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COMPUTATION OF $K_2Z[\frac{1+\sqrt{-35}}{2}]$**

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Abstract

The author shows that $K_2Z[\frac{1+\sqrt{-35}}{2}] \cong Z/2Z$. The method of proof is a generalization of the Tate's method.

Keywords K_2 group, Tate's method, Imaginary quadratic field
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§1. Introduction

In general, it is not easy to determine the structure of K_2O_F for a number field F with the ring of integers O_F, even for a quadratic field. Let $F = Q(\sqrt{d})$ be an imaginary quadratic field. We know that K_2O_F is trivial for $d = -1, -2, -3, -11$ and $K_2O_F \cong Z/2Z$ for $d = -7$, -15 (see [10]). And K_2O_F is also trivial for $d = -5$, -19 (see [4]). In [7], the auther shows that K_2O_F is trivial too for $d = -6$. In this paper, we show that $K_2O_F \cong Z/2Z$ for $d = -35$.

§2. Preliminaries

Let F be a number field, O_F be its ring of integers. Denote by S_∞ the set of Archimedean places of F. If $S \supseteq S_\infty$, we denote by O_S the ring of S-integers. For any $v \notin S$, $k(v) = O_S/P$, where P is the maximal ideal corresponding to the place v. Suppose that $v_1, v_2, \cdots, v_n, \cdots$ with $N(v_i) \le N(v_{i+1})$ for all i are all finite places of F, where $N(v) = \#(k(v))$. Let $S_n = \{v_1, v_2, \cdots, v_n\} \bigcup S_\infty$. H. Bass and J. Tate[1] show that there exists a positive integer m such that

$$K_2O_F = \operatorname{Ker}\Big(K_2^{S_m}F \xrightarrow{(\tau_v)} \coprod_{v\in S_m\setminus S_\infty} k^{\cdot}(v)\Big),$$

where $K_2^{S_m}F =$ the subgroup of K_2F generated by $\{x, y\}$ with $x, y \in O^{\cdot}_{S_m} = U$. Recall that

$$\tau_v\{x, y\} = (-1)^{v(x)v(y)}x^{v(y)}y^{-v(x)}(\mathrm{mod} P)$$

where P is the maximal ideal corresponding to v.

Suppose that the ideal P (corresponding to v) is principal, say $P = \pi O_S$. Let β be the map from U to $k^{\cdot}$ given by $\beta(u) = u(\mathrm{mod}\pi)$. Denote by U_1 the subgroup of U generated by $(1 + \pi U) \cap U$. J. Tate[10] gives the following result.

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Lemma 2.1. *Suppose that W, C and G are subsets of U such that*

(1) $W \subset CU_1$ and W generates U,

(2) $CG \subset CU_1$ and $\beta(G)$ generates k,

(3) $1 \in C \bigcap \mathrm{Ker}\beta \subset U_1$.

Then τ_v is bijective.

For the following lemma also refer to [10].

Lemma 2.2. *Let F be an imaginary quadratic field. Let M be an ideal in O_F, the prime factorization of which involves only primes in S. Suppose $a, b \in U \bigcap M$ and $|a| + |b| < N(v)(NM)^{\frac{1}{2}}$. If $\beta(a) = \beta(b)$, then $a \in bU_1$. Especially, if $a, b \in U \bigcap O_F$, $|a| + |b| < N(v)$ and $\beta(a) = \beta(b)$, then $a \in bU_1$.*

Lemma 2.3. *Suppose that we are given subsets $D \subset O_F$ and $W \subset O_F \bigcap U$. Put*

$$E = \{d - d' | d, d' \in D, \quad d \neq d'\}.$$

If the ideal P (corresponding to v) is principal, then τ_v is bijective provided that D and W satisfy the following conditions:

(1) $(\#(D))^3 > N(v)^2$,

(2) $E \subset U$,

(3) $1 \in W$ and W generates U,

(4) If $e_1, e_2, e_3, e_4 \in E$ and $w \in W$, then

(i) $N(e_1e_2 - e_3e_4) < N(v)^2$, (ii) $N(e_1w - e_2) < N(v)^2$ or $e_1w/e_2 \notin \mathrm{Ker}\beta$.

Proof. It follows from the proof of Lemma 3.4 in [1] that the conditons (1), (2) and (3) imply that $\mathrm{Ker}\beta = \mathrm{Ker}(U \to k^{\cdot}(v))$ can be generated by the following elements:

(I) $\frac{e_1e_2}{e_3e_4}$ $(e_1, e_2, e_3, e_4 \in E)$,

(II) $\frac{e_1w}{e_1}$ $(e_1, e_2 \in E, \quad w \in W)$.

Now, in view of Lemma 3.2 in [1], the result follows.

From now on, we suppose that $F = Q(\sqrt{-35})$. In this case, the class number $h = 2, 2$ is intert in O_F, $3O_F = Q_1Q_2$, where $Q_1 = (3, \frac{1+\sqrt{-35}}{2})$ and $Q_2 = (3, \frac{1-\sqrt{-35}}{2})$. Obviously, $Q_1 \neq Q_2$ and neither Q_1 nor Q_2 is principal. View Q_1 as a lattice in C. Then the maximum distance from Q_1 to C is $\sqrt{\frac{27}{7}}$. We will prove that τ_v is bijective if $N(v) > 7$. To do this, we divide all cases into three parts.

§3. Case One: $N(v) \geq 2801$

By a discusion similar to that in [10], we can easily show the following

Lemma 3.1. *Let $W = \{u \in O_F \bigcap U| \quad |u|^2 \leq 3N(v)\}$. Then W generates U.*

Lemma 3.2. *Choose d such that $d > 0$ and $d^2 = N(v)/13$. Put $D = \{x \in O_F| \quad |x| \leq d\}$ and $E = \{d - d' | d, d' \in D, d \neq d'\}$. Then E satisfies* (2) *and* (4) *of Lemma* 2.3 *if* $N(v) > 199$.

Proof. For any $e \in E$, there exist d and $d' \in D$ such that $e = d - d'$. Then $N(v) \leq (|d| + |d'|)^2 \leq (4/13)N(v) < N(v)$. Consequently, $e \in U$.

On the other hand, if $e_1, e_2, e_3, e_4 \in E$, $w \in W$, then

$$N(e_1e_2 - e_3e_4) \leq (|e_1e_2| + |e_3e_4|)^2 \leq \Big(\Big(\frac{8}{13}\Big)N(v)\Big)^2 < N(v)^2$$

and

$$N(e_1w - e_2) \leq (|e_1w| + |e_2|)^2 \leq \Big(\frac{4}{13}\Big)N(v)(\sqrt{3N(v)} + 1)^2.$$

Note that if $N(v) > 199$, then $(\sqrt{3N(v)}+1)^2 < (13/4)N(v)$. Hence, $N(e_1w-e_2) < N(v)^2$.

Lemma 3.3. *Let $d > 0$ and $D = \{x \in O_F | \ |x| \le d\}$. Then*

$$\begin{aligned}\#(D) \ge\ & 1+2[d]+2\Big[\frac{d}{35}\Big]+4[\sqrt{d^2-35\cdot 1^2}]+4[\sqrt{d^2-35\cdot 2^2}]+\cdots\\ &+4\left[\sqrt{d^2-35\cdot\Big[\sqrt{\frac{d}{35}}\Big]^2}\right]+2[\sqrt{4d^2-35\cdot 1^2}]\\ &+2[\sqrt{4d^2-35\cdot 3^2}]+\cdots+2[\sqrt{4d^2-35\cdot\theta^2}],\end{aligned}$$

where $[x]$ denotes the greatest integer which $\le x$ and

$$\theta=\begin{cases}\Big[\frac{2d}{\sqrt{35}}\Big], & \text{if } \Big[\frac{2d}{\sqrt{35}}\Big]\equiv 1(\text{mod}2),\\ \Big[\frac{2d}{\sqrt{35}}\Big]-1, & \text{if } \Big[\frac{2d}{\sqrt{35}}\Big]\equiv 0(\text{mod}2).\end{cases}$$

Proof. In D, there are $1+2[d]$ rational integers; there are $2[d/\sqrt{35}]$ elements of the forms $x\sqrt{-35}$ $(x\in Z, x\ne 0)$; there are

$$4[\sqrt{d^2-35\cdot 1^2}]+4[\sqrt{d^2-35\cdot 2^2}]+\cdots+4\left[\sqrt{d^2-35\Big[\frac{d}{\sqrt{35}}\Big]^2}\right]$$

elements of the forms $x+y\sqrt{-35}$ $(x,y\in Z, x\cdot y\ne 0)$; there are at least

$$2[\sqrt{4d^2-35\cdot 1^2}]+2[\sqrt{4d^2-35\cdot 3^2}]+\cdots+2[\sqrt{4d^2-35\cdot\theta^2}]$$

elements of the forms $\frac{1}{2}(x+y\sqrt{-35})$ $(x\equiv y\equiv 1(\text{mod}2))$, where the definition of θ is the same as above.

Lemma 3.4. *If $N(v)\ge 4693$, then τ_v is bijective.*

Proof. Choose d such that $d^2=N(v)/13$. We prove that if $N(v)\ge 4963$, then $(\#(D))^3 > N(v)^2$, so by Lemma 2.3 the result follows.

By Lemma 3.3,

$$\begin{aligned}\#(D) \ge\ & 1+2[d]+2\Big[\frac{d}{35}\Big]+4[\sqrt{d^2-35\cdot 1^2}]\\ &+4[\sqrt{d^2-35\cdot 2^2}]+\cdots+4\left[\sqrt{d^2-35\Big[\frac{d}{\sqrt{35}}\Big]^2}\right]\\ &+2[\sqrt{4d^2-35\cdot 1^2}]+2[\sqrt{4d^2-35\cdot 3^2}]+\cdots+2[\sqrt{4d^2-35\cdot\theta^2}]\\ >\ & 1+2[d]+2\Big[\frac{d}{35}\Big]\\ &+4\left(\sqrt{d^2-35\cdot 1^2}+\sqrt{d^2-35\cdot 2^2}+\cdots+\sqrt{d^2-35\Big[\frac{d}{\sqrt{35}}\Big]^2}-\Big[\frac{d}{\sqrt{35}}\Big]\right)\\ &+2\Big(\sqrt{4d^2-35\cdot 1^2}+\sqrt{4d^2-35\cdot 3^2}+\cdots+\sqrt{4d^2-35\cdot\theta^2}-\Big[\frac{2d}{35}\Big]\Big)\\ >\ & 1+2[d]-2\cdot\frac{d}{\sqrt{35}}-2\cdot\frac{2d}{\sqrt{35}}+4\int_1^{\frac{d}{\sqrt{35}}}\sqrt{d^2-35x^2}dx+\int_1^{\frac{2d}{\sqrt{35}}}\sqrt{4d^2-35x^2}dx\\ >\ & 1+0.98(d-1)+4\int_1^{\frac{d}{\sqrt{35}}}\sqrt{d^2-35x^2}dx+\int_1^{\frac{2d}{\sqrt{35}}}\sqrt{4d^2-35x^2}dx,\end{aligned}$$

where the definition of θ is the same as in Lemma 3.3.

Write

$$f(d)=4\int_1^{\frac{d}{\sqrt{35}}}\sqrt{d^2-35x^2}dx+\int_1^{\frac{2d}{\sqrt{35}}}\sqrt{4d^2-35x^2}dx+0.98(d-1)+1.$$

Then

$$\begin{aligned}f(d)&=4\Big(\frac{x}{2}\sqrt{d^2-35x^2}+\frac{d^2}{2\sqrt{35}}\arcsin\frac{\sqrt{35}}{d}x\Big)\Big|_1^{\frac{d}{\sqrt{35}}}\\&\quad+\Big(\frac{x}{2}\sqrt{4d^2-35x^2}+\frac{2d^2}{\sqrt{35}}\arcsin\frac{\sqrt{35}}{2d}x\Big)\Big|_1^{\frac{2d}{\sqrt{35}}}+0.98(d-1)+1\\&=4\Big(\frac{d^2}{2\sqrt{35}}\arcsin 1-\frac{1}{2}\sqrt{d^2-35}-\frac{d^2}{2\sqrt{35}}\arcsin\frac{\sqrt{35}}{d}\Big)\\&\quad+\frac{2d^2}{\sqrt{35}}\arcsin 1-\frac{1}{2}\sqrt{4d^2-35}-\frac{2d^2}{\sqrt{35}}\arcsin\frac{\sqrt{35}}{2d}+0.98(d-1)+1\\&>\frac{2\pi}{\sqrt{35}}d^2-\frac{1}{2}\sqrt{4d^2-35}-2\sqrt{d^2-35}-3d-\frac{\sqrt{35}}{3d}\cdot\frac{1}{1-35/d^2}\\&\quad-\frac{35}{24d}\cdot\frac{1}{1-35/4d^2}+0.98(d-1)+1,\end{aligned}$$

we have used the formula

$$\arcsin x=x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}\quad(|x|\le 1).$$

Suppose $d\ge 19$. Then

$$1-\frac{35}{3d}\cdot\frac{1}{1-35/d^2}-\frac{35}{24d}\cdot\frac{1}{1-35/4d^2}>0.$$

Thus

$$f(d)>\frac{2\pi}{\sqrt{35}}d^2-6d+0.98(d-1)=\frac{2\pi}{\sqrt{35}}d^2-5.02d-0.98.$$

Put $g(d)=\frac{2\pi}{\sqrt{35}}d^2-5.02d-0.98-(13d^2)^{\frac{2}{3}}$. Then

$$\begin{aligned}g''(d)>0,&\quad\text{if}\ \ d\ge 1.3;\\g'(d)>0,&\quad\text{if}\ \ d\ge 10;\\g(d)>0,&\quad\text{if}\ \ d\ge 19.\end{aligned}$$

But $d\ge 19$ implies $N(v)\ge 4963$. This completes the proof.

Lemma 3.5. *Suppose $d>0$ and $d^2<\frac{N(v)^2}{4(1+\sqrt{3N(v)})^2}$. Let*

$$D=\{x\in O_F|\ \ |x|\le d\},\quad E=\{d-d'|\ \ d,d\in D,d\ne d'\},$$

and $W=\{w\in O_F\bigcap U|\ \ |w|^2\le 3N(v)\}$. Then for $e_1,e_2,e_3,e_4\in E$ and $w\in W$,

$$N(e_1e_2-e_3e_4)<N(v)^2,\qquad N(e_1w-e_2)<N(v)^2.$$

The proof is analogous to that of Lemma 3.2.

Lemma 3.6. *When $2081\le N(v)\le 5379$, τ_v is bijective.*

Proof. When $N(v)\ge 2081$, $\frac{N(v)^2}{4(1+\sqrt{3N(v)})^2}>13^2$. For brevity, we let $D(n)=\{x\in O_F|\ \ |x|\le n\}$. We have $\#(D(13))=187$, $187^{3/2}>2557$.

When $N(v)\ge 2557$, $\frac{N(v)^2}{4(1+\sqrt{3N(v)})^2}>14^2$, we have $\#(D(14))=209$, $209^{3/2}>3021$.

When $N(v)\ge 3021$, $\frac{N(v)^2}{4(1+\sqrt{3N(v)})^2} > 15^2$, we have $\#(D(15)) = 243$, $243^{3/2} > 3787$.

When $N(v)\ge 3787$, $\frac{N(v)^2}{4(1+\sqrt{3N(v)})^2} > 17^2$, we have $\#(D(17)) = 307$, $307^{3/2} > 5379$.

Then by Lemma 3.1, Lemma 3.5 and Lemma 3.6, τ_v is bijective when $2081 \le N(v) \le 5379$.

§4. Case Two: $11\le N(v)\le 1037$ and $N(v)=1369$

The following two lemmas are analogous to Lemma 15.2 and Lemma 15.3 in [10] respectively.

Lemma 4.1. *Suppose that M is a non-principal ideal. Then every residue class* (mod M) can be represented by an integer c with $N(c)\le (9/7)NM$.

Lemma 4.2. *Suppose that* (b) *is a principal ideal prime to* $Q_1 = (3, \frac{1}{2}(1+\sqrt{-35}))$. *Then every residue class* (mod(b)) can be represented by an element $c\in Q_1$ with $N(c)\le (27/7)N(v)$.

For any non-principal ideal P, we put

$$C' = \{c\in O_F \mid \ |c|^2 \le (9/7)N(v)\}, \qquad W = \{w\in O_F\bigcap U \mid \ |w|^2\le 3N(v)\}.$$

Let

$$T = \{t_1,\cdots,t_r \ \mid\ t_i\in C',\ \ t_i\notin U,\ \ 1\le i\le r\},$$
$$S = \{s_1,\cdots,s_r \ \mid\ s_i\equiv t_i(\mathrm{mod}P),\ \ s_i\in U,\ \ 1\le i\le r\}$$

and $C = (C'\backslash T)\bigcup S$. Assume $m = \max\limits_{c\in C}|c|$. Then by Lemma 2.2, we have

(1) If $\sqrt{3N(v)} + m < N(v)$, then $W\subset CU_1$;

(2) If $G=\{g\}$ and $m|g| + m < N(v)$, then $CG\subset CU_1$ and $1\in C\bigcap \mathrm{Ker}\beta\subset U_1$.

It is easy to know that for each $N(v)$ we only need to discuss a prime ideal. For any principal prime ideal P, we choose g such that g is a primitive root (mod$P$) with least value of $|g|$. First, we see a few examples.

$$N(v)=13,\ \ P=(13,\frac{1}{2}(11+\sqrt{-35})),\ \ T=S=\emptyset,\ \ g=2,\ \ m\le\sqrt{\frac{117}{7}}.$$
$$N(v)=17,\ \ P=(17,\frac{1}{2}(13+\sqrt{-35})).$$

In this case, we give C directly. Let

$$C=\Big\{\pm1,\pm2,\pm3,\pm4,\pm\frac{1}{2}(-1-\sqrt{-35}),\ \pm\frac{1}{2}(1-\sqrt{-35}),\pm\frac{1}{2}(-3-\sqrt{-35}),\ \pm\frac{1}{2}(3-\sqrt{-35})\Big\}.$$

Then $m=4$. Take $g=3$, then (1) and (2) are satisfied.

$$N(v)=47,\ \ P=\Big(47,\frac{1}{2}(23+\sqrt{-35})\Big),\ \ T=S=\emptyset,\ \ g=5,\ \ m<\sqrt{\frac{423}{7}}.$$
$$N(v)=73,\ \ P=\Big(73,\frac{1}{2}(29+\sqrt{-35})\Big),\ \ T=\{\frac{1}{2}(1+3\sqrt{-35}),\ \ \frac{1}{2}(1-3\sqrt{35})\},$$
$$S=\{1-\sqrt{-35},\sqrt{-35}\},\ \ g=5,\ \ m<\sqrt{\frac{657}{7}}.$$
$$N(v)=83,\ \ P=\Big(83,\frac{1}{2}(31+\sqrt{-35})\Big),\ \ T=S=\emptyset,\ \ g=2,\ \ m<\sqrt{\frac{747}{7}}.$$

In most cases, we can take S as the following. Write P (corresponding to v) $=(N(v),\frac{1}{2}(a+b\sqrt{-35}))$ $(a\equiv b(\mathrm{mod}2))$ and $T=\{t_1,\cdots,t_r\}$. Suppose that either $t_i+\frac{1}{2}(a+b\sqrt{-35})$ or

$t_i - \frac{1}{2}(a+b\sqrt{-35}) \in U$. Then choose s_i to be one of them such that $s_i \in U$ and $N(s_i)$ is the less one. Such $N(v)$ are the following :

$$\begin{array}{ccccccccccc} 157, & 167, & 173, & 223, & 227, & 257, & 283, & 293, & 307, & 313, & 367, \\ 383, & 397, & 433, & 467, & 503, & 523, & 563, & 577, & 587, & 647, & 677, \\ 727, & 733, & 773, & 787, & 797, & 853, & 857, & 937, & 983, & 997, & 1013. \end{array}$$

For the rest, we list $N(v)$, g and m.

$$\begin{array}{llllll} N(v)=97, & g=5, & m<\sqrt{873/7}; & N(v)=103, & g=5, & m<\sqrt{927/7}; \\ N(v)=353, & g=3, & m\le\sqrt{6265}; & N(v)=593, & g=3, & m\le\sqrt{5265}; \\ N(v)=607, & g=3, & m\le\sqrt{10311}; & N(v)=643, & |g|\le 11, & m\le\sqrt{3375}; \\ N(v)=887, & g=5, & m\le\sqrt{8841}. & & & \end{array}$$

Now, we turn to principal prime ideals.

First, suppose that

$$N(v)=11,\quad P=\Big(\frac{1}{2}(3+\sqrt{-35})\Big),\quad \overline{P}=\Big(\frac{1}{2}(3-\sqrt{-35})\Big).$$

For P, we take

$$\begin{aligned} W &= \Big\{\pm1, 2, 3, \frac{1}{2}(1+\sqrt{-35}), \sqrt{-35}, 5\Big\}, \\ C &= \Big\{\pm1, \pm2, \pm3, \pm\frac{1}{2}(5-\sqrt{-35}), \pm\frac{1}{2}(7-\sqrt{-35})\Big\}, \\ G &= \{2\}. \end{aligned}$$

For $\overline{P}=(\frac{1}{2}(3-\sqrt{-35}))$, we take

$$\begin{aligned} W &= \Big\{\pm1, 2, 3, \frac{1}{2}(1+\sqrt{-35}), \sqrt{-35}, 5, \frac{1}{2}(3+\sqrt{-35})\Big\}, \\ C &= \Big\{\pm1, \pm2, \pm3, \pm\frac{1}{2}(5+\sqrt{-35}), \pm\frac{1}{2}(7+\sqrt{-35})\Big\}, \\ G &= \{2\}. \end{aligned}$$

It is easy to check that $W \subset CU_1$, W generates U, $CG \subset CU_1$ and $\{1\} = C \cap \mathrm{Ker}\beta$. Then by Lemma 2.1, τ_v is bijective.

For other principal prime ideals, we have the following method.

As done in the situation of non-principal prime ideals, we introduce the following notations:

$$\begin{aligned} C' &= \Big\{\alpha \in Q_1 |\ \ |\alpha|^2 \le \Big(\frac{27}{7}\Big)N(v)\Big\}, \\ T &= \{t_1, \cdots, t_r |\ \ t_i \in C',\ \ t_i \notin U,\ \ 1 \le i \le r\}, \\ S &= \{s_1, \cdots, s_r |\ \ s_i \in C',\ \ s_i \in U,\ \ s_i \equiv t_i (\mathrm{mod} P),\ \ 1 \le i \le r\}. \end{aligned}$$

Let $C = (C' \backslash T) \cup S \cup \{1\}$, $W = \{w \in O_F \cap U |\ \ |w|^2 \le 3N(v)\}$ and $G = \{g\}$. Assume $m = \max\limits_{c \in C} |c|$.

Suppose that $P = (a+b\sqrt{-35})$ is a principal prime ideal. In all cases below, for any $t_i \in T$, we can always choose β_i such that $t_i - \beta_i \in C'$ together with $t_i - \beta_i \in U$, where

$$\begin{aligned} \beta_i = \text{either } &\frac{1}{2}(1+\sqrt{-35})(a+b\sqrt{-35}) \ \text{ or } \ \frac{1}{2}(1+\sqrt{-35})(a+b\sqrt{-35}) \\ \text{or } \ &3(a+b\sqrt{-35}) \ \text{ or } \ -3(a+b\sqrt{-35}). \end{aligned}$$

We choose $s_i = t_i - \beta_i$ so that $s_i \in C' \bigcap U$ and $N(s_i)$ is the least one. The principle of choice of g is the same as in non-principal cases. The following is a list of $N(v)$.

29,	71,	79,	109,	149,	151,	179,	191,
211,	239,	281,	331,	359,	379,	389,	401,
421,	431,	449,	491,	499,	541,	569,	571,
599,	631,	641,	659,	701,	709,	739,	751,
809,	821,	911,	919,	991,	1009,	1019,	1031.

There are six intert prime ideals satisfying $11 \le N(v) \le 2069$. They are (19), (23), (31), (37), (41), (43).

For

$$P = (19), \quad N(v) = 19^2,$$
$$T = \Big\{31 - 2\sqrt{-35}, 34 + \sqrt{-35}, \frac{1}{2}(61 - 5\sqrt{-35}), \frac{1}{2}(59 - 7\sqrt{-35})\Big\},$$
$$S = \Big\{-26 - 2\sqrt{-35}, -23 + \sqrt{-35}, \frac{1}{2}(-53 - 5\sqrt{-35}), \frac{1}{2}(-55 - 7\sqrt{-35})\Big\},$$

in this time, $m < \sqrt{9747/7}$. Note that $\#(O_F/(19)) = 361$. Hence there are $\varphi(360) = 96$ primitive roots in $O_F/(19)$, where $\varphi(\ \ )$ is the Euler's function. Suppose that $g \in C$ is a primitive root. Then $(g) = Q_1 B$, where B is an integral ideal. If there is a primitive root $g \in C$ with $|g| < 15.7$, then by Lemma 2.1, $gU \subset CU_1$. Therefore, τ_v is bijective. If $|g| > 15.7$, we may assume B is not a prime ideal, since there are at most 14 prime ideals satisfying the assumpation. Clearly, B is not a principal ideal. Thus there are only the following possibilities:

(i) $B = (2)P$ (P is a non-principal ideal),

(ii) $B = QP$, $Q|3$ or $Q|5$,

(iii) $B = (b)P$, $3 < |b|^2 < 15.7^2$.

For case (i), take $\alpha = 1 + \sqrt{-35}$ and $M = 2Q_1^2$. For case (ii), take $\alpha = 9$, $M = 3Q_1$, if $Q = \bar{Q_1}$, or $M = Q_1^3$, if $Q = Q_1$; or $\alpha = \frac{1}{2}(5 - \sqrt{-35})$, $M = Q_1^2 Q'$, where $Q'^2 = (5)$, if $Q|5$. For case (iii), take $\alpha = b$ and $M = bQ_1$. Then, by Lemma 2.1, τ_v is bijective.

The same method can be applied to treat (23), (31), (37), (41), and (43). But we have a simpler method for (41) and (43).

§5. Case Three: 1037<$N(v)$<2081

Throughout this section, notation $D(n)$ always denotes the set $\{x \in O_F|\ \ |x| \le n\}$. Let $P = (41)$. Then $N(v) = 41^2 = 1681$. Take

$$W = \{\alpha \in U|\ \ |\alpha|^2 \le 3 \cdot 1657\},$$
$$D = D(11) \bigcup \Big\{\pm 10 \pm \sqrt{-35}, \frac{1}{2}(\pm 15 \pm 3\sqrt{-35}), 2\sqrt{-35}\Big\}.$$

Then W and D satisfy the assumption of Lemma 2.3. Therefore τ_v is bijective.

Similarly, for $P = (43)$, we take $W = \{\alpha \in U|\ \ |\alpha|^2 \le 3 \cdot 1847\}$, $D = D(12)$.

In general, the above method is simpler than the methods which we have used. But we need $N(v)$ to be quite great when we use the above method. Next, we give D for any $N(v)$ with $1031 < N(v) < 2081$ except $N(v) = 1369$. For any $N(v)$, $W = \{w \in O_F \bigcap U|\ \ |w|^2 \le$

$3N(v)\}$. It is easy to check that for any $N(v)$, D and W satisfy the assumptions of Lemma 2.3.

For $N(v)=1051,1061,1063,1123$, take $D=D(10)\backslash\{\pm 8\pm\sqrt{-35}\}$. Then $\#(D)=111$.

For $N(v)=1171$, take $D=D(10)\backslash\{\pm(8+\sqrt{-35})\}$. Then $\#(D)=113$.

For $N(v)=1097,1129,1153,1193,1201,1217,1223,1229$, take $D=D(10)$. Then $\#(D)=115$.

For $N(v)=1277,1289,1307$, take $D=D(10)\cup\{\pm 11\}\cup\{\frac{1}{2}(\pm 21\pm\sqrt{-35})\}$. Then $\#(D)=121$.

For $N(v)=1433$, take $D=D(11)\backslash\{\frac{1}{2}(13\pm 3\sqrt{-35})\}$. Then $\#(D)=131$.

For $N(v)=1477$, take $D=D(11)\backslash\{\pm 11\}$. Then $\#(D)=131$.

For $N(v)=1237,1381,1409,1427,1429,1439,1451,1471,1481,1483,1487,1499$, take $D=D(11)$. Then $\#(D)=133$.

For $N(v)=1543,1549,1553,1567$, take $D=D(11)\cup\{10\pm\sqrt{-35}\}$. Then $\#(D)=135$.

For $N(v)=1579,1613,1619,1621,1627,1637$, take $D=D(11)\cup\{\pm 10\pm\sqrt{-35},\ \pm 12\}$. Then $\#(D)=139$.

For $N(v)=1657,1693$, take $D=D(11)\cup\{\pm 10\pm\sqrt{-35},\pm 12,\frac{1}{2}(\pm 15+3\sqrt{-35})\}$. Then $\#(D)=143$.

For $N(v)=1697$, 1709, 1753, 1759, 1777, 1783, 1789, 1801, 1823, $1831,1847,1867,1871,1901,1907$, take $D=D(12)$. Then $\#(D)=157$.

For $N(v)=1973,1987,1993,1999,2011$, take $D=D(12)\cup\{\pm 11\pm\sqrt{-35}\}$. Then $\#(D)=161$.

For $N(v)=2039,2063,2069$, take $D=D(13)$. Then $\#(D)=187$.

From the results we have obtained, we have

Theorem 5.1. *Suppose $S=\{\infty,2O_F,Q_1,Q_2,Q,Q'\}$, where $Q_1Q_2=3O_F$, $Q^2=5O_F$, ${Q'}^2=7O_F$. Then*

$$K_2Z[\frac{1}{2}(1+\sqrt{-35})]\subset K_2^SF.$$

§6. Determining the Structure of $K_2Z[\frac{1}{2}(1+\sqrt{-35})]$

In this section, we will prove the following main result of the present paper.

Theorem 6.1. *Let $F=Q(\sqrt{-35})$ with the ring of integers $O_F=Z[\frac{1}{2}(1+\sqrt{-35})]$. Then $K_2O_F\cong Z/2Z$.*

Let S be the same as in Theorem 5.1. Then $U=O_S^{\cdot}$ can be generated by

$$V=\Big\{\pm 1,2,\frac{1}{2}(1+\sqrt{-35}),3,\frac{1}{2}(5+\sqrt{-35}),\frac{1}{2}(7+\sqrt{-35})\Big\}.$$

Hence K_2^SF can be generated by $\{x,y\}$ with $x,y\in V$. Let us observe below some relations among the generators of K_2^SF.

By [3], $2-\mathrm{rank}(K_2O_F)=1$. Let $\varepsilon=\{-1,5\}$. We claim that $\varepsilon\neq 1$. Otherwise $\{2,5\}^2=\varepsilon=1$, then $\{2,5\}$ is an element of order 2, then there exists a $z=x+\sqrt{-35}y\in F(x,y\in Q)$ such that $\{-1,z\}=\{2,5\}$, and therefore

$$\tau_v\{-1,z\}=\begin{cases}-1, & \text{if } N(v)=5,\\ 1, & \text{otherwise.}\end{cases}$$

This is impossible, since Diophantine equations $X^2+35Y^2=5N^2$ and $X^2+35Y^2=10N^2$ have no answers in Z. For general results, see [8]. The above discussion also shows that the 2-Sylow subgroup of K_2O_F is isomorphic to $Z/2Z$. As done above, we can show $\{-1,-1\}\neq 1$, hence $\{-1,-1\}=\{-1,-5\}$, furthermore, $\{-1,7\}=1$. Let $\Delta=\{z\in F^{\cdot}|\{-1,z\}=1\}$. Then $[\Delta : F^{\cdot 2}]=4$, and thus

$$\Delta = F^{\cdot 2}\cup 2F^{\cdot 2}\cup 7F^{\cdot 2}\cup 14F^{\cdot 2}.$$

By [2], for any $x\in F^{\cdot}$,

$$\{x,x+1\}^2=1,\quad \{x,x^2+1\}^4=1,\quad \{x,x^2+x+1\}^3=1.$$

Then we have

$$\begin{aligned}&\Big\{\frac12(1+\sqrt{-35}),\frac12(1-\sqrt{-35})\Big\}=1,\\ &\Big\{\frac12(1+\sqrt{-35}),\frac12(1+\sqrt{-35})\Big\}^2=1,\\ &\Big\{\frac12(1-\sqrt{-35}),\frac12(1-\sqrt{-35})\Big\}^2=1.\end{aligned}$$

Let $x=-\frac12(1+\sqrt{-35})$. Then $x^2+x+1=-8$, and hence $\{-8,-\frac12(1+\sqrt{-35})\}^3=1$. Similarly,

$$\Big\{\frac25,-\frac{5-\sqrt{-35}}{10}\Big\}^3=1,\quad \Big\{\frac47,-\frac{7+\sqrt{-35}}{14}\Big\}^3=1.$$

But $\{-1,1+\sqrt{-35}\}=1$, since $1+\sqrt{-35}=(1/14)(7+\sqrt{-35})^2$. Hence $\{2,1+\sqrt{-35}\}^9=\varepsilon$. On the other hand,

$$\Big\{\frac12\Big(\frac12(1+\sqrt{-35})\Big),\frac12\Big(\frac12(5+\sqrt{-35})\Big)\Big\}^2=1,$$

and

$$\begin{aligned}\Big\{\frac12(1+\sqrt{-35}),\frac12(5+\sqrt{-35})\Big\}^2&=\Big\{\frac12(1+\sqrt{-35}),-5\Big(\frac12(1-\sqrt{-35})\Big)\Big\}\\ &=\{1+\sqrt{-35},5\}\{5,2\},\Big\{\frac12,\frac12(5+\sqrt{-35})\Big\}^2\\ &=\Big\{\frac12,-5\Big(\frac12(1-\sqrt{-35})\Big)\Big\}\\ &=\{5,2\}\Big\{\frac12,\frac12(1-\sqrt{-35})\Big\}.\end{aligned}$$

Clearly,

$$\{2,1+\sqrt{-35}\}\{2,1-\sqrt{-35}\}=\Big\{2,\frac12(1+\sqrt{-35})\Big\}\Big\{2,\frac12(1-\sqrt{-35})\Big\}=1.$$

Then we have $\{1+\sqrt{-35},5\}=\{1+\sqrt{-35},2\}^3$. It follows from

$$\{2/5,-5(-\sqrt{-35})/10\}^6=1 \ \text{ and } \ \{2/5,-1/10\}=\varepsilon$$

that $\{2,1+\sqrt{-35}\}^3=\{5,1+\sqrt{-35}\}^3$. Hence

$$\{2,1+\sqrt{-35}\}^3=\varepsilon,\quad \{5,1+\sqrt{-35}\}=1.$$

Furthermore, $\{7,1+\sqrt{-35}\}=1$.

Note that

$$3^2 = \frac{1}{2}(1+\sqrt{-35})\cdot\frac{1}{2}(1-\sqrt{-35}),$$
$$\Big(\frac{1}{2}(5+\sqrt{-35})\Big)^2 = \frac{1}{2}\cdot 5(-1+\sqrt{-35}),$$
$$\Big(\frac{1}{2}(7+\sqrt{-35})\Big)^2 = \frac{1}{2}\cdot 7(1+\sqrt{-35}).$$

Clearly,

$$\{2, 1+\sqrt{-35}\} \notin K_2O_F, \quad \{2, 1+\sqrt{-35}\}^3 \in K_2O_F.$$

Hence, considering all relations we have obtained, we immediately conclude that $K_2O_F \cong Z/2Z$.

Remarks. 1. In [4], J. Browkin gives some conjectural values of K_2O_F for imaginary quadratic fields F; one of those is just what we have confirmed above, namely, $\#(K_2O_F) = 2$.

2. Our method can be used to determine K_2O_F for other imaginary quadratic fields F.

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