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THE COMPLETE LANGLANDS PARAMETERS CORRESPONDING TO THE REPRESENTATIONS OF THE CLASSICAL GROUPS WITH INTEGRAL REGULAR INFINITESIMAL CHARACTERS***

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Abstract

The authors determine all the complete Langlands parameters corresponding to the representations of the classical groups with integral regular infinitesimal characters, and get all L-packets in that case.

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§1. Complete Langlands Parameters and L-Packets

In this section we review some basic facts from [1]. We denote the sets of real numbers, complex numbers and integers by **R**, **C** and **Z**, respectively.

Suppose that G is a connected complex reductive algebraic group. We have an extended group G^Γ for G, and ${}^\vee G^\Gamma$ is an E-group for G^Γ. We denote the set of Langlands parameters and the set of geometric parameters for ${}^\vee G^\Gamma$ by $P({}^\vee G^\Gamma)$ and $X({}^\vee G^\Gamma)$, respectively. There is a natural map from $P({}^\vee G^\Gamma)$ to $X({}^\vee G^\Gamma)$. It induces a bijection from equivalence classes of Langlands parameters to geometric parameters (see [1, Proposition 6.17]). Write $\Phi({}^\vee G^\Gamma)$ for the set of equivalence classes of them.

Fix $\phi \in P({}^\vee G^\Gamma)$. Associated with ϕ we have a pair $(y,\lambda) \in ({}^\vee G^\Gamma - {}^\vee G) \times {}^\vee g$ satisfying certain conditions (see [1, Proposition 5.6]). We write $\phi = (y,\lambda)$. Define

$$L(\lambda) = \{g \in {}^\vee G \mid Ad(g)\lambda = \lambda\}, \quad K(y) = \text{centralizer in } {}^\vee G \text{ of } y,$$
$${}^\vee G_\phi = K(y) \cap L(\lambda).$$

Then $A_\phi^{\text{loc}} = \frac{{}^\vee G_\phi}{({}^\vee G_\phi)_0}$ is the Langlands component group for ϕ, and $A_\phi^{\text{loc,alg}} = \frac{{}^\vee G_\phi^{\text{alg}}}{({}^\vee G_\phi^{\text{alg}})_0}$ is the universal component group for ϕ. Here ${}^\vee G^{\text{alg}}$ is the algebraic universal cover of ${}^\vee G$, i.e., there is a short exact sequence $1 \longrightarrow \pi_1({}^\vee G)^{\text{alg}} \longrightarrow {}^\vee G^{\text{alg}} \longrightarrow {}^\vee G \longrightarrow 1$ and ${}^\vee G_\phi^{\text{alg}}$ is the inverse imagine of ${}^\vee G_\phi$ in ${}^\vee G^{\text{alg}}$.

Definition 1.1. *A complete Langlands parameter for* ${}^\vee G^\Gamma$ *is a pair* (ϕ,τ) *with* τ *an irreducible representation of* $A_\phi^{\text{loc,alg}}$. *Two such parameters are called equivalent if they are*

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conjugate under the obvious action of ${}^\vee G^{\mathrm{alg}}$. *We write* $\Xi({}^\vee G^\Gamma) = \Xi^z(G/\mathbf{R})$ *for the set of equivalence classes of complete Langlands parameters. Here* $z \in Z({}^\vee G)$ *is the second invariant of* ${}^\vee G^\Gamma$.

For an extended group G^Γ of G, we have the canonical covering G^{can}, i.e., there is a short exact sequence $1 \longrightarrow \pi_1(G)^{\mathrm{can}} \longrightarrow G^{\mathrm{can}} \longrightarrow G \longrightarrow 1$. The group of continuous characters of $\pi_1(G)^{\mathrm{can}}$ is naturally isomorphic to the set of elements of finite order in $Z({}^\vee G)^{\theta_Z}$ (see [1, Lamma 10.2]). Here θ_Z is the automorphism of $Z({}^\vee G)$ induced by any automorphism of ${}^\vee G$ coresponding to the first invariant a of ${}^\vee G^\Gamma$.

A canonical projective represetation of a strong real form of G^Γ is a pair (π, δ), with δ a strong real form of G^Γ and π an admissible representation of $G(\mathbf{R}, \delta)^{\mathrm{can}}$. Suppose $z \in Z({}^\vee G)^{\theta_Z,\mathrm{fin}}$. We define $\Pi^z(G^\Gamma) = \Pi^z(G/\mathbf{R})$ to be the set of infinitesimal equivalence classes irreducible canonical projective representation of type z.

An important consequence of [1] is the following theorem:

Theorem 1.1 (see [1, Theorem 10.4]). *Suppose that* ${}^\vee G^\Gamma$ *is an E-group for* G^Γ. *Write* z *for the second invariant of the E-group. Then there is a natural bijection* p *between* $\Xi^z(G/\mathbf{R})$ *and* $\Pi^z(G/\mathbf{R})$.

Suppose that $(\pi(\xi), \delta(\xi))$ is the imagine of $\xi \in \Xi^z(G/\mathbf{R})$ under the bijection p in the theorem. Let $M(\xi)$ be the standard representation with the Langlands quotient $\pi(\xi)$, and $K\Pi^z(G/\mathbf{R})$ free $\mathbf{Z}$-module with the base $\Pi^z(G/\mathbf{R})$. Then $M(\xi) = \sum_{\eta \in \Xi^z(G/\mathbf{R})} m_r(\eta, \xi)\pi(\eta)$ in $K\Pi^z(G/\mathbf{R})$, where $0 \le m_r(\eta, \xi) \in \mathbf{Z}$.

Definition 1.2. *Fix an equivalence class* $\phi \in \Phi^z(G/\mathbf{R})$ *of Langlands parameters. We define the L-packet of* ϕ *as*

$$\Pi^z_\phi = \{(\pi(\xi), \delta(\xi)) \mid \xi = (\phi, \tau),\ \tau \in \widehat{A}_\phi^{\mathrm{loc,alg}}\}.$$

Here z *is the second invariant of* ${}^\vee G^\Gamma$.

According to Chapter 6 in [1], $X({}^\vee G^\Gamma) = \cup_O X(O, {}^\vee G^\Gamma)$ with O the semisimple ${}^\vee G$-orbits in ${}^\vee g$. $X(O, {}^\vee G^\Gamma)$ has a smooth complex algebraic variety structure in a natural way. The open and closed ${}^\vee G$-orbits on $X({}^\vee G^\Gamma)$ have special meanings.

Theorem 1.2 (see [1, Proposition 1.11, 22.9, (7.11) and Corollory 15.13]). *Let* $\phi, \phi' \in \Phi^z(G/\mathbf{R})$ *be two Langlands parameters, and* $s_\phi, s_{\phi'} \subset X({}^\vee G^\Gamma)$ *the corresponding* ${}^\vee G$*-orbits. Then*

(a) ϕ *is tempered if and only if* s_ϕ *is open in* $X({}^\vee G^\Gamma)$. *In this case*

$$M(\xi) = \sum_{\pi(\eta) \in \Pi^z_\phi} m_r(\eta, \xi)\pi(\eta)$$

for any $\pi(\xi) \in \Pi^z_\phi$. *If* $s_\phi, s_{\phi'}$ *belong to the same* $X_j(O, {}^\vee G^\Gamma)$ (*see* [1, *Proposition* 6.24]), *there exists* $\pi(\xi) \in \Pi^z_\phi$ *and* $\pi(\eta) \in \Pi^z_{\phi'}$ *such that* $\pi(\xi)$ *is a composition factor of* $M(\eta)$.

(b) *If* s_ϕ *is closed,* $\pi(\xi)$ *must not be any composition factor of* $M(\eta)$ *for any* $\pi(\xi) \in \Pi^z_\phi$ *and* $\pi(\eta) \notin \Pi^z_\phi$.

In the sense of the theorem, we can say that tempered representations are “small” and representations described by closed ${}^\vee G$-orbits on $X({}^\vee G^\Gamma)$ are “large.” “small” representations must be unitary.

§2. Geometric Parameters of Representations of the Classical Groups

Write $\mathrm{Diag}[A_1, A_2, \cdots, A_n]$ for the block diagonal matrix

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}$$

and I_n for the unit matrix of order n. Define

$$\begin{array}{ll} a_p = (\det a'_p)^{-\frac{1}{n}} a'_p, & a'_p = \mathrm{Diag}[I_p, -I_{n-p}], \\ b_p = \mathrm{Diag}[-I_{2p}, I_{2(n-p)+1}], & c_p = \mathrm{Diag}[I_p, -I_{n-p}, I_p, -I_{n-p}], \\ d_p = \mathrm{Diag}[I_{2p}, -I_{2(n-p)}], & d''_p = \mathrm{Diag}[I_{2p+1}, -I_{2(n-p)-1}], \\ d'_p = (\det d''_p)^{-\frac{1}{2n}} d''_p, & d'_{nn} = d_{nn}\mathrm{Diag}[-1, I_{2n-1}], \end{array}$$

$$a_{kk} = c_{kk} = d_{kk} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}.$$

Suppose that G is a complex connected semisimple Lie group. Let σ be a real form of G, and θ be the corresponding Cartan involution. Write $K = G^\theta$ and $G(\mathbf{R}) = G^\sigma$ for the set of fixed points of G under the actions of θ and σ, respectively.

Theorem 2.1. *The sets of equivalence classes of Cartan involutions for the classical groups are listed as the following Table* I.

Table I

The proof is immediately by [2] and direct verifications. For the complex classical groups, it is well-known that

$${}^\vee SL(n, \mathbf{C}) = PGL(n, \mathbf{C}), \quad {}^\vee SO(2n+1, \mathbf{C}) = SP(n, \mathbf{C}), \quad {}^\vee SO(2n, \mathbf{C}) = SO(2n, \mathbf{C}).$$

From the knowledge of based root data (see [1, Definiition 2.10]) of the classical groups and definitions, we have

Theorem 2.2. *All of the extended groups and E-groups for the classcal groups are*

$$A_n(\pm 1, \omega_i I) \to^{\vee} A_n(\pm 1, I), \quad B_n(1, I) \to^{\vee} B_n(1, \pm I),$$
$$C_n(1, \pm I) \to^{\vee} C_n(1, I), \qquad D_n(\pm 1, \pm I) \to^{\vee} D_n(\pm 1, \pm I).$$

Here $A_n(1, \omega_i I)$ and ${}^{\vee}A_n(1, I)$ denote the extended groups with invariants $(1, \omega_i I)$ and the E-group with invariants $(1, I)$ for $SL(n, \mathbf{C})$, respectively. $A_n(1, \omega_i I) \to^{\vee} A_n(1, I)$ means that ${}^{\vee}A_n(1, I)$ is the E-group for $A_n(1, \omega_i I)$, others are similar. ${}^{\vee}A_n(\pm 1, I)$, ${}^{\vee}B_n(1, I)$, ${}^{\vee}C_n(1, I)$, ${}^{\vee}D_n(\pm 1, I)$ are L-groups.

It is easy to know that the regular dominant coweight set of the groups $G = SL(n, \mathbf{C})$, $SO(2n+1, \mathbf{C})$ and $SP(n, \mathbf{C})$ are

$$P_*^{++}(SL(n, \mathbf{C})) = \Big\{\sum_{i=1}^{n} \frac{\lambda_1}{n} + \lambda_i - \delta_{i1}\lambda_i)E_{ii} \mid \lambda_i \in \mathbf{Z},$$
$$\sum_{i=1}^{n} \lambda_i = 0, \quad 0 > \lambda_2 > \lambda_3 > \cdots > \lambda_n\Big\},$$
$$P_*^{++}(SO(2n+1, \mathbf{C})) = \Big\{\sum_{i=1}^{n} \sqrt{-1}\lambda_i(E_{2i-1,2i} - E_{2i,2i-1}) \mid \lambda_i \in \mathbf{Z},$$
$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0\Big\},$$
$$P_*^{++}(SP(n, \mathbf{C})) = \Big\{\sum_{i=1}^{n} (\frac{\lambda_1}{2} + \lambda_i - \delta_{i1}\lambda_i)(E_{ii} - E_{n+i,n+i}) \mid \lambda_i \in \mathbf{Z},$$
$$\frac{\lambda_1}{n} + \lambda_n > 0 > \lambda_2 > \lambda_3 > \cdots > \lambda_n\Big\}.$$

Here E_{ij} is the square matrix with entry 1 where the i-th row and the j-th column meet, all other entries being 0.

Fix an E-group $({}^{\vee}G^{\Gamma}, \mathcal{S})$ for the complex classical group G, ${}^{\vee}\delta \in \mathcal{S}$, and $\lambda \in P_+^{**}({}^{\vee}G)$. Then $P(\lambda)$ is a Borel subgroup of ${}^{\vee}G(\lambda) =^{\vee} G$ (here we used the nations of [1], Theorem 6.16). Now $O =^{\vee} G \cdot \lambda \subset^{\vee} g$ is an integral regular semisimple orbit. We will give a detailed description of $X(O, {}^{\vee}G^{\Gamma})$ in this case. Choose the Weyl group W of ${}^{\vee}G$ and the function $\omega(w)$, $w \in W$ as in §1 of [4] for certain Cartan involution θ of ${}^{\vee}G$.

From now on, we set

$$I\text{diag}[A_1, A_2, \cdots, A_n] = \begin{pmatrix} & & & A_1 \\ & & A_2 & \\ & \cdots & & \\ A_n & & & \end{pmatrix},$$

$$X = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, \quad Y = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

$$Y_n = \text{Diag}[\underbrace{Y, Y, \cdots, Y}_{n}], \quad W_n = I\text{diag}[\underbrace{1, 1, \cdots, 1}_{n}].$$

We define some sets for various $({}^{\vee}G^{\Gamma}, \mathcal{S})$ as follows.

1) $({}^\vee G^\Gamma, \mathcal{S}) = {}^\vee A_n(1, I)$, $\theta = a_p^{(1)}$. Set

$$A_{p,r}^{(1)}(\lambda) = \{(a_p a_{[n/2]}^{-1}\,{}^\vee\delta, \sqrt{a_p^{(1)}(\omega)\omega_r a_r \omega^{-1}} \cdot \lambda \;\mid\; \omega = \omega(w), w \in W\},$$
$$0 \le p \le [n/2], \quad 0 \le r \le p,$$
$$\omega_r = \mathrm{Diag}[Y_r, I_{n-2r-1}, (-1)^r],$$

$$a_r = \begin{cases} \mathrm{Diag}[I_{2r}, -\epsilon_1, -\epsilon_2, \cdots, -\epsilon_{p-2r}, \epsilon_{p-2r+1}, \\ \quad \epsilon_{p-2r+2}, \cdots, \epsilon_{n-2r-1}, (-1)^r \epsilon_{n-2r}] & \text{if } p = 2k, \quad p > 2r, \\ \mathrm{Diag}[I_{2r}, \epsilon_1, \epsilon_2, \cdots, \epsilon_{n-2r-1}, (-1)^r \epsilon_{n-2r}] & \text{if } p = 2k, \quad p \le 2r, \\ \mathrm{Diag}[I_{2r}, -\epsilon_1, -\epsilon_2, \cdots - \epsilon_{p-2r}, \epsilon_{p-2r+1}, \\ \quad \epsilon_{p-2r+2}, \cdots, \epsilon_{n-2r-1}, (-1)^r \epsilon_{n-2r}] & \text{if } p = 2k+1, \quad p > 2r, \\ \mathrm{Diag}[I_{p-1}, \sqrt{-1} I_2, I_{2r-p+1}, \epsilon_1, \\ \quad \epsilon_2, \cdots, \epsilon_{n-2r-1}, (-1)^r \epsilon_{n-2r}] & \text{if } p = 2k+1, \quad p < 2r, \end{cases}$$

where $\epsilon_i = \pm 1$ and the number of ϵ_i equal to -1 is $p - r$.

Choose two subsets of $\{A_{p,r}^{(1)}(\lambda) \mid 0 \le r \le p\}$:

$$A_p^{(1),o}(\lambda) = (y, g^o \cdot \lambda), \quad A_p^{(1),c}(\lambda) = \{(y, g^c \cdot \lambda)\},$$
$$(g^o)^2 = I\mathrm{diag}[\sqrt{-1} W_r, 1, \sqrt{-1} W_{n-r-1}],$$
$$(g^c)^2 = \mathrm{Diag}[-\epsilon_1, -\epsilon_2, \cdots, -\epsilon_p, \epsilon_{p+1}, \epsilon_{p+2}, \cdots, \epsilon_n],$$

where $\epsilon_i = \pm 1$, and the number of ϵ_i equal to -1 is p.

2) $({}^\vee G^\Gamma, \mathcal{S}) = {}^\vee A_n(-1, I)$, $\theta = a_1^{(2)}$. Set

$$A_{1,r}^{(2)}(\lambda) = \{({}^\vee\delta, \sqrt{a_1^{(2)}(\omega)\omega_r a_r \omega^{-1}} \cdot \lambda) \;\mid\; \omega = \omega(w), w \in W\}(0 \le r \le [n/2]),$$
$$a_r = \begin{cases} I \text{ or } \mathrm{Diag}[-I_2, I_{n-2}] & \text{if } n = 2k, \ r = k, \\ I & \text{if not}, \end{cases}$$
$$\omega_r = \mathrm{Diag}[\sqrt{-1} Y_r, I_{n-r}].$$

Choose two subsets of $\{A_{1,r}^{(2)}(\lambda), \quad 0 \le r \le [n/2]\}$:

$$A_1^{(2),c} = \{({}^\vee\delta, g^c \cdot \lambda)\}, \qquad A_1^{(2),o}(\lambda) = ({}^\vee\delta, \lambda),$$
$$(g^c)^2 = I\mathrm{diag}[\sqrt{-1} W_k, 1, \sqrt{-1} W_k](if \quad n = 2k+1),$$
$$(g^c)^2 = \sqrt{-1} W_n \text{ or } \ (g^c)^2 = I\mathrm{diag}[-\sqrt{-1} W_2, \sqrt{-1} W_{n-2}].$$

If $n = 2k$, set

$$A_2^{(2)}(\lambda) = \{(a_{kk}\,{}^\vee\delta, \sqrt{a_2^{(2)}(\omega)\omega^{-1}} \cdot \lambda \;\mid\; \omega = \omega(w), w \in W\},$$
$$A_2^{(2),o}(\lambda) = (a_{kk}\ {}^\vee\delta, g^o \cdot \lambda) \in A_2^{(2)}(\lambda), \quad A_2^{(2),c}(\lambda) = (a_{kk}^\vee \delta, \lambda) \in A_2^{(2)}(\lambda),$$

$$(g^o)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

3) $({}^\vee G^\Gamma, \mathcal{S}) = {}^\vee B_n(1, I)$, $\theta = c_p^{(1)}$. Set

$$B^{(1)}_{p,r,s}(\lambda) = \{(c_p c_{nn}^{-1}{}^\vee\delta, \sqrt{c_p^{(1)}(\omega)\omega_{r,s}a_{r,s}\omega^{-1}}\cdot\lambda \quad | \quad \omega = \omega(w), w \in W\}$$
$$(0 \le p \le [n/2], \quad 0 \le r+s \le p),$$

$$\omega_{r,s} = \begin{pmatrix} Y_r & & & & & \\ & O_{2s} & & & I_{2s} & \\ & & I_{n-2s-2r} & & & \\ & & & Y_r & & \\ & -I_{2s} & & & O_{2s} & \\ & & & & & I_{n-2r-2s} \end{pmatrix},$$

$$a_{r,s} = \mathrm{Diag}[a'_{r,s}, a'_{r,s}] \ (p \text{ is even or } p \text{ is odd but } p \ge 2r),$$

$$a'_{r,s} = \begin{cases} \mathrm{Diag}[I_{2r+2s}, -\epsilon_1, -\epsilon_2, \cdots, -\epsilon_{p-2r-2s}, \\ \quad \epsilon_{p-2r-2s+1}, \epsilon_{p-2r-2s+2}, \cdots, \epsilon_{n-2r-2s}] & \text{if } \ p > 2r+2s, \\ \mathrm{Diag}[I_{2r+2s}, \epsilon_1, \epsilon_2, \cdots, \epsilon_{n-2r-2s}] & \text{if } \ 2r \le p \le 2r+2s, \\ & \text{or } p < 2r, \ p = 2k, \end{cases}$$

$$a_{r,s} = \mathrm{Diag}[a''_{r,s}, a'''_{r,s}] \quad (p \text{ is odd and } p < 2r),$$
$$a''_{r,s} = \mathrm{Diag}[I_{p-1}, \sqrt{-1}I_2, I_{2r+2s-p-1}, \epsilon_1, \epsilon_2 \cdots, \epsilon_{n-2r-2s}],$$
$$a'''_{r,s} = \mathrm{Diag}[I_{p-1}, -\sqrt{-1}I_2, I_{2r+2s-p-1}, \epsilon_1, \epsilon_2, \cdots, \epsilon_{n-2r-2s}].$$

Here $\epsilon_i = \pm 1$, and the number of ϵ_i equal to -1 is $p - r - s$.

Choose two subsets of $\{B^{(1)}_{p,r,s}(\lambda) \quad 0 \le p \le [n/2], \quad 0 \le r+s \le p\}$:

$$B_p^{(1),o}(\lambda) = (y, g^o \cdot \lambda), \qquad B_p^{(1),c}(\lambda) = \{(y, g^c \cdot \lambda)\},$$

$$(g^o)^2 = \begin{pmatrix} O_p & & I_p & \\ & I_{n-p} & & \\ -I_p & & O_p & \\ & & & I_{n-p} \end{pmatrix},$$

$$(g^c)^2 = \mathrm{Diag}[g', g'], \quad g' = \mathrm{Diag}[-\epsilon_1, -\epsilon_2, \cdots, -\epsilon_p, \epsilon_{p+1}, \epsilon_{p+2}, \cdots, \epsilon_n],$$

where the number of ϵ_i equal to -1 is p.

If $\theta = c_n^{(1)}$, set

$$B^{(1)}_{n,r,s}(\lambda) = \{({}^\vee\delta, \sqrt{c_n^{(1)}(\omega)\omega_{r,s}a_{r,s}\omega^{-1}}\cdot\lambda \quad | \quad \omega = \omega(w), w \in W\}(0 \le 2r+s \le n),$$
$$a_{r,s} = \mathrm{Diag}[I_{2r}, \epsilon_1, \epsilon_2, \cdots, \epsilon_s, I_{n-2r-s}, I_{2r}, \epsilon_1, \epsilon_2, \cdots, \epsilon_s, I_{n-2r-s}](\epsilon_i = \pm 1),$$

$$\omega_{r,s} = \begin{pmatrix} & & & \sqrt{-1}Y_r & & \\ & & & & \sqrt{-1}I_s & \\ & & I_{n-2r-s} & & & O_{n-2r-s} \\ \sqrt{-1}Y_r & & & & & \\ & \sqrt{-1}I_s & & & & \\ & & O_{n-2r-s} & & & I_{n-2r-s} \end{pmatrix}.$$

Choose two subsets of $\{B^{(1)}_{n,r,s}(\lambda) \ \ 0 \le r+s \le p\}$:

$$B_n^{(1),o}(\lambda) = ({}^\vee\delta, \lambda), \qquad B_n^{(1),c}(\lambda) = \{({}^\vee\delta, g^c \cdot \lambda)\},$$
$$(g^c)^2 = I\mathrm{diag}[\sqrt{-1}\mathrm{Diag}[\epsilon_1, \epsilon_2, \cdots, \epsilon_n], \sqrt{-1}\mathrm{Diag}[\epsilon_1, \epsilon_2, \cdots, \epsilon_n]](\epsilon_i = \pm 1).$$

4) $({}^\vee G^\Gamma, \mathcal{S}) = {}^\vee C_n(1, I)$, $\theta = b_p^{(1)}$. Set

$$C^{(1)}_{p,r,s}(\lambda) = \{(b_p b^{-1}_{[(n+1)/2]} {}^\vee\delta, \sqrt{b_p^{(1)}(\omega)\omega_{r,s}a_{r,s}\omega^{-1}} \cdot \lambda \quad | \quad \omega = \omega(w), w \in W\}$$

$$(0 \le p \le [n/2], \quad 0 \le r + [(s+1)/2] \le p),$$

$$\omega_{r,s} = \mathrm{Diag}[\underbrace{X, \cdots, X}_{r}, \underbrace{Z_1, \cdots, Z_1}_{s}, I_{2n-4r-2s}, (-1)^s],$$

$$a_{r,s} = \begin{cases} \mathrm{Diag}[I_{4r+2s}, \epsilon_1 I_2, \epsilon_2 I_2, \cdots, \epsilon_{n-2r-2s} I_2, 1] & \text{if} \quad p \le 2r+s, \\ \mathrm{Diag}[I_{4r+2s}, -\epsilon_1 I_2, -\epsilon_2 I_2, \cdots, -\epsilon_{p-2r-s} I_2, \epsilon_{p-2r-s+1} I_2, & \\ \quad \epsilon_{p-2r-s+2} I_2, \cdots, \epsilon_{n-2r-s} I_2, 1] & \text{if} \quad p > 2r+s, \end{cases}$$

where $\epsilon_i = \pm 1$, and the number of ϵ_i equal to -1 is $p - r - [(s+1)/2]$.

Choose two subsets of $\{C^{(1)}_{p,r,s}(\lambda), \quad 0 \le p \le [n/2], \quad 0 \le r + [(s+1)/2] \le p\}$:

$$C_p^{(1),o}(\lambda) = (y, g^o \cdot \lambda), \qquad C_p^{(1),c}(\lambda) = \{(y, g^c \cdot \lambda)\}$$

$$(g^o)^2 = \mathrm{Diag}[\underbrace{Z_1, \cdots, Z_1}_{2p}, I_{2n+1-4p}], \quad (g_j^c)^2 = \mathrm{Diag}[\epsilon_1 I_2, \epsilon_2 I_2, \cdots, \epsilon_n I_2, 1],$$

where the number of $\epsilon_i = \pm 1$ equal to -1 is p.

Theorem 2.3. *Suppose* $O = {}^\vee G \cdot \lambda \subset {}^\vee g$ *for* $\lambda \in P_*^{++}({}^\vee G)$. *Then* $e(O) = \exp(2\pi i \lambda)$ *is independent of* $\lambda \in O$. *For the E-group* $({}^\vee G^\Gamma, \mathcal{S})$ *in Theorem* 2.2, *the set* $\mathcal{A}$ *of the* ${}^\vee G$*-orbits on* $X(O, {}^\vee G^\Gamma)$ *are in Table* II, *where* ${}^\vee\delta \in \mathcal{S}$. *In paticular, we have given the open and closed orbits.*

Table II

Proof. Since the proof is fairly similar for all the cases, we shall give the details only in the case $({}^\vee G^\Gamma, \mathcal{S}) = {}^\vee C_n(1, I)$. Then ${}^\vee G = SO(2n+1, \mathbf{C})$. First we calculate the K-orbits of

Borel subgroups when $\theta = b_p^{(1)}$ by the theory of [4]. Now in the nations of [4], $G_u = SO(2n+1)$. Choose $t = \{(t_1, t_2, \cdots, t_n) \mid t_i \in \mathbf{R}\}$, where $(t_1, t_2, \cdots, t_n) = \mathrm{Diag}[t_1 Z_2, t_2 Z_2, \cdots, t_n Z_2]$. The root system of G with respect to t is $\Delta = \{e_i - e_j, \pm(e_j + e_k), \pm e_i \mid 1 \le i \ne j \le n, 1 \le j < k \le n\}$. Here e_i acts on t as follows: $e_i(t_1, t_2, \cdots, t_n) = -\sqrt{-1} t_i$.

Set $P = \{e_i - e_j, e_i + e_j, e_k \mid 1 \le i < j \le n, 1 \le k \le n\}$ for the set of positive roots and write B for the corresponding Borel subgroup. The Weyl group is $W = S_n \cdot (\mathbf{Z}_2)^n$ (semi-direct, while $(\mathbf{Z}_2)^n$ is the direct sum of n cyclic groups of order 2). Set $P = (i_1, i_2, \cdots, i_n) \in S_n$ and $A = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in (\mathbf{Z}_2)^n (\epsilon_i = \pm 1)$. Then $w = PA$ operates on t as follows: $w(t_1, t_2, \cdots, t_n) = (\epsilon_{i_1} t_{i_1}, \epsilon_{i_2} t_{i_2}, \cdots, \epsilon_{i_n} t_{i_n})$.

For $\theta = b_p^{(1)} (0 \le p \le n)$, we have $\theta \mid t = id$ and $\theta \circ w = w$ for $w \in W$. One can choose a representative in each conjugacy class of elements of order two in W from [5] and get the set $W_0 = \{w_{r,s} = P_r A_s \mid 0 \le 2r + s \le n\}$, where $P_r = (12)(34) \cdots (2r-1, 2r)$.

$$A_s = (\underbrace{1, \cdots, 1}_{2r}, \underbrace{-1, \cdots, -1}_{s}, 1, \cdots, 1).$$

For $w_{r,s} \in W_0$, choose $\omega_{r,s} = \omega(w_{r,s}) \in N(T)$ as 4) before the theorem. Then

$$a_0 = \theta(\omega_{r,s})\omega_{r,s} = \begin{cases} \mathrm{Diag}[I_{4k}, -I_4, I_{2n-4k-3}] & \text{if } p = 2k+1,\ r > k, \\ I & \text{if not,} \end{cases}$$

$$h_u^+ = \{(t_1, t_1, t_2, t_2, \cdots, t_r, t_r, \underbrace{0, \cdots, 0}_{s}, t_{2r+s+1}, t_{2r+s+2}, \cdots, t_n) \mid t_i \in \mathbf{R}\},$$

$$h_u^- = \{(t_1, -t_1, t_2, -t_2, \cdots, t_r, -t_r, t_{2r+1}, t_{2r+2}, \cdots, t_{2r+s}, \underbrace{0, \cdots, 0}_{n-2r-s}) \mid t_i \in \mathbf{R}\},$$

$$\begin{aligned} T_w^+ = \{&\mathrm{Diag}[\alpha(t_1), \alpha(t_1), \alpha(t_2), \alpha(t_2), \cdots, \alpha(t_r), \\ &\alpha(t_r), I_{2s}, \alpha(t_{2r+s+1}), \alpha(t_{2r+s+2}), \ldots, \alpha(t_n), 1]\}, \\ T_w^- = \{&\mathrm{Diag}[\alpha(t_1), \alpha(-t_1), \alpha(t_2), \alpha(-t_2), \cdots, \alpha(t_r), \\ &\alpha(-t_r), \alpha(t_{2r+1}), \alpha(t_{2r+2}), \cdots, \alpha(t_{2r+s}), I_{2n-4r-2s+1}]\}, \end{aligned}$$

where $\alpha(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$,

$$\begin{aligned} \widehat{T}_w^+ &= \{a \in T_w^+ \mid a^2 = a_0^{-1}\} \\ &= \begin{cases} \mathrm{Diag}[\underbrace{\pm I_4, \cdots, \pm I_4}_{r}, I_{2s}, \underbrace{\pm I_2, \cdots, \pm I_2}_{n-2r-s}, 1] & \text{if } p = 2k \text{ or } p = 2k+1, p > 2r, \\ \emptyset & \text{if not.} \end{cases} \end{aligned}$$

The set of elements of $\widehat{T}_w^+$ which are not congruent modulo T_w^- to each other is

$$\widehat{T}_0 = \{\mathrm{Diag}[I_{4r+2s}, \underbrace{\pm I_2, \cdots, \pm I_2}_{n-2r+s}, 1]\}.$$

We must make sure if $g \in \widehat{T}_0$ lies in $\exp \sqrt{-1}p$. From the knowledge of quadratic forms we know that $B(Adg_1 X, Y)$ and $B(Adg_2 X, Y)$ have the same siganatures iff the Jordan canonical forms of g_1 and g_2 have the same numbers of diagnal elements equal to -1, where $g_1, g_2 \in SO(2n+1)$, and $B(\cdot, \cdot)$ is the Killing form. So $\omega_{r,s} a_{r,s} \in \exp(\sqrt{-1}p)$ iff $0 \le r + [(s+1)/2] \le p$ and $a_{r,s}$ as 4) shows by Proposition 8 and Remark of [4].

Since t is a fundamental Cartan subalgebra, the closed K-orbits, equivalently, the open

B-orbits on the flag manifold $\mathcal{B}$ (see [3, Corollary of Proposition 2]) correspond to $w_{0,0} = 1$, and the number of them is $\binom{n}{p}$.

For the unique open K-orbit, the corresponding $\dim \mathfrak{n}_1$ (see (44) of [4]) should be the maximal. Suppose $p \le [n/2]$. Then for $w_{0,2p} = A_{2p} \in W$, $P_+^{\sigma_1} = \{e_i - e_j, e_i + e_j, e_k \mid 1 \le i \le 2p, i < j \le n, 1 \le k \le 2p\}$ (see (42) of [4]). It is easy to see $\dim \mathfrak{n}_1 = 4p(n-p)$. By comparing the dimensions and ranks we know that the corresponding K-orbit is open.

Now we can come to the conclusion by the proof of Proposition 6.16 in [1].

§3. The Complete Langlands Parameters and L-Packets of the Classical Groups

Suppose that G is a connected semisimple complex classical group. ${}^\vee G^\Gamma$ is an E-group for an extended group G^Γ. Choose a Langlands parameter $\phi = (y, g \cdot \lambda)$ in Table II with $y \in {}^\vee G^\Gamma - {}^\vee G$ and $\lambda \in P_+^{**}({}^\vee G)$. Then $\theta_y = Ady|{}^\vee G$ is a Cartan involution of ${}^\vee G$ and $K(y) = K$ is the set of the fixed points. $T^d = L(\lambda)$ (section 1) is a Cartan subgroup of ${}^\vee G$ and $g \cdot \lambda$ represents a K-orbit of Borel subgroups. In the nations of [4], we have $T = K(y) \cap {}^dT$ and $g = \sqrt{\omega a} \in \sqrt{[\Phi_0]}$, $\omega = \omega(w)$, for some $w \in W$, $a = (\theta_y(\omega)\omega)^{-1}$. In fact, we have made $\theta_y(g) = g^{-1}$ by the appropriate choice of g.

Theorem 3.1. *In the setting of descriptions above, set $\theta_w = \theta_y \circ w|{}^dT$ and ${}^dT_w$ is the set fixed points of θ_w in ${}^dT$. Then $A_\phi^{loc,alg} = {}^dT_w^{alg}/({}^dT_w^{alg})_0$.*

Proof. Since λ is regular and semisimple, we have ${}^\vee G_\phi = K(y) \cap L(g \cdot \lambda) = (g\,{}^dT g^{-1})^{\theta_y}$. If $x = gtg^{-1} \in {}^\vee G_\phi$ for some $t \in {}^dT_w$, then $\theta_y \circ Adg^2(t) = t$, i.e., $\theta_y \circ w(t) = t$ because of $\theta_y(x) = x$ and $\theta_y(g) = g^{-1}$. So ${}^\vee G_\phi = g\,{}^dT_w g^{-1}$ and we come to the conclusion.

According to [1], the Langlands parameter $\phi = (y, g \cdot \lambda)$ in Table II describes the representations with the infinitesimal character λ.

Theorem 3.2. *Suppose that G is a semisimple connected complex classical group, and ${}^\vee G^\Gamma$ is an E-group for an extended group G^Γ. z is the second invariant of ${}^\vee G^\Gamma$. Choose $\lambda \in P_+^{**}({}^\vee G)$, $O = {}^\vee G \cdot \lambda$. Then the set of infinitesimal equivalence classes irreducible canonical projective representation of G of type z with the infinitesimal character λ is in the following Table* III. *Obviously this table specifically describes all the the L-packets in this case.*

Proof. We only give the proof in the case of $C_n(1, \pm I) \to {}^\vee C_n(1, I)$. Others are similar.

Now ${}^\vee G = SO(2n+1, \mathbf{C})$. Choose $\lambda \in P_+^{**}({}^\vee G)$ and $\phi = (y, g \cdot \lambda) \in B_{p,r,s}^{(1)}$. Then $e(\lambda) = I$ and $L(\lambda) = \exp{}^d\mathfrak{t}$. Here $\mathfrak{t} = \{(t_1, t_2, \cdots, t_n) \mid t_i \in \mathbf{C}\}$, $(t_1, t_2, \cdots, t_n) = \mathrm{Diag}[t_1 Z_2, t_2 Z_2, \cdots, t_n Z_2]$.

It is easy to prove the following fact:

a) In the setting of Theorem 3.1, write ${}^d\mathfrak{t}_w^{\pm} = \{X \in {}^d\mathfrak{t} \mid \theta_w(X) = \pm X\}$, ${}^d\widetilde{T}_w^- = \{\exp X \mid X \in {}^d\mathfrak{t}_w^-, \ 2X \in X_*({}^dT)\}$. Then ${}^dT_w$ is generated by $\exp{}^d\mathfrak{t}_w^+$ and ${}^d\widetilde{T}_w^-$.

b) For a Langlands parameter $\phi = (y, g \cdot \lambda)$, if g has the form $g = \sqrt{\theta_y(\omega)\omega_0 a_0 \omega^{-1}} \in \sqrt{[\Phi_0]}$, then $A_\phi^{\mathrm{loc,alg}} \simeq A_{\phi_0}^{\mathrm{loc,alg}}$, where $\phi_0 = (y, g_0 \cdot \lambda)$, $g_0 = \sqrt{\omega_0 a_0} \in \sqrt{[\Phi_0]}$.

Because of b), it is sufficient to consider only the case of $g = \sqrt{\omega_{r,s} a_{r,s}}$.

$$
{}^d\mathfrak{t}_{w_{r,s}}^+ = \{(t_1, t_1, \cdots, t_r, t_r, \underbrace{0, \cdots, 0}_{s}, t_{2r+s+1}, t_{2r+s+2}, \cdots, t_n)\},
$$

$$ {}^{d}t^{-}_{w_{r,s}} = \{(t_1, -t_1, t_2, -t_2, \cdots, t_r, -t_r, t_{2r+1}, t_{2r+2}, \cdots, t_{2r+s}, 0, \cdots, 0)\}. $$

Table III

By a) we have

$$ {}^{d}T^{-}_{w_{r,s}} = \{\mathrm{Diag}[\alpha(t_1), \alpha(t_1), \alpha(t_2), \alpha(t_2), \cdots, \alpha(t_r), \alpha(t_r), \underbrace{\pm I_2, \cdots, \pm I_2}_{s}, \alpha(t_{2r+s+1}), \cdots, \alpha(t_n), 1] \mid t_i \in \mathbf{C}\}. $$

Here

$$ \alpha(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. $$

But $\pi_1({}^{\vee}G)^{\mathrm{alg}} = \{\pm I\}$, so

$$ {}^{d}T^{\mathrm{alg}}_{w_{r,s}} = \{\mathrm{Diag}[\alpha(t_1), \alpha(t_1), \alpha(t_2), \alpha(t_2), \cdots, \alpha(t_r), \alpha(t_r), \underbrace{\pm I_2, \cdots, \pm I_2}_{s}, \alpha(t_{2r+s+1}), \cdots, \alpha(t_n), \pm 1] \mid t_i \in \mathbf{C}\}. $$

Obviously $A^{\mathrm{loc,alg}}_{\phi} = {}^{d}T^{\mathrm{alg}}_{w_{r,s}}/({}^{d}T^{\mathrm{alg}}_{w_{r,s}})_0 \simeq \mathbf{Z}_2^{s+1}$. Now by the representation theory of finite groups we know there is a bijection between $\hat{A}^{\mathrm{loc,alg}}_{w_{r,s}}$ and $\mathbf{Z}_2^{s+1}$.

References

[1] Adams, J., Barbasch, D. & Vogan, D., The langlands classification and irreducible characters for real reductive groups, Progress in Math., vol. 104.

[2] Helgason, S., Differential geometry, Lie groups and symmetric Spaces, Academic Press, 1978.

[3] Matsuki, T., The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, *J. Math. Soc. Japan,* **31:**2(1979), 331-357.

[4] Aomoto, K., On some double coset decompositions of complex semisimple Lie groups, *J. Math. Soc. Japan,* **18**(1966), 1-44.

[5] Spécht, W., Gruppentheorie, Springer, 1956.