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ON THE LEAST DOMINANT CONTINUOUS MODULUS AND ITS APPLICATION**

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Abstract

This paper discusses pointwise error estimates for the approximation by bounded linear operators of continuous functions defined on compact metric spaces (X, d). The authors introduce a new majorant of the modulus of the continuity which is the smallest among those $g(\xi)'$s which have the following properties $\omega(f, \epsilon) \le g(f, \epsilon)$ and $g(f, \lambda\epsilon) \le (1+\lambda)g(f, \epsilon)$ and by this majorant a new quantitative Korovkin type theorem on any compact metric space is proved.

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§0. Introduction

In this paper we deal with quantitative Korovkin type theorems for the approximation by bounded linear operators defined on $C(X)$, and in particular by positive ones. Here $C(X) = C_R(X, d)$ denotes the Banach lattice of real-valued continuous functions defined on the compact metric space (X, d) with norm given by $\|f\|_X = \max |f(x)|, x \in X$. We also assume that X has diameter $d(X) > 0$. The first such theorem for general positive linear operators and $X = [a, b]$ equipped with the euclidian distance is due to R. Mamedov[4]. For spaces (X, d) being metrically convex in the sense of K. Menger[5], D. Newman and M. Shapiro proved a theorem similar to that of Mamedov[6]. More recently M. Jimenez Pozo introduced the compact spaces having a coefficient of convex deformation $\rho < \infty$ and generalized the theorem to these spaces[3]. Furthermore, H.Gonska proved the theorem of this type in terms of the least concave majorant of the modulus of continuity on any compact metric space as follows.

Theorem A.[1] *Let A be of the form $A(f, y) = \psi_A(y)$ and let L be a bounded linear operator, both mapping $C(X)$ into $B(Y)$. Then for $f \in C(X), y \in Y$ and $0 < \epsilon$, we have*

$$|(L-A)(f, y)| \le \max\{\frac{1}{2}(\|L\| + \|L(1_X)\|_Y), \epsilon^{-1}(d(X)(\|\Phi_y \circ L\| - L(1_X, Y)))\} + |L(d(\cdot, g_A(y)))|\tilde{\omega}(f, \epsilon) + |(L-A)(1_X, y)||f(g_A(Y))|.$$

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Here 1_X *denotes the function* $X \ni x \mapsto 1 \in R, \Phi_y$ *denotes the operator* $C(Y) \ni f \mapsto f(y) \in R$, *and* $\tilde{\omega}(f,\epsilon)$ *denotes the least concave majorant of* ω *given by*

$$\tilde{\omega}(f,\epsilon) := \begin{cases} \sup\limits_{\substack{0\le x\le\epsilon\le y\le d(X)\\ x\ne y}} \dfrac{(\epsilon-x)\omega(f,y)+(y-\epsilon)\omega(f,x)}{y-x} & \text{for } 0\le\epsilon\le d(X), \\ \omega(f,d(X)) & \text{for } \epsilon > d(X). \end{cases}$$

Gonska's results are achieved via the use of a certain K-functional, and the least concave majorant he used is not the best majorant in terms of the property $\tilde{\omega}(f,\lambda\epsilon) \le (1+\lambda)\tilde{\omega}(f,\epsilon)$. So in this paper we will introduce a new majorant of the modulus of continuity and prove a new quantitative theorem by direct method.The new majorant and its properties will be given in §1, and in §2 we prove the new quantitative theorem in terms of it.

§1. The Majorant and Its Properties

Definition 1.1. *The* n*-majorant of* $\omega(f,\epsilon)$ *is defined by*

$$\hat{\omega}_n(f,\epsilon) := \max(\omega_n(f,\epsilon),\omega(f,\epsilon)),$$

where

$$\omega_n(f,0) := \omega(f,0),$$

$$\omega_n\Big(f,\frac{d(X)}{2^n}\Big) := K_1\frac{d(X)}{2^n}, \text{ where } K_1 = \inf\Big\{K : K\epsilon \ge \omega(f,\epsilon) \text{ for } \epsilon \ge \frac{d(X)}{2^n}\Big\},$$

$$\omega_n(f,\epsilon) := \Big(\omega_n\Big(f,\frac{d(X)}{2^n}\Big)-\omega_n(f,0)\Big)2^n\frac{\epsilon}{d(X)} \text{ for } 0\le\epsilon\le\frac{d(X)}{2^n}.$$

More general, if we have defined $\omega_n\Big(f,\frac{kd(X)}{2^n}\Big)$, *we let*

$$\omega_n\Big(f,\frac{(k+1)d(X)}{2^n}\Big) := \max\Big\{K_k\frac{(k+1)}{2^n},\omega_n\Big(f,\frac{kd(X)}{2^n}\Big)\Big\},$$

where

$$K_k := \inf\Big\{K : K\epsilon+\omega_n\Big(f,\frac{kd(X)}{2^n}\Big) \ge \omega(f,\epsilon) \text{ for } \epsilon \ge \frac{(k+1)d(X)}{2^n}\Big\},$$

$$\omega_n(f,\epsilon) = \omega_n\Big(f,\frac{kd(X)}{2^n}\Big)+\frac{2^n}{d(X)}\Big(\omega_n\Big(f,\frac{(k+1)d(X)}{2^n}\Big)-\omega_n\Big(f,\frac{kd(X)}{2^n}\Big)\Big)\Big(\epsilon-\frac{kd(X)}{2^n}\Big)$$

for $\frac{kd(X)}{2^n}\le\epsilon\le\frac{(k+1)d(X)}{2^n}$.

Lemma 1.1. $\omega_n(f,\epsilon)$ *is a Cauchy sequence.*

Proof. We will prove it by three steps.

In step 1, we prove that for any $\epsilon'>0$ there exists an N such that for any $n>N$,

$$\Big|\omega_n\Big(f,\frac{(k+1)d(X)}{2^n}\Big)-\omega_n\Big(f,\frac{kd(X)}{2^n}\Big)\Big| < \epsilon'.$$

Without loss of generality we can assume

$$\omega(f,0)=0,\ \ \lim_{\epsilon\mapsto 0}(f,\epsilon)=\omega(f,0),\ \ \omega(f,\epsilon)\le 1 \text{ and } d(X)=1.$$

Thus there exists a δ, and if $0<\epsilon<\delta$, we have $\omega(f,\epsilon)<\epsilon'$. Let N satisfy $\frac{1}{\delta 2^N}<\frac{\epsilon'}{2}$. If $n\ge N$, we have $\omega\big(f,\frac{k}{2^n}\big)<\frac{\epsilon'}{2}$ for $\frac{k}{2^n}\le\frac{\epsilon'\delta}{2}$. Thus

$$\Big|\omega_n\Big(f,\frac{(k+1)}{2^n}\Big)-\omega_n\Big(f,\frac{k}{2^n}\Big)\Big| \le \epsilon'.$$

For $\frac{k}{2^n} > \frac{\epsilon'\delta}{2}$, we have $K_{k+1} \le \frac{1}{\delta}$, where K_k is as in Definition 1.1. Thus

$$\Big|\omega_n\Big(f, \frac{(k+1)}{2^n}\Big) - \omega_n\Big(f, \frac{k}{2^n}\Big)\Big| \le K_{k+1}\Big(\frac{1}{2^n}\Big) \le \frac{1}{\delta 2^N} \le \epsilon',$$

and all these conclude the proof.

In step 2, we prove that for the N in step 1 and any $n > N$ we have

$$\Big|\omega_n\Big(f, \frac{k}{2^n}\Big) - \omega_N\Big(f, \frac{k}{2^n}\Big)\Big| < \epsilon'.$$

We prove it by inductive method. First it is obvious that $|\omega_n(f,0) - \omega_N(f,0)| = 0$. Now assuming

$$\Big|\omega_n\Big(f, \frac{k}{2^n}\Big) - \omega_N\Big(f, \frac{k}{2^n}\Big)\Big| \le \epsilon,$$

by Definition 1.1 we know

$$\omega_n\Big(f, \frac{k2^{n-N} + 2^{n-N} - 1}{2^n}\Big) - \omega_N\Big(f, \frac{k}{2^n}\Big) > -\epsilon'$$

and

$$\omega_n\Big(f, \frac{k2^{n-N} + 2^{n-N} - 1}{2^n}\Big) - \omega_N\Big(f, \frac{(k+1)}{2^n}\Big) < \epsilon'.$$

So by Definition 1.1 we have

$$\Big|\omega_n\Big(f, \frac{(k+1)}{2^n}\Big) - \omega_N\Big(f, \frac{(k+1)}{2^n}\Big)\Big| < \epsilon'.$$

In step 3, we prove

$$|\omega_n(f,\epsilon) - \omega_N(f,\epsilon)| < 2\epsilon' \quad \text{for } \frac{k}{2^n} \le \epsilon \le \frac{(k+1)}{2^n}.$$

From

$$\omega_n\Big(f, \frac{k}{2^n}\Big) \le \omega_n(f,\epsilon) \le \omega_n\Big(f, \frac{(k+1)}{2^n}\Big)$$

and

$$\omega_N\Big(f, \frac{k}{2^n}\Big) \le \omega_N(f,\epsilon) \le \omega_N\Big(f, \frac{(k+1)}{2^n}\Big),$$

it can be seen that

$$\begin{aligned} &|\omega_n(f,\epsilon) - \omega_N(f,\epsilon)| \\ &\le \max\Big\{\Big|\omega_N\Big(f, \frac{(k+1)}{2^n}\Big) - \omega_n\Big(f, \frac{k}{2^n}\Big)\Big|, \Big|\omega_n\Big(f, \frac{(k+1)}{2^n}\Big) - \omega_N\Big(f, \frac{k}{2^n}\Big)\Big|\Big\}. \end{aligned}$$

Because

$$\begin{aligned} &\Big|\omega_n\Big(f, \frac{(k+1)}{2^n}\Big) - \omega_N\Big(f, \frac{k}{2^n}\Big)\Big| \\ &\le \Big|\omega_N\Big(f, \frac{(k+1)}{2^n}\Big) - \omega_N\Big(f, \frac{k}{2^n}\Big)\Big| + \Big|\omega_N\Big(f, \frac{k}{2^n}\Big) - \omega_n\Big(f, \frac{k}{2^n}\Big)\Big| \le 2\epsilon, \end{aligned}$$

and similarly

$$\Big|\omega_n\Big(f, \frac{(k+1)}{2^n}\Big) - \omega_n\Big(f, \frac{k}{2^n}\Big)\Big| \le 2\epsilon',$$

we arrive at the estimate of step 3.

Finally, if $n, m \ge N$,

$$\begin{aligned} |\omega_n(f,\epsilon) - \omega_m(f,\epsilon)| &\le |\omega_n(f,\epsilon) - \omega_N(f,\epsilon)| + |\omega_M(f,\epsilon) - \omega_N(f,\epsilon)| \\ &\le 2\epsilon' + 2\epsilon' = 4\epsilon', \end{aligned}$$

and this concludes the proof.

Lemma 1.2. *For any linear function* $f(x)$, *if* $f(x) \le (1+\frac{x}{\epsilon})f(\epsilon)$ *and* $f(y) \le (1+\frac{y}{\epsilon})f(\epsilon)$, *then for any* $z = ax+by, a+b=1$, *we have* $f(z) \le (1+\frac{z}{\epsilon})f(\epsilon)$.

Proof.

$$\begin{aligned} f(z) &= f(ax+by) = af(x)+bf(y) \\ &\le a(1+\frac{x}{\epsilon})f(\epsilon)+b(1+\frac{y}{\epsilon})f(\epsilon) = (1+\frac{z}{\epsilon})f(\epsilon). \end{aligned}$$

Lemma 1.3. *For* $\omega_n(f,\epsilon)$ *as in Definition* 1.1, *we have*

$$\omega_n(f,\lambda\epsilon) \le (1+\lambda)\omega_n(f,\epsilon).$$

Proof. For $\lambda \le 1$, it is the consequence of the nondecreasing of $\omega_n(f,\epsilon)$. For $\lambda > 1$, we assume $\epsilon \in \Big[\frac{l}{2^n}, \frac{l+1}{2^n}\Big]$ and $\lambda\epsilon \in \Big[\frac{k}{2^n}, \frac{(k+1)}{2^n}\Big]$. If $l = k$, the proof is simple, so let $l < k$. By Definition 1.1, we know $K_l \ge K_{l+1}$. Thus

$$\omega_n\Big(f, \frac{k}{2^n}\Big) \le \frac{k\epsilon}{2^n}\omega_n(f,\epsilon),$$

and

$$\omega_n\Big(f, \frac{(k+1)}{2^n}\Big) \le \frac{(k+1)\epsilon}{2^n}\omega_n(f,\epsilon).$$

By Lemma 1.2 we arrive at the estimate.

Corollary 1.1. $\hat{\omega}_n(f,\epsilon)$ *is a Cauchy sequence and it satisfies*

$$\hat{\omega}_n(f,\lambda\epsilon) \le (1+\lambda)\hat{\omega}_n(f,\epsilon).$$

Proof. It can be easily seen from Lemma 1.1 and Lemma 1.3.

Definition 1.2. *We define the least* 1*-majorant of the modulus of the continuity* $\omega(f,\epsilon)$ *as* $\hat{\omega}(f,\epsilon) = \lim \hat{\omega}_n(f,\epsilon)$.

Theorem 1.1. $\hat{\omega}(f,\lambda\epsilon) \le (1+\lambda)\hat{\omega}(f,\epsilon)$.

Proof. It is trivial from Corollary 1.1.

Lemma 1.4. *If* $\omega(f,\epsilon)$ *satisfies* $\omega(f,\lambda\epsilon) \le (1+\lambda)\omega(f,\epsilon)$, *we have* $\hat{\omega}(f,\epsilon) \doteq \omega(f,\epsilon)$.

Proof. By Lemma 1.1, it can be seen that if n is large enough,we have

$$\Big|\omega_n\Big(f, \frac{(k+1)d(X)}{2^n}\Big) - \omega_n\Big(f, \frac{kd(X)}{2^n}\Big)\Big| < \epsilon' \quad \text{for any} \quad \epsilon' > 0.$$

We also use the inductive method. First $\omega(f,0) = \omega_n(f,0)$. Now assume

$$\Big|\omega_n\Big(f, \frac{kd(X)}{2^n}\Big) - \omega\Big(f, \frac{kd(X)}{2^n}\Big)\Big| < \epsilon'.$$

For $\omega_n\Big(f, \frac{kd(X)}{2^n}\Big) \le \omega(f, \frac{kd(X)}{2^n})$, it can be easily seen from Definition 1.1 that

$$\begin{aligned} \omega_n\Big(f, \frac{(k+1)d(X)}{2^n}\Big) &\le \omega_n\Big(f, \frac{kd(X)}{2^n}\Big) + \epsilon' \le \omega\Big(f, \frac{kd(X)}{2^n}\Big) + \epsilon' \\ &\le \omega\Big(f, \frac{(k+1)d(X)}{2^n}\Big) + \epsilon'. \end{aligned}$$

For $\omega_n\Big(f, \frac{kd(X)}{2^n}\Big) > \omega\Big(f, \frac{kd(X)}{2^n}\Big)$, we also have

$$\omega_n\Big(f, \frac{(k+1)d(X)}{2^n}\Big) \le \omega\Big(f, \frac{(k+1)d(X)}{2^n}\Big) + \epsilon'.$$

All these show that

$$\omega\Big(f, \frac{kd(X)}{2^n}\Big) \le \hat{\omega}_n\Big(f, \frac{kd(X)}{2^n}\Big) \le \omega\Big(f, \frac{kd(X)}{2^n}\Big) + \epsilon'.$$

When $n \to \infty$, we have $\hat{\omega}(f,\epsilon) = \omega(f,\epsilon)$ almost everywhere.

Theorem 1.2. *If $g(f,\epsilon)$ satisfies*

(1) *$g(f,\epsilon)$ is non-decreasing,*

(2) *$g(f,\lambda\epsilon) \le (1+\lambda)g(f,\epsilon)$,*

(3) *$\omega(f,\epsilon) \le g(f,\epsilon)$.*

Then we have $\hat{\omega}(f,\epsilon) \le g(f,\epsilon)$ almost everywhere.

Proof. We can also define $\hat{g}(f,\epsilon)$ as in Definition 1.1, and by Lemma 1.4 $\hat{g}(f,\epsilon) \doteq g(f,\epsilon)$, so we can use the following Lemma 1.5 to get the result.

Lemma 1.5. *If $g_1(f,\epsilon) \le g_2(f,\epsilon)$ then we have $\hat{g}_1(f,\epsilon) \le \hat{g}_2(f,\epsilon)$ almost everywhere.*

Proof. The spirit is the same as that of Lemma 1.4, so we omit it here.

Lemma 1.6. *$\tilde{\omega}(f,\epsilon)$ satisfies $\tilde{\omega}(f,\lambda\epsilon) \le (1+\lambda)\tilde{\omega}(f,\epsilon)$.*

Proof. By Brundyi's Lemma,

$$\tilde{\omega}(f,\epsilon) = 2K(\epsilon/2, f, C(X), \mathrm{Lip}1),$$

where K satisfies $K(\lambda t, f) \le \max(1,\lambda)K(t,f)$, which yields the proof.

Lemma 1.4, Lemma 1.5 and Lemma 1.6 show that $\hat{\omega}(f,\epsilon)$ can be estimated above by $\tilde{\omega}(f,\epsilon)$, the converse estimate is also true.

Theorem 1.3. *$\hat{\omega}(f,\lambda\epsilon) \le \tilde{\omega}(f,\lambda\epsilon) \le (1+\lambda)\hat{\omega}(f,\epsilon)$. In particular if $\lambda = 1$ this reduces to*

$$\hat{\omega}(f,\epsilon) \le \tilde{\omega}(f,\epsilon) \le 2\hat{\omega}(f,\epsilon).$$

Proof. If one of λ, ϵ is equal to zero, it is obviously true. So let $\lambda, \epsilon > 0$, and $0 < \lambda\epsilon \le d(X)$. From the definition,

$$\hat{\omega}(f,\lambda\epsilon) = \sup_{\substack{0\le x\le\lambda\epsilon\le y\le d(X)\\ x\ne y}} \frac{(\lambda\epsilon - x)\omega(f,y) + (y-\lambda\epsilon)\omega(f,x)}{y-x},$$

so we have

$$\begin{aligned}\frac{\lambda\epsilon - x}{y-x}\omega(f,y) + \frac{y-\lambda\epsilon}{y-x}\omega(f,x) &\le \frac{\lambda\epsilon - x}{y-x}\hat{\omega}(f,y) + \frac{y-\lambda\epsilon}{y-x}\hat{\omega}(f,x)\\ &\le \frac{\lambda\epsilon - x}{y-x}(1+\frac{y}{\epsilon})\hat{\omega}(f,\epsilon) + \frac{y-\lambda\epsilon}{y-x}(1+\frac{x}{\epsilon})\hat{\omega}(f,\epsilon)\\ &= (1+\lambda)\hat{\omega}(f,\epsilon).\end{aligned}$$

For $\lambda\epsilon > d(X)$, we have

$$\hat{\omega}(f,\lambda\epsilon) = \hat{\omega}(f,d(X)) \le \Big(1 + \frac{d(X)}{\epsilon}\Big)\hat{\omega}(f,\epsilon) \le (1+\lambda)\hat{\omega}(f,\epsilon).$$

Remark 1.1. We can also define the least ρ-majorant which satisfies $\hat{\omega}(f,\lambda\epsilon) \le (1+\rho\lambda)\hat{\omega}(f,\epsilon)$ corresponding to the compact metric space with a coefficient of convex deformation $\rho < \infty$ and get the similar results as in Theorem 1.1, Theorem 1.2 and Theorem 1.3.

§2. Quantitative Theorem and Its Proof

Using $\hat{\omega}(f,\epsilon)$, we can now give the quantitative theorem on any compact metric space (X,d). The proof is as that of Pozo.

Theorem 2.1. *Let (X,d) be a compact metric space, A be of the form $A(f,y)=\psi_A(y)f(g_A(y))$ and L be a bounded linear operator, both mapping $C(X)$ in $B(Y)$. If $y\in Y$ is such that $L(1_X,y)\neq 0$, then for all $f\in C(X)$ and all $\epsilon>0$ we have*

$$\begin{aligned}|(L-A)(f,y)|\le{}&\|\Phi_y\circ L\|\Big(1-\frac{L(1_X,y)}{\|\Phi_y\circ L\|}\Big)(1+\epsilon^{-1}d(X))\\&+\frac{|L(1_X,y)|}{L(1_X,y)\|\Phi_y\circ L\|}(L(1)+\epsilon^{-1}L(d(\dot{;}g_A(y)),y))\hat{\omega}(f,\epsilon)\\&+|(L-A)(1_X,y)||f(g_A(y))|.\end{aligned}$$

Proof. If $f\in C(X)$, then for all $t\in X$,

$$\begin{aligned}|f(t)-f(g_A(y))|&\le\omega(f,d(t,g_A(y)))\le\hat{\omega}(f,d(t,d(t,g_A(Y)))\\&\le\Big(1+\frac{d(t,g_A(y))}{\epsilon}\Big)\hat{\omega}(f,\epsilon).\end{aligned}$$

For fixed $y\in Y$, if $t\in X$, we define

$$\begin{aligned}h_1(t):&=f(g_A(y))-\Big(1+\frac{d(t,g_A(y))}{\epsilon}\Big)\hat{\omega}(f,\epsilon),\\h_2(t):&=f(g_A(y))+\Big(1+\frac{d(t,g_A(y))}{\epsilon}\Big)\hat{\omega}(f,\epsilon).\end{aligned}$$

The continuous functions $h_i, i=1,2$, satisfy $h_1(t)\le f(t)\le h_2(t)$ and

$$\begin{aligned}|f(t)-h_1(t)|&=f(t)-h_1(t)\\&\le|f(t)-f(g_A(y))|+\Big(1+\frac{d(t,g_A(y))}{\epsilon}\Big)\hat{\omega}(f,\epsilon)\\&\le 2\Big(1+\frac{d}{\epsilon}\Big)\hat{\omega}(\epsilon).\end{aligned}$$

Here d denotes $d(t,g_A(y)),\hat{\omega}(\epsilon)$ denotes $\hat{\omega}(f,\epsilon)$. Also we have

$$|h_2(t)-f(t)|\le 2\Big(1+\frac{d}{\epsilon}\Big)\hat{\omega}(\epsilon).$$

Hence

$$\max\{\|f-h_i\|,i=1,2\}\le 2\Big(1+\frac{d(X)}{\epsilon}\Big)\hat{\omega}(\epsilon).$$

The assumption $L(1_X,y)\neq 0$ allows one to introduce the auxilary function T.

$$T(f):=T_y(f)=\frac{|L(1_X,y)|}{L(1_X,y)\|\Phi_y\circ L\|}L(f,y).$$

For fixed $y\in Y$ this is a continuous linear functional on $C(X)$. Hence by Riesz's representation theorem, there exists a $\mu=\mu^+-\mu^-$, where μ^+,μ^- are positive measures such that

$$T(f,y)=\int_X fd\mu^-=\int_X fd\mu^+-\int_X fd\mu^+.$$

We have

$$T(f,y)+\int_X fd\mu^-=\int_X fd\mu^+\ge 0.$$

We estimate $\int_X fd\mu^-$ as follows:

$$\begin{aligned}\int_X fd\mu^-\le\|f\|_X\int_X 1_Xd\mu^-&=\|f\|_X\int_X 1_X\frac{1}{2}(-\mu+|\mu|)\\&=\|f\|_X\frac{1}{2}(-T(1_X,y)+\|\mu\|).\end{aligned}$$

Moveover

$$\|\mu\| = \sup\Big\{\Big|\int_X f d\mu\Big| : f \in C(X), \|f\|_X \le 1\Big\} = \|\Phi_y \circ T\|.$$

Thus $0 \le T(f,y) + \int_X f d\mu^-$ or $T(f,y) \ge -\|f\|_X \frac{1}{2}(-T(1_X,y) + \|\Phi_y \circ T\|)$. This implies $T(f,y) + \|f\|_X \frac{1}{2}(1-M) \ge 0$, where $M := \dfrac{|L(1_X,y)|}{\|\Phi_y \circ L\|} \le 1$. Applying the latter inequality to $f - h_1$ and $h_2 - f$ shows that

$$T(f-h_1) + \|f-h_1\|\frac{1}{2}(1-M) \ge 0,$$

and

$$T(h_2-f) + \|h_2-f\|\frac{1}{2}(1-M) \ge 0.$$

Consequently

$$\begin{aligned}T(f) - f(g_A(y))T(1_X) &\ge -\|f-h_i\|\frac{1}{2}(1-M) - T(1+\frac{d}{\epsilon})\hat{\omega}(f,\epsilon)\\ &\ge -\max(\|f-h_i\|, i=1,2)\frac{1}{2}(1-M) - T(1+\frac{d}{\epsilon})\hat{\omega}(f,\epsilon).\end{aligned}$$

Similarly

$$T(f) - f(g_A(y))T(1_X) \le \max(\|f-h_i\|, i=1,2)\frac{1}{2}(1-M) + T(1+\frac{d}{\epsilon})\hat{\omega}(f,\epsilon).$$

Thus

$$|T(f) - f(g_A(y))T(1_X)| \le (1-M)(1+\frac{d}{\epsilon})\hat{\omega}(f,\epsilon) + T(1+\frac{d}{\epsilon})\hat{\omega}(f,\epsilon).$$

Recalling the definition of T, we obtain

$$|(L-A)(f,y)| \le |L(f,y) - L(1_X,y)f(g_A(y))| + |(L-A)(1_X,y)||f(g_A(y))|$$

and

$$\begin{aligned}&|L(f,y) - L(1_X,y)f(g_A(y))|\\ &\le \|\Phi_y \circ L\|(T(f,y) - f(g_A(y))T(1_X)|\\ &\le \|\Phi_y \circ L\|(1-M)(1+\epsilon^{-1}d(X)) + \frac{|L(1_X,y)|}{L(1_X,y)\|\Phi_y \circ L\|}L\Big(1+\frac{d}{\epsilon}\Big)\hat{\omega}(f,\epsilon)\\ &\le (\|\Phi_y \circ L\| - L(1_X,y))(1+\epsilon^{-1}d(X)) + \frac{|L(1_X,y)|}{L(1_X,y)\|\Phi_y \circ L\|}L\Big(1+\frac{d}{\epsilon}\Big)\hat{\omega}(f,\epsilon),\end{aligned}$$

which yields the theorem.

Corollary 2.1. *As is immediately seen from Theorem* 2.1 *we have*

(1) *Under the conditions of Theorem* 2.1, *we have*

$$\begin{aligned}\|(L-A)(f,y)| &\le (\|\Phi_y \circ L\| - L(1_X,y))\Big(1+\frac{d(X)}{\epsilon}\Big)\\ &\quad + |L(1) + \epsilon^{-1}L(d(t,g_A(y)),y)|\hat{\omega}(\epsilon) + |(L-A)(1_X,y)||f(g_A(y))|.\end{aligned}$$

(2) *If L is a positive linear operator, and hence $\|\Phi_y \circ L\| = |L(1_X,y)|$, then the above estimate reduces to*

$$|(L-a)(f,y)| \le L(1+\epsilon^{-1}d(t,g_A(y)))\hat{\omega}(\epsilon) + |(L-A)(1_X,y)||f(g_A(y))|.$$

(3) *If (X,d) is metrically convex, we can use $\omega(f,\epsilon)$ instead of $\hat{\omega}(f,\epsilon)$ in all the above estimates.*

Remark 2.1. Comparing Theorem 2.1 with Theorem A, we consider the case $Y = X,\ A = Id,\ L(1_X, x) = 1$. Theorem A implies that

$$|L(f,x) - f(x)| \leq \max(1, \epsilon^{-1} L(d(t, g_A(y)), x)\tilde{\omega}(f, \epsilon)$$

and Corollary 2.2 implies that

$$|L(f,x) - f(x)| \leq (1 + \epsilon^{-1} L(d(t, g_A(y)), x))\hat{\omega}(f, \epsilon).$$

If f satisfies $\tilde{\omega}(f,\epsilon) = \hat{\omega}(f,\epsilon)$, the first estimate is sharper. If f satisfies $\tilde{\omega}(f,\epsilon) = 2\hat{\omega}(f,\epsilon)$, the second estimate is sharper.

Remark 2.2. We can apply Theorem 2.1 on many operators. For example, we can consider Berstain-type operator on $X = [0,1]\bigcup[2,3]$ which is not metrically convex.

$$(L_n f)(x) := \sum_{i=0}^{n-1} f\big(\frac{i}{n}\big)\omega_{3n,i}(x,\alpha) + \sum_{i=n}^{2n-1}\Big(f(1)\big(2 - \frac{i}{n}\big) + f(2)\big(\frac{i}{n} - 1\big)\Big)\omega_{3n,i}(x,\alpha) + \sum_{i=2n}^{3n} f\big(\frac{i}{n}\big)\omega_{3n,i}(x,\alpha).$$

Here we have

$$\omega_{p,k}(f,\alpha) = \binom{n}{k}\frac{\prod_{r=0}^{k-1}(t+\alpha r)\prod_{r=0}^{p-k-1}(1-t+\alpha r)}{\prod_{r=0}^{p-1}(1+\alpha r)}.$$

Then we have

$$|L_n(f,x) - f(x)| \leq (1 + \epsilon^{-1}(3n)^{-1}(1+\alpha)^1(1+3\alpha n)x_i(1-x_i))^{\frac{1}{2}}\hat{\omega}(f,\epsilon).$$

The other applications will appear elsewhere.

References

[1] Gonska, H., On approximation by linear operators, Improved estimate, *Anal. Numér Théor. Approx.,* **14** (1985), 7-32.
[2] Gonska, H., Quantitative approximation in $C(X)$, Schriftenreihe des Fachbereichs Mathematik, 1985.
[3] Pozo, M. J., Déformation de la convexité et théorèmes du type Korovkin, *C. R. Acad. Sci. Paris Sér.,* **A 290** (1980), 213-215.
[4] Mamedov, R., On the order of approximation of functions by linear positive operators, *Dokl. Akad. Nauk. SSSR,* **128** (1959), 674-676.
[5] Menger, K., Unterschungen uber allgemeine metrik, *Math. Ann.,* **100** (1928).
[6] Newman, D. & Shapiro, H., Jackson's theorem in higher dimension, On approximation theory (ed. by P. Butzer and J. Korevaar), 1964, 208-219.