal{A}_3.$$

In what follows we denote by $\mathcal{S}$ the set $\mathcal{A}_1.\mathcal{A}_3 \cup \mathcal{A}_2.\mathcal{A}_2$.

Consider the mid-points $\frac{1}{2}(r+s)$ with $r,\ s \in \mathcal{A}_2$ and r on the x-axis and s on the y-axis. Two of such points $\frac{1}{2}(r_1+s_1)$ and $\frac{1}{2}(r_2+s_2)$ are different if $r_1 \neq r_2$ or $s_1 \neq s_2$. Consider the mid-points $\frac{1}{2}(p+q)$ with $p = (0,\ 0,\ 1) \in \mathcal{A}_1$ and q in the plane $z = 0$. Therefore $\mathcal{S}$ contains at least $m_{x2}m_{y2} + \#\mathcal{A}_{30}$ points. Hence we can assume $m_{x2}m_{y2} \leq \frac{13}{4}\#\mathcal{A}_2$. Consequently,

$$m_{x2} + m_{y2} \leq \frac{1}{4}\#\mathcal{A}_2 + 12, \tag{2.1}$$

as we have $\sqrt[3]{\#\mathcal{A}_2} \leq m_{x2} \leq \frac{1}{4}\#\mathcal{A}_2$, $m_{y2} \leq \frac{1}{4}\#\mathcal{A}_2$.

First, we have

$$\#(\mathcal{A}_{20}.\mathcal{A}_{20}) = 4\#\mathcal{A}_{20} - 2(m_{x2} + m_{y2}) + 1, \tag{2.2}$$

$$\#\mathcal{S} \geq \#(\mathcal{A}_2.\mathcal{A}_2) = (t_2 - 1)(m_{y2} - 3) + 5\#\mathcal{A}_2 - 2(m_{x2} + m_{y2}) - 3. \tag{2.3}$$

See [5, Lemma 6] for the proofs of (2.2) and (2.3).

Second, we estimate $\#\mathcal{S}$ by considering not only the set $\mathcal{A}_2.\mathcal{A}_2$ but also the set $\mathcal{A}_1.\mathcal{A}_3$.

Since $p := (0,\ 0,\ 1)$ is in $\mathcal{A}_1$, the mid-points in

$$\mathcal{B}_1 := \left\{\frac{1}{2}(p+q), q \text{ is in } \mathcal{A}_{30}\right\}$$

are in $\mathcal{A}_1.\mathcal{A}_3$; clearly, the mid-points in $\mathcal{B}_2 := \mathcal{A}_{30}.\mathcal{A}_{30}$ and

$$\mathcal{B}_3 := \left\{\frac{1}{2}((a,\ 0,\ 1)+(b,\ 0,\ 1)), 0 \le a,\ b \le t_2-1, a,\ b \in \mathbb{Z}\right\}$$

are in $\mathcal{A}_2.\mathcal{A}_2$. It is easy to see that $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for $i \ne j$. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then

$$\#\mathcal{B} = \#\mathcal{A}_{30} + \#(\mathcal{A}_{20}.\mathcal{A}_{20}) + 2t_2 - 1.$$

By (2.1) and (2.2), we have

$$\#\mathcal{S} \ge \#\mathcal{B} = \#\mathcal{A}_3 + 4\#\mathcal{A}_2 - 2(m_{x2}+m_{y2}) - t_3 - 2t_2 \tag{2.4}$$

$$\ge \#\mathcal{A}_3 + \frac{7}{2}\#\mathcal{A}_2 - t_3 - 2t_2 - 24. \tag{2.5}$$

Now we consider the following cases separately.

Case I1. $\frac{1}{2}t_3 < t_2 \le \frac{1}{8}\#\mathcal{A}_2$. By (2.5), we have

$$\#\mathcal{S} \ge \#\mathcal{A}_3 + \frac{7}{2}\#\mathcal{A}_2 - 4t_2 - 23 \ge \#\mathcal{A}_3 + 3\#\mathcal{A}_2 - 23.$$

Case I2. $t_2 \le \min\{\frac{1}{2}t_3,\ \frac{1}{8}\#\mathcal{A}_2\}$. Let $a = t_3 - 2t_2 + 1$. Since $p := (0,\ 0,\ 1)$ is in $\mathcal{A}_1$, the mid-points in

$$\mathcal{C} := \left\{\frac{1}{2}(p+(t_3-1,\ 0,\ 1)), \frac{1}{2}(p+(t_3-2,\ 0,\ 1)), \cdots, \frac{1}{2}(p+(t_3-a,\ 0,\ 1))\right\}$$

are in $\mathcal{S}$; it is clear that $\mathcal{B} \cap \mathcal{C} = \emptyset$, and $\#\mathcal{C} = a = t_3 - 2t_2 + 1$.

Now by (2.5) we have

$$\#\mathcal{S} \ge \#\mathcal{B} + \#\mathcal{C} \ge \#\mathcal{A}_3 + 3\#\mathcal{A}_2 - 23.$$

Case II1. $t_2 \ge \frac{1}{8}\#\mathcal{A}_2$ and $m_{y2} \ge 7$. By (2.1) and (2.3), we have

$$\#\mathcal{S} \ge 4(t_2-1) + \frac{9}{2}\#\mathcal{A}_2 - 27 \ge 5\#\mathcal{A}_2 - 31.$$

Case II2. $t_2 \ge \frac{1}{8}\#\mathcal{A}_2$ and $m_{y2} \le 6$. Note that we have $m_{y2} \ge 3$ since $m_{x2} \le \frac{1}{4}\#\mathcal{A}_2$. Let $e = m_{y2}m_{x2} + t_2 - \#\mathcal{A}_2 \ge 0$, and let n_i be the number of the points of $\mathcal{A}_2$ with $y = i,\ z = 0$. Clearly, $m_{x2} = n_0$. The mid-points in

$$\mathcal{E} := \Big\{\frac{1}{2}((n_i-1,\ i,\ 0)+(n_{i+1}-1,\ i+1,\ 0)), \cdots, \frac{1}{2}((n_{i+1},\ i,\ 0)+(n_{i+1}-1,\ i+1,\ 0)), \text{provided } n_i > n_{i+1},\ i = 0, \cdots, 3.\Big\}$$

are in $\mathcal{S}$, $\mathcal{B} \cap \mathcal{E} = \emptyset$, and $\#\mathcal{E} = e$. By (2.4), we have

$$\#\mathcal{S} \ge \#\mathcal{B} + \#\mathcal{E} \ge \#\mathcal{A}_3 + 4\#\mathcal{A}_2 - 2(m_{x2}+m_{y2}) - t_3 - 2t_2 + e. \tag{2.6}$$

Case II2.1. $m_{y2} = 5$ or 6. If $m_{y2} = 6$, then $m_{x2} \ge \frac{1}{7}(\#\mathcal{A}_2 + e)$. By (2.6), we have

$$\#\mathcal{S} \ge \#\mathcal{A}_3 + 2\#\mathcal{A}_2 + 10m_{x2} - t_3 - e - 12$$

$$\ge \frac{5}{6}\#\mathcal{A}_3 + \frac{24}{7}\#\mathcal{A}_2 - 12 \ge \frac{5}{6}\#\mathcal{A}_3 + \frac{10}{3}\#\mathcal{A}_2 - 10.$$

Similarly, if $m_{y2}=5$, then we have $\#\mathcal{S}\geq\frac{5}{6}\#\mathcal{A}_3+\frac{10}{3}\#\mathcal{A}_2-10$.

Case II2.2. $m_{y2}=4$. By (2.6), we have

$$\#\mathcal{S}\geq\frac{5}{6}\#\mathcal{A}_3+\frac{13}{4}\#\mathcal{A}_2-2+\frac{e}{4},$$

provided $m_{x2}\geq\frac{5}{24}(\#\mathcal{A}_2+e)+1$. This allows us to assume

$$\frac{1}{5}(\#\mathcal{A}_2+e)\leq m_{x2}\leq\frac{5}{24}(\#\mathcal{A}_2+e).$$

Therefore, $t_2\geq\frac{1}{6}(\#\mathcal{A}_2+e)$. Let w be the integer part of $\frac{1}{12}\#\mathcal{A}_3$. Then the point $p=(w-1,\ 0,\ 1)$ is in $\mathcal{A}_2$ since $\#\mathcal{A}_3\leq 2\#\mathcal{A}_2$ by the hypothesis, and the mid-points in

$$\mathcal{F}:=\left\{\frac{1}{2}(p+(a,\ b,\ 0)),\ a\in\mathbb{Z},\ a\geq w-1,\ b=0,\cdots,\ 3,(a,\ b,\ 0)\text{ is in }\mathcal{A}_{20}\right\}$$

are in $\mathcal{S}$, $\mathcal{B}\cap\mathcal{F}=\emptyset$, and

$$\#\mathcal{F}=4(m_{x2}-w+1)-e\geq 4(m_{x2}-\frac{1}{12}\#\mathcal{A}_3+1)-e.$$

Now by (2.6) we have

$$\begin{aligned}\#\mathcal{S}&\geq\#\mathcal{A}_3+4\#\mathcal{A}_2-2(m_{x2}+m_{y2})-t_3-2t_2+e+\#\mathcal{F}.\\&\geq\frac{1}{2}\#\mathcal{A}_3+2\#\mathcal{A}_2+10m_{x2}-2e-4\geq\frac{1}{2}\#\mathcal{A}_3+4\#\mathcal{A}_2-4.\end{aligned}$$

Case II2.3. $m_{y2}=3$. In this case $m_{x2}=t_2=\frac{1}{4}\#\mathcal{A}_2$. The mid-points in

$$\mathcal{G}:=\left\{\frac{1}{2}((a,\ 0,\ 1)+(a,\ b,\ 0)),\quad\text{where}\quad\frac{1}{12}\#A_3\leq a\leq\frac{1}{4}\#\mathcal{A}_2,\ a\in\mathbb{Z},\ b=0,\cdots,\ 2.\right\}$$

are in $\mathcal{S}$, $\mathcal{B}\cap\mathcal{G}=\emptyset$ since $2a>m_{x2}$, and $\#\mathcal{G}=3(\frac{1}{4}\#\mathcal{A}_2-\frac{1}{12}\#\mathcal{A}_3)$. Then by (2.5), we have

$$\#\mathcal{S}\geq\#\mathcal{B}+\#\mathcal{G}\geq\frac{7}{12}\#\mathcal{A}_3+\frac{15}{4}\#\mathcal{A}_2-6.$$

Summing up above inequalities, we get what we wanted.

Corollary 2.1. *Suppose that $\chi\geq 8$ and S has no pencils of curves of genus 2. Then H_6 is not uniquely decomposable.*

Proof. We can suppose H_n is uniquely decomposable for $n\leq 4$, for otherwise the corollary is trivially true. Let Σ_i be a basic set in P_i ($i=2,\ 3,\ 4$). By Lemma 2.2 and Lemma 2.3, the condition of Lemma 2.5 is satisfied for $\Sigma_2\subset\Sigma_3\subset\Sigma_4$. Then this corollary results from the relations

$$\begin{aligned}\#\Sigma_i&=\dim H_i=\frac{i(i-1)}{2}K^2+\chi\quad(i=2,\ 3,\ 4),\\K^2&\leq 9\chi\quad(\text{ Bogomolov-Miyaoka-Yau's inequality}),\end{aligned}$$

that $\#(\Sigma_2.\Sigma_4\cup\Sigma_3.\Sigma_3)>\dim H_6$. In particular, there are more than dim H_6 semi-invariants in H_6.

The following lemma is a modification of [5, Lemma 7].

Lemma 2.6. *Suppose that $K^2\geq 10$ and S has no pencils of curves of genus 2. Let G be an Abelian group of automorphisms of S such that $|6K|$ contains a pencil Λ whose general members are fixed under the action of G. Then*

$$\#G\leq 36K^2+24.$$

Proof. Blowing up the base points of Λ, we get a surface S' such that Λ is associated to a fibration $f: S' \to C$. Let F be a general fibre of f, and let k be the number of fibres of f contained in a general member of the moving part of Λ. We have

$$g(F) - 1 \le \frac{3k+18}{k^2} K^2.$$

Let H be the stabiliser of F. Then the index of H in G is at most k. Because H is an Abelian group of automorphisms of curve F, we have $\#H \le 4g(F) + 4$. Therefore if $k \ge 3$, we get $\#G \le 36K^2 + 24$. This allows us to assume $k \le 2$. Consequently there is no divisor in $|2K|$ whose pull-back on S' contains F.

Let $\pi: F \to B := F/H$ be the projection. Let $O_1, \cdots, O_l$ be the orbits of the action of H on F contained in pull-backs of fixed divisors in $|2K|$, n_i the number of points in O_i. If $l < 4$, then the image of the bicanonical map of S is a rational surface (see e.g. [5, p. 624] for a proof); therefore S has a pencil of curves of genus 2 (cf. [1]), contrary to the hypothesis. So we can assume $l \ge 4$. Using Hurwitz formula to π, we get

$$(l-2)\#H \le 2g(F) - 2 + \sum_{i=1}^{l} n_i.$$

Because $n_i \le 2KF \le 12K^2/k$, we see that $\#G \le 34K^2$, once $l \ge 5$, and $\#G \le 33K^2 + 24$ when $l = 4$ as in [5, Lemma 7].

Proof of Theorem 1.1. If S has a relatively minimal genus 2 fibration, then $\#G \le 12.5K^2 + 100$ (cf. [2, Theorem 0.2]). Hence we can suppose that S has no pencil of curves of genus 2. Then the corollary to Lemma 2.5 guarantees that there is a pencil Λ in $|6K|$ each of whose elements is a fixed divisor by G. Hence by Lemma 2.6 we get the result.

§3. Abelian Subgroups for Small Numerical Invariants

As a consequence of Theorem 1.1, we have that the order of an Abelian subgroup of $\mathrm{Aut}(S)$ is at most $\#G \le 36K^2 + 24$, provided $K^2 \ge 64$. Here we give a similar estimation for surfaces with $K^2 < 64$.

Lemma 3.1. *Suppose that $K^2 \ge 4$ (resp. $K^2 \ge 2$). Then H_{12} (resp. H_{16}) is not uniquely decomposable.*

Proof. We may consider the natural map

$$H_4 \otimes H_8 \oplus H_6 \otimes H_6 \to H_{12} \quad (\text{resp. } H_5 \otimes H_{11} \oplus H_8 \otimes H_8 \to H_{16})$$

instead of

$$H_2 \otimes H_4 \oplus H_3 \otimes H_3 \to H_6.$$

Note that if there is a chain in Σ_6 (resp. Σ_8, Σ_{11}) of length $\ge \frac{1}{4}\#\Sigma_6$ (resp. $\ge \frac{1}{4}\#\Sigma_8$, $\ge \frac{1}{6}\#\Sigma_{11}$) then S has a pencil of curves of genus 1, contradicting the fact that S is of general type (cf. [5, Lemma 4] for a proof). We choose and fix a semi-invariant u' in H_2 (resp. u'' in H_3) instead of u in H_1, as in §1. Then one checks immediately that the first half of §1 goes for the new pair, and the lemma is a counterpart of the corollary to Lemma 2.5.

Theorem 3.1. *Let S be a minimal smooth surface of general type over the complex number field, and K the canonical divisor of S. Let G be an abelian group of automorphisms* *of* S *(i.e.,* $G \subset \mathrm{Aut}(S)$*). Then*

$$\#G \leq \begin{cases} 114K^2 + 24, & \textit{provided } \ 4 \leq K^2 \leq 63, \\ 200K^2 + 22, & \textit{provided } \ 2 \leq K^2 \leq 3, \\ 270, & \textit{provided } \ K^2 = 1. \end{cases}$$

Proof. If $K^2 \geq 4$ (resp. $K^2 \geq 2$), by Lemma 3.1, H_{12} (resp H_{16}) is not uniquely decomposable. We modify the proof of [5, Lemma 7] to get our results. We can assume that there is no divisor in $|3K|$ (resp. $|4K|$) whose pull-back contains a general fibre F of the fibration $f: S' \to C$ induced by a pencil Λ of G-invariant divisors in $|12K|$ (resp. $|16K|$). Let $O_1, \cdots, O_l$ be the orbits on F contained in pull-backs of fixed divisors in the moving parts of the pull-backs of $|3K|$ (resp. $|4K|$). As $K^2 \geq 4$ (resp. $K^2 \geq 2$), the tricanonical map (resp. 4-canonical map) is birational by Bombieri's theorem, and so $l \geq 4$. And as

$$g(F) - 1 \leq \frac{6k+72}{k^2} K^2 \quad \Big(\text{resp. } g(F) - 1 \leq \frac{8k+128}{k^2} K^2\Big),$$

$n_i \leq 36K^2/k$ (resp. $\leq 64K^2/k$), we get $\#G \leq 112K^2$ (resp. $\leq 198K^2$), once $l \geq 5$, and $\#G \leq 114K^2 + 24$ (resp. $\leq 200K^2 + 22$) when $l = 4$ as in the proof of Lemma 2.5.

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