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A CONSTRUCTIVE PROOF OF THE INVERSION FORMULA FOR ZONAL FUNCTIONS ON $SL(2, R)$

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Abstract

A constructive proof is given for the inversion formula for zonal functions on $SL(2,R)$. A concretely constructed sequence of zonal functions are proved to satisfy the inversion formula obtained by Harish-Chandra for compact supported infinitely differentiable zonal functions. Making use of the property of this sequence somehow similar to that of approximation kernels, the authors deduce that the inversion formula is true for continuous zonal functions on $SL(2,R)$ under some condition. The classical result can be viewed as a corollary of the results here.

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§1. Introduction

Let $SL(2,R)$ denote the multiplicative group of all 2×2 real matrices with determinant 1. In this paper, we use G to denote both $SL(2,R)$ and the linear Lie group

$$SU(1,1)=\left\{\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}:|\,\alpha\,|^2-|\,\beta\,|^2=1\right\} \tag{1.1}$$

because they are isomorphic to each other. For $j=\{0,1/2\},\ s=\frac{1}{2}+i\lambda$ (where $\lambda\in R$, and R is the set of all real numbers), let $V^{j,s}$ be the principal continuous series of unitary representations of G (cf. [4]).

Set

$$SK=\left\{u_s=\begin{pmatrix}\exp(is/2) & 0\\ 0 & \exp(-is/2)\end{pmatrix}: s\in R\right\}, \tag{1.2}$$

$$SA=\left\{a_t=\begin{pmatrix}\cosh(t/2) & \sinh(t/2)\\ \sinh(t/2) & \cosh(t/2)\end{pmatrix}: t\in R\right\}, \tag{1.3}$$

and

$$SN=\left\{n_r=\begin{pmatrix}1+ir/2 & -ir/2\\ ir/2 & 1-ir/2\end{pmatrix}: r\in R\right\}. \tag{1.4}$$

By the Iwasawa decomposition, any $g\in G$ can be uniquely written as

$$g=u_sa_tn_r,\quad u_s\in SK,\quad a_t\in SA,\quad n_r\in SN. \tag{1.5}$$

Also any g in G has a Cartan decomposition as follows:

$$g=u_xa_tu_y,\quad 0\le x<4\pi,\quad 0\le t,\quad 0\le y<2\pi. \tag{1.6}$$

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A function f on G is said to be a zonal function if it satisfies $f(kgk') = f(g)$ for each $g \in G$ and $k, k' \in SK$. The set of all complex valued zonal functions on G is denoted by A.

The following inversion formula for a function in $C_c^\infty(G) \cap A$ is well known (cf. [2-6]).

Proposition 1.1. *If $f \in C_c^\infty(G) \cap A$, i.e., f is an infinitely differentiable, compact supported and zonal function on G, then*

$$f(g) = \frac{1}{2\pi} \int_0^{+\infty} \hat{f}\big(\frac{1}{2} + i\lambda\big)\phi\big(g, \frac{1}{2} + i\lambda\big)\lambda \tanh \pi\lambda d\lambda, \quad for\ each\ g \in G, \tag{1.7}$$

where

$$\phi(g, s) = (V_g^{0,s} f_0, f_0), \quad f_0 \equiv 1, \tag{1.8}$$

and

$$\hat{f}(s) = \int_G f(g)\phi(g^{-1}, s)dg. \tag{1.9}$$

In this paper, we give a constructive proof for the inversion formula. A concretely constructed sequence of zonal functions are proved to satisfy the inversion formula. Making use of the property of this sequence somehow similar to that of approximation identity kernels, we can deduce that the inversion formula is true for continuous zonal functions on G under the condition $f \in L^1(R, \lambda \tanh \pi\lambda)$. Proposition 1.1 can be viewed as a corollary of our result.

§2. Construction

For any $f \in A$, set

$$f^\star(t) = f(a_t), \quad t \in R. \tag{2.1}$$

Since $a_{-t} = u_\pi a_t u_\pi^{-1}$ for any $t \in R$, $f^\star$ is an even function on R. Therefore the following definition is meaningful:

$$f^0(x) = f^\star(t) = f(a_t), \quad x = \cosh t. \tag{2.2}$$

For $f \in L^1(G) \cap A$, it can be proved that (cf. [4])

$$\hat{f}(\frac{1}{2} + i\lambda) = \int_{-\infty}^{\infty} F_f(t)e^{-i\lambda t}dt, \quad t \in R, \tag{2.3}$$

where

$$F_f(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(a_t n_r)dr. \tag{2.4}$$

If $a_t n_r = u_x a_{t'} u_y$ is the Cartan decomposition, we can prove

$$\cosh t' = \cosh t + \frac{1}{2}e^t r^2. \tag{2.5}$$

It follows that

$$\begin{aligned}(F_f)^0(x) &= e^{t/2} \int_{-\infty}^{\infty} f^0(\cosh t + \frac{1}{2}e^t r^2)dr \\ &= \int_{-\infty}^{\infty} f^0(x + \frac{1}{2}s^2)ds.\end{aligned} \tag{2.6}$$

For $n \geq 1, g \in G$, set

$$h_n(g) = \begin{cases} \frac{1}{M_n}(L_n - n\cosh t)^2, & \text{for } 0 \leq t \leq 1/n \\ 0, & \text{for } t > 1/n, \end{cases} \tag{2.7}$$

where $g = u_x a_t u_y$ is the Cartan decomposition,

$$K_n = \cosh(1/n), \quad L_n = nK_n, \tag{2.8}$$

$$M_n = 2\pi \int_0^{\frac{1}{n}} (L_n - n\cosh t)^2 \sinh t dt > 0. \tag{2.9}$$

It follows from the continuity of the Cartan decomposition that $h_n \in C_c(g) \cap A$ and we can easily prove that

$$h_n \geq 0 \text{ and } \int_G h_n(g)dg = 1. \tag{2.10}$$

By the definition (2.7), for any $\delta > 0$ there is an $N = [\frac{1}{\delta}] + 1$ such that, when $n > N$, we have

$$2\pi \int_\delta^\infty h_n(a_t) \sinh t dt = 0. \tag{2.11}$$

Theorem 2.1. *If $f \in C(G) \cap A$, then*

$$\lim_{n\to\infty} (f * h_n)(e) = f(e), \tag{2.12}$$

where e is the identity of G.

Proof. Firstly, we note that the Haar integral on G is given by the formula

$$\int_G f(g)dg = 2\pi \int_{SK} \int_0^\infty \int_{SK} f(ka_t k') \sinh t dk dt dk', \tag{2.13}$$

so

$$\begin{aligned}(f * h_n)(e) - f(e) &= \int_G (f(g^{-1}) - f(e))h_n(g)dg \\ &= 2\pi \int_0^\infty (f(a_t) - f(e))h_n(a_t) \sinh t dt. \end{aligned} \tag{2.14}$$

For any $\epsilon > 0$, because f is continuous at e, there exists a $\delta > 0$ such that, when $0 \leq t < \delta$, we have

$$|f(a_t) - f(e)| < \frac{\epsilon}{2}. \tag{2.15}$$

It follows from (2.11) that when $n > N = [\frac{1}{\delta}] + 1$,

$$\begin{aligned} &| (f * h_n)(e) - f(e) | \\ \leq\ & 2\pi \int_0^\delta | f(a_t) - f(e) || h_n(a_t) | \sinh t dt \\ &+ 2\pi \int_\delta^\infty | f(a_t) - f(e) || h_n(a_t) | \sinh t dt \\ \leq\ & \frac{\epsilon}{2} 2\pi \int_0^\delta | h_n(a_t) | \sinh t dt + 2 \parallel f \parallel_\infty 2\pi \int_\delta^\infty | h_n(a_t) | \sinh t dt \\ \leq\ & \frac{\epsilon}{2} 2\pi \int_0^\infty | h_n(a_t) | \sinh t dt + 0 \\ =\ & \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Corollary 2.1. *For any $\lambda \in R$, we have*

$$\lim_{n\to\infty} \hat{h}_n(\frac{1}{2} + i\lambda) = 1. \tag{2.16}$$

Proof. By the definitions of ϕ and $\hat{h}_n$, we know

$$\phi\big(g, \frac{1}{2} + i\lambda\big) \in C(G) \cap A$$

and

$$\hat{h}_n\big(\frac{1}{2} + i\lambda\big) = \big(\phi\big(\cdot, \frac{1}{2} + i\lambda\big) * h_n\big)(e). \tag{2.17}$$

Therefore, (2.16) follows from Theorem 3.1.

Theorem 2.2. *For each $n \geq 1$, we have*

$$h_n(e) = \frac{1}{2\pi} \int_0^\infty \hat{h}_n\big(\frac{1}{2} + i\lambda\big)\lambda \tanh \pi\lambda d\lambda. \tag{2.18}$$

Proof. For $x = \cosh t$, by the definition of h_n, we have

$$h_n^0(x) = \begin{cases} \frac{1}{M_n}(L_n - nx)^2, & 1 \leq x \leq K_n, \\ 0, & x > K_n. \end{cases} \tag{2.19}$$

When $1 \leq x \leq K_n$, it follows from (2.6) that

$$\begin{aligned} F_{h_n}^0(x) &= \int_{-\infty}^\infty h_n^0(x + (s^2)/2) ds \\ &= \int_{-\sqrt{2(K_n - x)}}^{\sqrt{2(K_n - x)}} \frac{1}{M_n}\big(L_n - n\big(x + \frac{1}{2}s^2\big)\big)^2 ds \\ &= \frac{16\sqrt{2}n^2}{15M_n}(K_n - x)^{5/2}; \end{aligned}$$

and when $x > K_n$, we have $x + s^2/2 > K_n$, so $F_{h_n}^0(x) = 0$. Therefore

$$F_{h_n}^0(x) = \begin{cases} \frac{16\sqrt{2}n^2}{15M_n}(K_n - x)^{5/2}, & 1 \leq x \leq K_n, \\ 0, & x > K_n. \end{cases} \tag{2.20}$$

So we have

$$\frac{d}{dt}(F_{h_n}^0(x)) = \begin{cases} -\frac{8\sqrt{2}n^2}{3M_n}(K_n - x)^{3/2}, & 1 \leq x \leq K_n, \\ 0, & x > K_n. \end{cases} \tag{2.21}$$

Set

$$H_n = -\frac{1}{2\pi} \int_{-\infty}^\infty \big(F_{h_n}^0(x)\big)' \big(1 + \frac{1}{2}s^2\big) ds. \tag{2.22}$$

Then

$$\begin{aligned} H_n &= \frac{2n^2}{3M_n\pi} \int_{-\sqrt{2(K_n - x)}}^{\sqrt{2(K_n - x)}} ((\sqrt{2(K_n - 1)})^2 - s^2)^{3/2} ds \\ &= \frac{n^2}{M_n}(K_n - 1)^2 \\ &= h_n(e). \end{aligned} \tag{2.23}$$

On the other hand, setting $s = 2\sinh(t/2)$ in (2.22), we get

$$H_n = -\frac{1}{2\pi} \int_{-\infty}^\infty (F_{h_n}^0(x))'(\cosh t) \cosh(t/2) dt. \tag{2.24}$$

It is easy to see from (2.20) that

$$F_{h_n}(t) = \begin{cases} \frac{16\sqrt{2}n^2}{15M_n}(K_n - \cosh t)^{5/2}, & -\frac{1}{n} \leq t \leq \frac{1}{n} \\ 0, & |t| > \frac{1}{n}, \end{cases} \tag{2.25}$$

so $F_{h_n}(t)\in C_c^2(R)$, and it follows from (2.3) that $\hat{h}_n(\frac{1}{2}+i\lambda)$ is the Fourier transform of $F_{h_n}(t)$ on R. By the classical results about Fourier analysis on R, we have

$$\begin{aligned}F^0_{h_n}(\cosh t)&=F_{h_n}(t)\\&=\frac{1}{\pi}\int_0^\infty \hat{h}_n(\frac{1}{2}+i\lambda)\cos\lambda t d\lambda.\end{aligned}\tag{2.26}$$

Taking derivative with respect to $x=\cosh t$ in both sides of (2.26), we get

$$(F^0_{h_n}(x))'(\cosh t)=\frac{1}{\pi}\int_0^\infty \hat{h}_n\Big(\frac{1}{2}+i\lambda\Big)\Big(\frac{1}{\sinh t}\Big)\frac{d}{dt}(\cos(\lambda t))d\lambda.\tag{2.27}$$

Making use of (2.23), (2.24), Fubini theorem and the relation

$$\int_{-\infty}^\infty (\sinh(t/2))^{-1}\sin\lambda t dt=2\pi\tanh\pi\lambda,\tag{2.28}$$

we obtain

$$\begin{aligned}h_n(e)&=\frac{1}{2\pi^2}\int_0^\infty \hat{h}_n(\frac{1}{2}+i\lambda)\Big(\int_{-\infty}^\infty \frac{\lambda\sin\lambda t}{\sinh t}\cosh\frac{t}{2}dt\Big)d\lambda\\&=\frac{1}{2\pi}\int_0^\infty \hat{h}_n(\frac{1}{2}+i\lambda)\lambda\tanh\pi\lambda d\lambda.\end{aligned}\tag{2.29}$$

Thoerem 2.3. *For any $n\geq 1$, we have*

$$h_n(g)=\frac{1}{2\pi}\int_0^\infty \hat{h}_n(\frac{1}{2}+i\lambda)\phi(g,\frac{1}{2}+i\lambda)\lambda\tanh\pi\lambda d\lambda.\tag{3.30}$$

Proof. For $0\leq t_0<\infty$, set

$$h_n^c(g)=\int_{SK} h_n(a_{t_0}kg)dk.\tag{2.31}$$

It is easy to see that $h_n^c\in C_c(G)\cap A$. For $x=\cosh t$, we have

$$\begin{aligned}(h_n^c)^0(x)&=h_n^c(a_t)\\&=\frac{1}{4\pi}\int_0^{4\pi} h_n(a_{t_0}u_\theta a_t)d\theta.\end{aligned}\tag{2.32}$$

If $a_{t_0}u_\theta a_t=u_y a_{t'}u_z$ is the Cartan decomposition, then

$$\cosh t'=\cosh t_0\cosh t+\sinh t_0\sinh t\cos\theta.\tag{2.33}$$

Therefore

$$\begin{aligned}(h_n^c)^0(x)&=\frac{1}{4\pi}\int_0^{4\pi} h_n(a_{t'})d\theta\\&=\frac{1}{4\pi}\int_0^{4\pi} h_n^0(x\cosh t_0+\sqrt{x^2-1}\sinh t_0\cos\theta)d\theta.\end{aligned}\tag{2.34}$$

It follows that $(h_n^c)^0(x)\in C_c(R)$, and we can easily prove that the infinite integral

$$\int_0^\infty ((h_n^c)^0)'(x+\frac{1}{2}s^2)ds$$

converges uniformly with respect to x in the support of $(h_n^c)^0(x)$. Hence it follows from the definition of $(F_{h_n}c)^0(x)$ (cf. (2.6)) that

$$((F_{h_n^c})^0)'(x)=\int_{-\infty}^\infty ((h_n^c)^0)'(x+\frac{1}{2}s^2)ds.\tag{2.35}$$

Hence

$$
\begin{aligned}
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}((F_{h_n^c})^0)'\big(x+\frac{1}{2}s^2\big)ds\\
&=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}((h_n^c)^0)'\big(x+\frac{1}{2}s^2+\frac{1}{2}t^2\big)dtds\\
&=-\frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{2\pi}((h_n^c)^0)'\big(x+\frac{1}{2}r^2\big)rd\theta dr\\
&=(h_n^c)^0(x).
\end{aligned}
\tag{2.36}
$$

It follows that

$$
\begin{aligned}
h_n^c(e)&=(h_n^c)^0(1)\\
&=-\frac{1}{2\pi}\int_{-\infty}^{\infty}(F_{h_n}c^0)'\big(1+\frac{1}{2}s^2\big)ds.
\end{aligned}
\tag{2.37}
$$

Note that (2.36) is the same as (2.23) with h_n^c, so we can prove that (2.29) is true with h_n^c replacing h_n, i.e.,

$$
h_n^c(e)=\frac{1}{2\pi}\int_0^{\infty}\widehat{(h_n^c)}\big(\frac{1}{2}+i\lambda\big)\lambda\tanh\pi\lambda d\lambda. \tag{2.38}
$$

For any $x,y\in G$ and $\lambda\in R$, we can verify

$$
\int_{SK}\phi\big(xky,\frac{1}{2}+i\lambda\big)dk=\phi\big(x,\frac{1}{2}+i\lambda\big)\phi\big(y,\frac{1}{2}+i\lambda\big). \tag{2.39}
$$

By Fubini theorem, we have

$$
\begin{aligned}
\widehat{(h_n^c)}\big(\frac{1}{2}+i\lambda\big)&=\int_G h_n^c(g_1)\phi\big(g_1^{-1},\frac{1}{2}+i\lambda\big)dg_1\\
&=\int_G\int_{SK}h_n(a_{t_0}kg_1)\phi\big(g_1^{-1},\frac{1}{2}+i\lambda\big)dkdg_1\\
&=\int_{SK}\int_G h_n(g_1)\phi\big(g_1^{-1}a_{t_0}k,\frac{1}{2}+i\lambda\big)dg_1dk.
\end{aligned}
\tag{2.40}
$$

Since h_n and ϕ are zonal functions on G, it follows from (2.39) and (2.40) that

$$
\begin{aligned}
\widehat{(h_n^c)}(\frac{1}{2}+i\lambda)&=\int_G\int_{SK}h_n(k^{-1}g_1)\phi\big(g_1^{-1}a_{t_0},\frac{1}{2}+i\lambda\big)dkdg_1\\
&=\int_G\int_{SK}h_n(g_1)\phi\big(g_1^{-1}ka_{t_0},\frac{1}{2}+i\lambda\big)dkdg_1\\
&=\int_G h_n(g_1)\phi\big(g_1^{-1},\frac{1}{2}+i\lambda\big)\phi\big(a_{t_0},\frac{1}{2}+i\lambda\big)dg_1\\
&=\hat{h}_n\big(\frac{1}{2}+i\lambda\big)\phi\big(a_{t_0},\frac{1}{2}+i\lambda\big).
\end{aligned}
\tag{2.41}
$$

On the other hand, we have

$$
h_n^c(e)=\int_{SK}h_n(a_{t_0}k)dk=h_n(a_{t_0}). \tag{2.42}
$$

From (2.38), (2.41) and (2.42), it follows that

$$
h_n(a_{t_0})=\frac{1}{2\pi}\int_0^{\infty}\hat{h}_n\big(\frac{1}{2}+i\lambda\big)\phi(a_{t_0},\frac{1}{2}+i\lambda)\lambda\tanh\pi\lambda d\lambda. \tag{2.43}
$$

For any $g\in G$, let $g=u_xa_{t_0}u_y$ be the Cartan decomposition. Then $t_0\geq 0$, so it follows from (2.43) that

$$\begin{aligned} h_n(g) &= h_n(a_{t_0}) \\ &= \frac{1}{2\pi}\int_0^\infty \hat{h}_n\big(\frac{1}{2}+i\lambda\big)\phi\big(g,\frac{1}{2}+i\lambda\big)\lambda\tanh\pi\lambda d\lambda. \end{aligned} \tag{2.44}$$

§3. Main Result

Theorem 3.1. *Let $f\in C(G)\cap A$. If $\hat{f}\in L^1(R,\lambda\tanh\pi\lambda)$, then*

$$f(g) = \frac{1}{2\pi}\int_0^\infty \hat{f}\big(\frac{1}{2}+i\lambda\big)\phi\big(g,\frac{1}{2}+i\lambda\big)\lambda\tanh\pi\lambda d\lambda. \tag{3.1}$$

Proof. It is easy to see that

$$\mid \phi(g,\frac{1}{2}+i)\mid\le 1,\ \ g\in G,$$

so the integral on the right side of (3.1) is well defined.

Making use of Theorem 2.4 and Fubini theorem, we get

$$(f*h_n)(e) = \frac{1}{2\pi}\int_0^\infty \hat{f}\big(\frac{1}{2}+i\lambda\big)\hat{h}_n\big(\frac{1}{2}+i\lambda\big)\lambda\tanh\pi\lambda d\lambda. \tag{3.2}$$

Hence it follows from Theorem 2.1, Corollary 2.1 and (3.2) that

$$f(e) = \frac{1}{2\pi}\int_0^\infty \hat{f}\big(\frac{1}{2}+i\lambda\big)\lambda\tanh\pi\lambda d\lambda. \tag{3.3}$$

For any $g\in G$, if

$$g = u_x a_{t_0} u_y$$

is the Cartan decomposition, set

$$f^c(g) = \int_{SK} f(a_{t_0}kg)dk. \tag{3.4}$$

It can be shown that f^c satisfies the conditions demanded for f, so (3.3) is also true with f^c replacing f. Therefore we get

$$\begin{aligned} f^c(e) &= f(a_{t_0}) = f(g) \\ &= \int_0^\infty \hat{f}^c\big(\frac{1}{2}+i\lambda\big)\lambda\tanh\pi\lambda d\lambda. \end{aligned} \tag{3.5}$$

We can prove (cf. the proof of (2.41)) that

$$\hat{f}^c\big(\frac{1}{2}+i\lambda\big)\phi\big(g,\frac{1}{2}+i\lambda\big). \tag{3.6}$$

And (3.1) follows from (3.4) and (3.5).

Remark. If $f\in {C_c}^\infty(G)\cap A$, then f satisfies the conditions required in Theorem 3.1. So Proposition 1.1 (the known result) can be viewed as a corollary of our Theorem 3.1.

References

[1] Chevally, C., Thoery of Lie groups I, Princeton University Press, 1946.

[2] Gangolli, R., On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semi-simple Lie groups, *Ann. of Math.,* **93** (1971), 150-165.

[3] Harish-Chandra, Spherical functions on a semi-simple Lie group I, II, *Amer. J. Math.,* **80** (1958), 241-310, 553-613.

[4] Sugiura, M., Unitary representations and harmonic analysis, An introduction, Kokansha Ltd., 1975.

[5] Wallach, N., Harmonic analysis on homogeneous spaces, Marcel Dekker, 1973.

[6] Xiao Changbai & Zheng Weixing, About the inversion formula on the Lie group $SL(2, R)$, *Acta Math. Scientia,* **13:**4 (1993), 413-420.