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OSCILLATION CRITERIA FOR NONLINEAR SECOND ORDER ELLIPTIC DIFFERENTIAL EQUATIONS****

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Abstract

The second order elliptic differential equations

$$L_1(y;x)=\sum_{i,j=1}^{n} D_i[A_{ij}(x)D_jy]+p(x)f(y)=0 \tag{1.1}$$

and

$$L_2(y;x)=\Delta y+p(x)|y|^{\gamma}\text{sign}y=0, 1\neq\gamma>0 \tag{1.2}$$

are considered in an exterior domain $\Omega\subset R^n$, $n\geq 2$, where p can chang sign. Some new sufficient conditions for the oscillation of solutions of (1.1) and (1.2) are established.

Keywords Oscillation, Nonlinear elliptic equation, Riccati inequality
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§1. Introduction

Oscillation theory for elliptic differential equations with variable coefficients has been extensively developed in recent years by many authors (see e.g., Allegretto[1,2], Bugir[2], Fiedler[5], Kitamura and Kusano[7], Kura[4] Kusano and Naito[10], Naito and Yoshida[12], Noussair and Swanson[13,14], Swanson[15,16] and the references cited therein). In this paper, we are concerned with the oscillatory behavior of solutions of second order elliptic differential equations

$$L_1(y;x)=\sum_{i,j=1}^{n} D_i[A_{ij}(x)D_jy]+p(x)f(y)=0 \tag{1.1}$$

and

$$L_2(y;x)=\Delta y+p(x)|y|^{\gamma}\text{sign}y=0, 1\neq\gamma>0 \tag{1.2 }$$

with alternating coefficients.

As usual, points in n-dimensional Euclidean space R^n will be denoted by $x=(x_1,x_2,\cdots,x_n)$, the Euclidean length of x is denoted by $|x|$, and differentiation with respect to x_i is denoted by $D_i, i=1,2,\cdots,n$.

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We denoted by Δ the n-dimensional Laplace operator.

Definition 1.1. *For* $\Omega \subset R^n, \mu \in (0,1)$, *a function* $y \in C^{2+\mu}_{\rm loc}(\Omega)$ *which satisfies* (1.1) *for all* $x \in \Omega$ *is called a solution of* (1.1) *in* Ω.

In this paper we will consider (1.1) in an exterior domain $\Omega \subset R^n$, that is, a domain which contains the complement of some n-ball. Regarding the question of existence of solutions of (1.1) we refer the reader to the monograph [6].

Definition 1.2. *A nontrivial solution* y *of* (1.1) *is said to be oscillatory in* Ω *if the set* $\{x \in \Omega : y(x) = 0\}$ *is unbounded. Then equation* (1.1) *is called oscillatory in* Ω *if every solution of* (1.1) *is oscillatory in* Ω.

The purpose of this paper is to establish some new sufficient conditions for the oscillation of solutions of (1.1) and (1.2) in Ω. In section 2 we use the method of Riccati transformation developed by Noussair and Swanson[13] to reduce the problem of oscillation for (1.1) to that for some ordinary differential equations. By doing so we obtain some new oscillation criteria for (1.1). Section 3 contains some new oscillation criteria for (1.2) when p has a variable sign. To the best of our knowledge very little is known about the oscillation of (1.2) when p has a variable sign, especially, in the superlinear case.

§2. Oscillation Criteria for Equation (1.1)

Throughout this section we assume that the following conditions holds:

(A$_1$) $f \in C(R) \bigcup C^1(R\backslash\{0\}), f(-r) = -f(r), r > 0, f(r) > 0, f'(r) \geq k > 0$ for $r > 0$;

(A$_2$) $p \in C^{\mu}_{\rm loc}(\Omega), 0 < \mu < 1$;

(A$_3$) $A(x) = (a_{ij}(x))$ is a real symmetric positive definite matrix function with $a_{ij} \in C^{1+\mu}_{\rm loc}(\Omega), i, j = 1, 2, \cdots, n$, and $\mu \in (0,1)$.

Denote by $\rho_{\max}(x)$ the largest eigenvalue of $A(x)$. We suppose that there is a positive function $\lambda \in C^1((0,\infty))$ such that $\lambda(r) \geq \max\limits_{|x|=r} \rho_{\max}(x)$. Let y be a positive solution of (1.1) in $\Omega \bigcup G_b$, where $G_b = \{x \in R^n : 0 \leq b < |x| < \infty\}$. Since Ω is an exterior domain, for suitably large b, $G_b \subset \Omega$. Define

$$w(x) = -\frac{a(|x|)}{f(y(x))}(A(x) \bigtriangledown y) \tag{2.1}$$

and

$$\hat{w}(x) = w(x) + \frac{\lambda(|x|)}{2k} a'(|x|)\mathbf{n}(x), \tag{2.2}$$

where $a \in C^2(0,\infty)$ is an arbitrary positive function, $\bigtriangledown y$ denotes the gradient of y, $\mathbf{n}(x) = \frac{x}{|x|}, x \neq 0$, denotes the outward unit normal. By direct calculation we get

$$\text{div } \hat{w}(x) \geq \text{div} \frac{\lambda(r)}{2k} a'(r)\mathbf{n}(x) + a(r)p(x) - \frac{\lambda(r)[a'(r)]^2}{4ka(r)} + \frac{k}{a(r)\lambda(r)}|\hat{w}(x)|^2, \tag{2.3}$$

where $r = |x|$.

Lemma 2.1. *Let* y *be a positive solution of* (1.1) *in* G_b . *Put*

$$z(r) = \int_{S_r} \hat{w}^*(x)\mathbf{n}(x) d\sigma, \quad r > b, \tag{2.4}$$

where $S_r = \{x \in R^n : |x| = r\}$, $r > 0$, $d\sigma$ *denotes the spherical integral element in* R^n, *and*

$\hat{w}^*(x)$ is the transpose of $\hat{w}(x)$. Then z satisfies the Riccati inequality

$$z'(r) \geq \hat{p}(r) + h(r)z^2(r), \tag{2.5}$$

where

$$\begin{aligned}
\hat{p}(r) &= \int_{S_r} \left[a(|x|)p(x) - \frac{\lambda(|x|)[a'(|x|)]^2}{4ka(|x|)} \right] d\sigma + L'(r),\\
L(r) &= \frac{\omega}{2k}\lambda(r)a'(r)r^{n-1},\\
h(r) &= \frac{k}{\omega}\frac{r^{1-n}}{a(r)\lambda(r)} > 0,\\
&\omega = \mathit{the\ surface\ measure\ of\ unit\ sphere.}
\end{aligned} \tag{2.6}$$

Proof. Using Green's formula in (2.4) we get

$$z'(r) = \frac{d}{dr}\left(\int_{S_r} \hat{w}^*(x)\mathbf{n}(x)d\sigma \right) = \int_{S_r} \text{div } \hat{w}^*(x)d\sigma.$$

In view of (2.3), it follows that

$$\begin{aligned}
z'(r) \geq &\int_{S_r} \text{div}\frac{\lambda(r)}{2k}a'(r)\mathbf{n}^*(x)d\sigma + \int_{S_r}\left[a(r)p(x) - \frac{\lambda(r)[a'(r)]^2}{4ka(r)}\right]d\sigma\\
&+ \int_{S_r} \frac{k|\hat{w}(x)|^2}{a(r)\lambda(r)}d\sigma.
\end{aligned} \tag{2.7}$$

By the Cauchy-Schwartz inequality

$$\int_{S_r} |\hat{w}(x)|^2 d\sigma \geq \frac{r^{1-n}}{\omega}\left[\int_{S_r} \hat{w}^*(x)\mathbf{n}(x)d\sigma\right]^2 = \frac{r^{1-n}}{\omega}z^2(r),$$

and by Green's formula

$$\begin{aligned}
\int_{S_r} \text{div}\frac{\lambda(r)}{2k}a'(r)\mathbf{n}^*(x)d\sigma &= \frac{d}{dr}\left[\int_{S_r} \frac{\lambda(r)}{2k}a'(r)\mathbf{n}^*(x)\mathbf{n}(x)d\sigma\right]\\
&= \frac{d}{dr}\left[\frac{\lambda(r)}{2k}a'(r)\omega r^{n-1}\right] = L'(r).
\end{aligned}$$

Thus, (2.7) reduces to (2.5).

We now establish the main result of this section.

Theorem 2.1. *If the second order differential equation*

$$\left[\frac{1}{h(r)}y'(r)\right]' + \hat{p}(r)y(r) = 0 \tag{2.8}$$

is oscillatory then so is (1.1) *in* Ω *,where $h(r)$ and $\hat{p}(r)$ are as defined in* (2.6).

Proof. Assume the contrary, and let y be a positive solution of (1.1) in G_b. Then z satisfies

$$z'(r) \geq \hat{p}(r) + h(r)z^2(r), \quad r > b. \tag{2.9}$$

Consequently (see [17]) (2.8) has a nonoscillatory solution, which is a contradiction.

Remark 2.1. Noussair and Swanson (see [13, Theorem 4]) have shown that the conditions

$$\int_\tau^\infty h(r)dr = \infty \tag{2.10}$$

and

$$\int_{\tau}^{\infty} \hat{p}(r)dr = \infty \tag{2.11}$$

are sufficient for the oscillation of (1.1) in Ω. Obviously, if (2.10) and (2.11) hold then (2.8) is oscillatory (see [11]), and hence it follows from Theorem 2.2 that (1.1) is oscillatory in Ω. Thus, Theorem 4 in [13] is included in Theorem 2.2.

Now we consider the case when (2.10) holds and

$$\lim_{t\to\infty} \int_{T}^{t} \hat{p}(r)dr < \infty. \tag{2.12}$$

We put

$$\begin{cases} Q(t) = \displaystyle\int_{t}^{\infty} \hat{p}(r)dr, \quad A_0(t) = Q(t), \\ A_1(t) = \displaystyle\int_{t}^{\infty} h(r)A_{0+}^2(r)dr, \\ \qquad \cdots\cdots, \\ A_m(t) = \displaystyle\int_{t}^{\infty} h(r)[\{A_0(r) + A_{m-1}(r)\}_+]^2 dr, \quad m = 2, 3, \cdots, \end{cases} \tag{2.13}$$

where $f_+(t) = \max\{f(t), 0\}$, and prove the following:

Theorem 2.2. *Suppose that* (2.10) *and* (2.12) *hold. Then each of the following conditions imply the oscillation of* (1.1) *in* Ω*:*

(i) *there exists an integer* $m \geq 1$ *such that* $A_0(t), A_1(t), \cdots, A_{m-1}(t)$ *exist and* $A_m(t)$ *does not exist;*

(ii) $A_i(t), i = 0, 1, 2, \cdots,$ *exists and for each* T *there exists a* $t^* \geq T$ *such that*

$$\lim_{m\to\infty} A_m(t^*) = \infty.$$

Proof. We can use Theorem 2.2 of Yan[19] to conclude that (2.8) is oscillatory in Ω.

Theorem 2.3. *In addition to* (2.10) *and* (2.12)*, suppose there exists a positive twice continuously differentiable function* a *defined on* $(0, \infty)$ *such that the inequality*

$$\int_{T}^{\infty} \left\{ \int_{S_r} \left[a(r)p(x) - \frac{\lambda(r)a'^2(r)}{4ka(r)} \right] d\sigma + L'(r) \right\} dr$$
$$\geq \frac{k_0}{\int_{\tau}^{T} \frac{kr^{1-n}}{\omega a(r)\lambda(r)} dr}, \quad k_0 > \frac{1}{4}, \quad 0 < \tau < T, \tag{2.14}$$

holds. Then (1.1) *is oscillatory in* Ω.

Proof. As in [19] we can show that $\lim\limits_{m\to\infty} A_m(T) = \infty$. Thus, Theorem 2.3 follows from Theorem 2.2.

Corollary 2.1. Suppose that (2.10) holds and that

$$\lim_{t\to\infty} \Big\{ \int_{G(T,t)} \Big[a(|x|)p(x) - \frac{\lambda(|x|)[a'(|x|)]^2}{4ka(|x|)} \Big] d\sigma + \frac{\omega}{2k} t^{n-1} a'(t)\lambda(t) - \frac{\omega}{2k} T^{n-1} a'(T)\lambda(T) \Big\}$$
$$\geq \frac{k_0}{\int_{\tau}^{T} \frac{kr^{1-n}}{\omega a(r)\lambda(r)} dr}. \tag{2.15}$$

Then (1.1) is oscillatory in Ω. Here $G(T, t) = \{x \in R^n : T < |x| < t\}$.

Remark 2.2. Corollary 2.1 improves Theorem 4 in [13].

The following is an oscillation criterion for the equation

$$\Delta y + p(x)y = 0. \tag{2.16}$$

Corollary 2.2. *Suppose that the conditions*

$$\lim_{t\to\infty}\int_T^t [r\ln r p_m(r) - \frac{1}{4r\ln r}]dr \geq \frac{k_0\omega}{\ln(\ln T)}, n = 2, \tag{2.17}$$

and

$$\lim_{t\to\infty}\int_T^t \Big[rp_m(r) - \frac{(n-2)^2}{4r}\Big]dr \geq \frac{k_0\omega}{\ln T}, n \geq 3, \tag{2.18}$$

where

$$p_m(r) = \frac{1}{\omega(S_r)}\int_{S_r} p(x)d\sigma, k_0 > \frac{1}{4} \tag{2.19}$$

hold. Then (2.16) *is oscillatory in* Ω.

Proof. The assertion of Corollary 2.2 follows from that of Corollary 2.1 if we choose

$$a(r) = \begin{cases} \ln r & \text{for } n = 2, \\ r^{2-n} & \text{for } n \geq 3. \end{cases} \tag{2.20}$$

Remark 2.3. Corollary 2.2 improves some results in [13] and [18] given for (2.16).

Remark 2.4. In case (2.10) holds and

$$\lim_{t\to\infty}\int_T^t \hat{p}(r)dr \quad \text{does not exist}, \tag{2.21}$$

we can obtain some oscillation criteria for (1.1) by using the method given in [4]. We omit the details.

Finally we consider the case when instead of (2.10) we have

$$\int_T^\infty \frac{r^{1-n}}{a(r)\lambda(r)}dr < \infty. \tag{2.22}$$

Let

$$u = s(t) = \left(\int_t^\infty h(r)dr\right)^{-1} = \left(\frac{k}{\omega}\int_t^\infty \frac{r^{1-n}}{a(r)\lambda(r)}dr\right)^{-1}.$$

Since s is a monotone increasing function, s^{-1} exists. Note that

$$\frac{du}{dt} = u^2 h(s^{-1}(u)).$$

Now set

$$v(u) = -\frac{1}{u} + \frac{1}{u^2}z(s^{-1}(u)), \tag{2.23}$$

where z satisfies (2.5). From (2.23) it follows that

$$\frac{dv}{du} = \frac{1}{u^2} - \frac{2z}{u^3} + \frac{1}{u^2}\frac{dz}{du}. \tag{2.24}$$

Since

$$\frac{dz}{du} = \frac{dz}{dt}\frac{dt}{du} \geq (\hat{p}(t) + h(t)z^2(t))\frac{1}{u^2 h(s^{-1}(u))}, \tag{2.25}$$

by substituting this into (2.24) we get the following Riccati inequality

$$\frac{dv(u)}{du} \geq v^2(u) + \frac{\hat{p}(s^{-1}(u))}{u^4 h(s^{-1}(u))}. \tag{2.26}$$

Now from (2.26) we deduce the following result.

Theorem 2.4. *Suppose that there exists a positive twice continuously differentiable function a on* $(0,\infty)$ *satisfying* (2.22). *If*

$$\lim_{t\to\infty}\int_T^t \frac{\hat{p}(s^{-1}(\tau))}{\tau^4 h(s^{-1}(\tau))}d\tau \ge \frac{k_0}{T}, \quad k_0 > \frac{1}{4}, \tag{2.27}$$

then (1.1) *is oscillatory in* Ω.

Remark 2.5. In case $\lim\limits_{t\to\infty}\int_T^t h(r)dr$ does not exist, we can derive some oscillation creteria for (1.1) by using the method given in [20]. We omit the details.

§3. Oscillation of Equation (1.2)

First we consider a generalized sublinear equation

$$\Delta y(x) + p(x)f(y) = 0, \quad x\in\Omega, \tag{3.1}$$

and prove the following

Lemma 3.1. *Assume that*

(i) $f\in C(R)\bigcup C^1(R\backslash\{0\}), \quad f(r)>0$ *for* $r>0,\ f(-r)=-f(r)$;

(ii) $p\in C^{\mu}_{\rm loc}(\Omega), \quad 0<\mu<1$;

(iii) $0<\int_0^{\varepsilon}\frac{du}{f(u)}<\infty, \quad 0<\int_{-\varepsilon}^0\frac{du}{f(u)}<\infty$ *for every* $\varepsilon>0$;

(iv) *there exists* $c>0$ *such that*

$$\frac{F''(t)F(t)}{[F'(t)]^2} \le -\frac{1}{c}, \tag{3.2}$$

where $F(t)=\int_0^t\frac{du}{f(u)}, t>0$. *For a positive solution y of* (3.1) *in* G_b *and an arbitrary function* $m\in C^2(0,\infty)$, *put* $u(x)=m^{\nu}(r)y(x)$ *with*

$$\nu=\frac{1}{1+c}<1. \tag{3.3}$$

Then u satisfies

$$\begin{aligned}\Delta u(x)+m^{\nu}(r)p(x) \le& -\frac{1}{c}\sum_{i=1}^n m^{\nu-2}(r)\frac{1}{F}\left[mF'\frac{\partial y}{\partial x_i}-(1-\nu)m'\frac{Fx_i}{r}\right]^2\\ &+\nu m^{\nu-1}F^{-1}\left[\frac{n-1}{r}m'+m''\right], x\in G_b.\end{aligned} \tag{3.4}$$

Proof. By directly calculating $\Delta u(x)$ we have

$$\begin{aligned}&\Delta u(x)+m^{\nu}(r)p(x)\\ =&\sum_{i=1}^n\Big\{m^{\nu}F''(\frac{\partial y}{\partial x_i})^2+\nu(\nu-1)m^{\nu-2}Fm'^2\frac{x_i^2}{r^2}+2\nu m^{\nu-1}F'm'\frac{x_i}{r}\frac{\partial y}{\partial x_i}\Big\}\\ &+\nu m^{\nu-1}F^{-1}\left(\frac{n-1}{r}m'+m''\right).\end{aligned}$$

From (3.2) and (3.3)

$$F''(t)\le -\frac{1}{c}\frac{1}{F(t)}F'^2(t), \quad c=\frac{1-\nu}{\nu}.$$

Therefore

$$\Delta u(x)+m^{\nu}(r)p(x)\leq -\frac{1}{c}\sum_{i=1}^{n}m^{\nu-2}\frac{1}{F}\Big[mF'\frac{\partial y}{\partial x_i}-(1-\nu)m'F\frac{x_i}{r}\Big]^2 + \nu m^{\nu-1}F^{-1}\Big(\frac{n-1}{r}m'+m''\Big).$$

Now we establish the following result.

Theorem 3.1. Suppose that (i)-(iv) and (3.3) hold. If there exists a function m such that

$$\begin{cases} m(r)>0, \quad m\in C^2(0,\infty), \\ \dfrac{n-1}{r}m'(r)+m''(r)\leq 0, \quad r>0 \end{cases} \tag{3.5}$$

and

$$\int^{\infty} rm^{\nu}(r)p_m(r)dr=\infty, \tag{3.6}$$

then equation (3.1) is oscillatory in Ω.

Proof. Suppose that y is a positive solution of (3.1) in G_b. Using Lemma 3.1 we have

$$\Delta u(x)+m^{\nu}(r)p(x)\leq 0, \quad u(x)>0, \quad x\in G_b.$$

If

$$u_m(r)=\frac{1}{\omega(s_r)}\int_{S_r}u(x)d\sigma, \tag{3.7}$$

then

$$\frac{d}{dr}\Big[r^{n-1}\frac{d}{dr}u_m(r)\Big]=r^{n-1}\frac{1}{\omega(s_r)}\int_{S_r}\Delta u(x)d\sigma \leq -r^{n-1}m^{\nu}(r)p_m(r). \tag{3.8}$$

First, we suppose that $n=2$, in which case equation (3.8) reduces to

$$-\frac{d}{dr}\Big[r\frac{d}{dr}u_m(r)\Big]\geq rm^{\nu}(r)p_m(r). \tag{3.9}$$

Integrating (3.9) from t_0 to t we have

$$-tu_m'(t)+t_0u_m'(t_0)\geq \int_{t_0}^{t} rm^{\nu}(r)p_m(r)dr.$$

From assumption (3.6), it follows that there exist $K_1>0$ and $T>t_0$ such that

$$-tu_m'(t)\geq K_1 \ \text{for} \ \ t\geq T,$$

which leads to $\lim\limits_{t\to\infty}u_m(t)=-\infty$, which contradicts the fact that $u_m(t)>0$.

Now we suppose that $n\geq 3$. We use the Liouville transformation

$$r=\beta(s)=(ls)^l, \quad U(s)=su_m(\beta(s)), \quad l=\frac{1}{n-2}. \tag{3.10}$$

By a straight forward calculation, from (3.8) we obtain

$$-U''(s)\geq \frac{l^2}{s}m^{\nu}(\beta(s))p_m(\beta(s))\beta^2(s). \tag{3.11}$$

Now we integrate (3.11) and get

$$-U'(s)+U'(s_0)\geq \int_{s_0}^{s}\frac{l^2}{\sigma}m^{\nu}(\beta(\sigma))p_m(\beta(\sigma))\beta^2(\sigma)d\sigma.$$

Setting $\beta(\sigma)=\tau$ in the integration, then we get

$$-U'(s)+U'(s_0)\geq l\int_a^r \tau m^{\nu}(\tau)p_m(\tau)d\tau, \tag{3.12}$$

where $a=\beta(s_0), r=\beta(s)$ and $l>0$.

In view of conditions (3.6), (3.12) implies

$$-U'(s)\geq 1 \text{ for all large } s,$$

and so

$$U(s)\to -\infty \quad \text{as} \quad s\to +\infty,$$

which is a contradiction. Thus equation (3.1) is oscillatory in Ω.

Corollary 3.1. *Suppose that* (iii) *holds and* $0<\gamma<1$. *If*

$$\int^{\infty} r\ln^{\gamma} rP_m(r)dr=\infty, \quad \text{for} \quad n=2,$$

and

$$\int^{\infty} rP_m(r)dr=\infty, \quad \text{for} \quad n\geq 3,$$

then equation (1.2) *is oscillatory in* Ω.

Proof. In (3.1) we choose $f(y)=|y|^{\gamma}\mathrm{sgn}y, \ \ \nu=\gamma, \ 0<\gamma<1$ and

$$m(r)=\begin{cases}\ln r & \text{for} \ \ n=2,\\ 1 & \text{for} \ \ n\geq 3.\end{cases}$$

Then $m(r)$ satisfies (3.5) and hence the conclusion follows from Theorem 3.1.

Remark 3.1. Corollary 3.1 extends Theorem 1 in [16] when $0<\gamma<1$, where $p(x)\geq 0$ is required.

Now we consider equation (1.2) with $\gamma>1$.

Let y be a positive solution of (1.2) in G_b and let

$$u(x)=[y(x)]^{1-\gamma}. \tag{3.13}$$

By a direct calculation, we get

$$\Delta u(x)=\gamma_0 p(x)+\gamma_1\sum_{i=1}^{n}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2, \tag{3.14}$$

where $\gamma_0=\gamma-1, \gamma_1=\frac{\gamma}{\gamma-1}>1$. Put

$$W_1(r)=\int_{S_r}u(x)d\sigma, \quad m(r,u)=\frac{W_1(r)}{\omega r^{n-1}}. \tag{3.15}$$

Then

$$\frac{d}{dr}[r^{n-1}m'(r,u)]=\frac{1}{\omega}\int_{S_r}\Delta u d\sigma$$

and hence

$$\frac{d}{dr}[r^{n-1}m'(r,u)]=\frac{\gamma_0}{\omega}\int_{S_r}p(x)d\sigma+\frac{\gamma_1}{\omega}\int_{S_r}\sum_{i=1}^{n}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma. \tag{3.16}$$

Lemma 3.2. *If y is a positive solution of equation* (1.2) *in* G_b, *then* W_1 *defined in* (3.15) *satisfies*

$$W_1'(r) \le \frac{n-1}{r}W_1(r) + \sqrt{W_1(r)}\sqrt{\sum_{i=1}^{n}\int_{S_r}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma}, \quad r > b \tag{3.17}$$

and

$$W_1(r) \le \Big(\frac{r}{r_0}\Big)^{n-1}\left[\sqrt{W_1(r_0)} + \frac{1}{2}\int_{r_0}^{r}\Big(\frac{r_0}{\tau}\Big)^{\frac{n-1}{2}}\sqrt{\sum_{i=1}^{n}\int_{S_\tau}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma d\tau}\right]^2. \tag{3.18}$$

Proof. Since $W_1(r) = \int_{S_r} u(x)d\sigma$, we have

$$W_1'(r) = \frac{n-1}{r}W_1(r) + \int_{S_r}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{x_i}{r}d\sigma.$$

By the Cauchy-Schwartz inequality

$$\int_{S_r}\frac{\partial u}{\partial x_i}\frac{x_i}{r}d\sigma \le \sqrt{\int_{S_r}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma}\sqrt{\int_{S_r}u\frac{x_i^2}{r^2}d\sigma}.$$

Hence

$$\sum_{i=1}^{n}\int_{S_r}\frac{\partial u}{\partial x_i}\frac{x_i}{r}d\sigma \le \sqrt{\sum_{i=1}^{n}\int_{S_r}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma}\sqrt{\sum_{i=1}^{n}\int_{S_r}u\frac{x_i^2}{r^2}d\sigma}$$
$$= \sqrt{W_1(r)}\sqrt{\sum_{i=1}^{n}\int_{S_r}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma}$$

and

$$W_1'(r) \le \frac{n-1}{r}W_1(r) + \sqrt{W_1(r)}\sqrt{\sum_{i=1}^{n}\int_{S_r}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma}.$$

We let $z^2(r) = W_1(r)$, and get

$$z'(r) \le \frac{n-1}{2r}z(r) + \frac{1}{2}\sqrt{\sum_{i=1}^{n}\int_{S_r}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma}$$

and

$$z(r) \le \Big(\frac{r}{r_0}\Big)^{\frac{n-1}{2}}\left[z(r_0) + \frac{1}{2}\int_{r_0}^{r}\Big(\frac{\tau}{r_0}\Big)^{-\frac{n-1}{2}}\sqrt{\sum_{i=1}^{n}\int_{S_\tau}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma}\; d\tau\right].$$

Thus

$$W_1(r) \le \Big(\frac{r}{r_0}\Big)^{n-1}\left[\sqrt{W_1(r_0)} + \frac{1}{2}\int_{r_0}^{r}\Big(\frac{\tau}{r_0}\Big)^{-\frac{n-1}{2}}\sqrt{\sum_{i=1}^{n}\int_{S_\tau}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma}\; d\tau\right]^2,$$

and the proof of the lemma is complete.

Theorem 3.2. *Let $n = 2$ and $p \in C^{\mu}_{\rm loc}(\Omega), \mu \in (0,1)$. If*

$$\liminf_{t\to\infty}\frac{1}{t}\int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\tau\ln\tau p_m(\tau)d\tau dr > 0, \tag{3.19}$$

then equation (1.2) *with $\gamma > 1$ is oscillatory in Ω .*

Proof. Suppose the contrary and assume without loss of generality that there exists a positive solution y of (1.2) in G_b. With $u(x), W_1(r)$ and $m(r)$ defined in (3.3) and (3.15) we get, for $n=2$, from (3.16) the following

$$\frac{d}{dr}[rm'(r,u)] = \frac{\gamma_0}{\omega}\int_{S_r} p(x)d\sigma + \frac{\gamma_1}{\omega}\int_{S_r}\sum_{i=1}^{2}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma, r>b. \tag{3.20}$$

Multipling both sides of (3.20) by $\ln r$ and integrating from r_0 to r we get

$$r\ln rm'(r,u) - m(r,u) - c_0$$
$$= \frac{\gamma_0}{\omega}\int_{r_0}^{r}\ln\tau\int_{S_\tau} p(x)d\sigma d\tau + \frac{\gamma_1}{\omega}\int_{r_0}^{r}\ln\tau\int_{S_\tau}\sum_{i=1}^{2}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma d\tau, \tag{3.21}$$

where $c_0 = r_0\ln r_0 m'(r_0,u) - m(r_0,u)$.

Divide both sides of (3.21) by r, integrate from r_0 to t and finally divide by t to get

$$\frac{\ln t}{t}m(t,u) - \frac{2}{t}\int_{r_0}^{t}\frac{m(r,u)}{r}dr - \frac{c_0}{t}\ln\frac{t}{r_0} - \frac{c_2}{t}$$
$$= \frac{\gamma_0}{\omega}\frac{1}{t}\int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\tau\ln\tau\int_{S_\tau} p(x)d\sigma d\tau dr$$
$$+ \frac{\gamma_1}{\omega}\frac{1}{t}\int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\ln\tau\int_{S_\tau}\sum_{i=1}^{2}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma d\tau dr, \tag{3.22}$$

where $c_2 = \ln r_0 m'(r_0,u)$.

Using the Cauchy-Schwartz inequality in (3.18) we have

$$W_1(r) \le \frac{r}{r_0}\left\{\sqrt{W_1(r_0)} + \frac{1}{2}\Big[\int_{r_0}^{r}\sqrt{\frac{r_0}{\tau}}\sqrt{\sum_{i=1}^{2}\int_{S_\tau}\frac{1}{u}(\frac{\partial u}{\partial x_i})^2 d\sigma d\tau}\Big]^2\right\}$$
$$\le \frac{r}{r_0}\left\{\sqrt{W_1(r_0)} + \frac{r_0}{2}\Big[\int_{r_0}^{r}\frac{1}{\sqrt{\tau\ln\tau}}\sqrt{\ln\tau\sum_{i=1}^{2}\int_{S_\tau}\frac{1}{u}(\frac{\partial u}{\partial x_i})^2 d\sigma d\tau}\Big]^2\right\}$$
$$\le \frac{r}{r_0}\left\{\sqrt{W_1(r_0)} + \frac{r_0}{2}\int_{r_0}^{r}\frac{d\tau}{\tau\ln\tau}\int_{r_0}^{r}\ln\tau\sum_{i=1}^{2}\int_{S_\tau}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma\ d\tau\right\}$$
$$= r\left\{\frac{\sqrt{W_1(r_0)}}{r_0} + \frac{1}{2}\ln\Big(\frac{\ln r}{\ln r_0}\Big)\int_{r_0}^{r}\ln\tau\sum_{i=1}^{2}\int_{S_\tau}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma\ d\tau\right\}, \quad r>b. \tag{3.23}$$

We now consider two cases:

Case (i) $\ln\tau\int_{s_\tau}\sum_{i=1}^{2}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma \in L(0,\infty)$.

Then there exists a constant $K>0$ such that

$$\int_{r_0}^{r}\ln\tau\int_{S_\tau}\sum_{i=1}^{2}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma d\tau < K, \tag{3.24}$$

and hence from (3.23) we have

$$W_1(r) \le r\left\{\frac{\sqrt{W_1(r_0)}}{r_0} + \frac{K}{2}\ln\Big(\frac{\ln r}{\ln r_0}\Big)\right\}.$$

So

$$0 \le m(r,u) = \frac{W_1(r)}{\omega r} \le \frac{1}{\omega}\left\{\frac{\sqrt{W_1(r)}}{r_0} + \frac{K}{2}\ln\Big(\frac{\ln r}{\ln r_0}\Big)\right\}.$$

Thus

$$\lim_{t\to\infty}\frac{1}{t}\ln t\cdot m(t,u) = 0.$$

From (3.24) it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\ln\tau\int_{S_\tau}\sum_{i=1}^{2}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma\ d\tau\ dr = 0.$$

From (3.22) we have

$$\lim_{t\to\infty}\frac{1}{t}\int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\tau\ln\tau p_m(\tau)d\tau\ dr = 0,$$

which contradicts (3.19).

Case (ii)

$$\ln\tau\int_{S_\tau}\sum_{i=1}^{2}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma \quad\notin L(0,\infty). \tag{3.25}$$

We define

$$\Phi(t) = \int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\ln\tau\sum_{i=1}^{2}\int_{S_\tau}\frac{1}{u}\Big(\frac{\partial u}{\partial x_i}\Big)^2 d\sigma\ d\tau\ dr.$$

From (3.22)

$$\begin{aligned}&\frac{\gamma_0}{\omega}\frac{1}{t}\int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\tau\ln\tau p_m(\tau)d\tau dr\\ =& -\frac{1}{t}\Big[\frac{\gamma_1}{\omega}\Phi(t) + 2\int_{r_0}^{t}\frac{m(r,u)}{r}dr\ln t\cdot m(t,u)\Big] - \frac{c_0}{t}\ln\frac{t}{r_0} - \frac{c_2}{t}.\end{aligned} \tag{3.26}$$

Now if

$$\limsup_{t\to\infty}\Big[\frac{\gamma_1}{\omega}\Phi(t) + 2\int_{r_0}^{t}\frac{m(r,u)}{r}dr - \ln t\cdot m(t,u)\Big] \ge 0, \tag{3.27}$$

then by (3.26)

$$\liminf_{t\to\infty}\frac{1}{t}\int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\tau\ln\tau p_m(\tau)d\tau dr \le 0,$$

which contradicts (3.19). And if (3.27) does not hold, then for t sufficiently large we have

$$\Big\{\frac{\gamma_1}{\omega}\Phi(t) + 2\int_{r_0}^{t}\frac{m(r,u)}{r}dr - \ln t\cdot m(t,u)\Big\} < 0. \tag{3.28}$$

Therefore

$$\gamma_1\Phi(t) \le \omega\ln t\cdot m(t,u) = \frac{\ln t}{t}W_1(t) \tag{3.29}$$

and

$$\sqrt{\frac{(t\Phi'(t))'}{t\Phi'(t)}}\sqrt{\frac{t\Phi'(t)}{t\Phi(t)}}=\sqrt{\frac{\ln t\int_{S_t}\sum\limits_{i=1}^{2}\frac{1}{u}\left(\frac{\partial u}{\partial x_i}\right)^2 d\sigma}{t\Phi(t)}}$$
$$\geq \sqrt{\gamma_1}\frac{W_1'(t)-\frac{1}{t}W_1(t)}{w_1(t)}$$
$$=\sqrt{\gamma_1}\Big(\frac{W_1'(t)}{W_1(t)}-\frac{1}{t}\Big). \tag{3.30}$$

From (3.25), it follows that $\Phi(t)\to\infty$ as $t\to\infty$. Thus, for a sufficiently large t_1 we get $\Phi(t_1)\geq 1$. Integrating (3.30) from t_1 to t and using the Cauchy-Schwartz inequality in the left side, we have

$$\sqrt{\ln\frac{t\Phi'(t)}{t_1\Phi'(t_1)}}\sqrt{\ln\Phi(t)}\geq\sqrt{\gamma_1}\ [\ln W_1(t)-\ln t]$$
$$\geq\sqrt{\gamma_1}\ [\ln\Phi(t)-\ln\ln t]$$

or

$$t\Phi'(t)\geq c_3\frac{\Phi^{\gamma_1}(t)}{(\ln t)^{2\gamma_1}}, c_3=t_1\Phi'(t_1). \tag{3.31}$$

Integrate (3.31) from T to t, $T\geq t_1$, to get

$$\frac{1}{1-\gamma_1}\Big[\frac{1}{\Phi^{\gamma_1-1}(t)}-\frac{1}{\Phi^{\gamma_1-1}(T)}\Big]\geq\frac{c_3}{1-2\gamma_1}\Big[\frac{1}{(\ln t)^{2\gamma_1-1}}-\frac{1}{(\ln T)^{2\gamma_1-1}}\Big].$$

Since $\gamma_1>1$ and $\Phi(t)\to\infty$ as $t\to\infty$,

$$\Phi(T)\leq c_4(\ln T)^{\frac{2\gamma_1-1}{\gamma_1-1}},\quad c_4=\Big[\frac{2\gamma_1-1}{c_3(\gamma_1-1)}\Big]^{\frac{1}{\gamma_1-1}}. \tag{3.32}$$

We want to show that

$$\liminf_{t\to\infty}\frac{\ln t}{t}m(t,u)=0. \tag{3.33}$$

In case (3.33) does not hold, there exists a $c_5>0$ such that

$$\frac{\ln t}{t}m(t,u)\geq c_5 \text{ for all sufficiently large } t.$$

Then

$$W_1(t)\geq c_5\omega\frac{t^2}{\ln t}.$$

From (3.23)

$$c_5\omega\frac{t^2}{\ln t}\leq W_1(t)\leq t\Big\{\frac{\sqrt{W_1(r_0)}}{r_0}+\frac{1}{2}\ln\Big(\frac{\ln t}{\ln r_0}\Big)t\Phi'(t)\Big\},$$

so

$$\Phi'(t)\geq\frac{c_5\omega}{\ln t\ln\big(\frac{\ln t}{\ln r_0}\big)},$$

or

$$\Phi(t)\geq\Phi(t_4)+\frac{c_5\omega}{\ln t\ln(\frac{\ln t}{\ln r_0})}(t-t_4), t\geq t_4. \tag{3.34}$$

This contradicts (3.32). Thus (3.33) must hold. From (3.26) we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_{r_0}^{t}\frac{1}{r}\int_{r_0}^{r}\tau\ln\tau p_m(\tau)d\tau dr\le 0,$$

which contradicts (3.19). Hence equation (1.2) is oscillatory in Ω.

§4. Discussion, an Open Problem

Example 4.1. Consider the linear equation

$$\Delta y+p(r)y=0,\quad n=2, \tag{4.1}$$

where

$$p(r)=\frac{1}{4r^2\ln^2 r}+\frac{\sin r}{r\ln r}.$$

Here no matter how we choose $a(r)$, it is impossible to have

$$\int^{\infty}\frac{dr}{ra(r)}=\infty$$

and

$$\int^{\infty}\hat{p}(r)dr=\infty \ \text{ simultaneously.}$$

Thus, the results in [13] cannot apply to (4.1) . But if we take $a(r)=\ln r$, then $\hat{p}(r)=\sin r$ is a periodic function. Moreover

$$-\infty<\liminf_{T\to\infty}\int^{T}\hat{p}(r)dr \ \text{ and } \ \int^{\infty}\frac{dr}{ra(r)}=\infty$$

hold. Thus by Theorem 2.1 and a known result for (2.8) in [4] equation (4 .1) is oscillatory in an exterior domain Ω.

Example 4.2. Consider the superlinear equation

$$\Delta y+\frac{2+\sin r+3r\cos r-r^2\sin r}{r\ln r}|y|^{\gamma}\text{sgn}y=0,\quad n=2,\quad \gamma>1. \tag{4.2}$$

Then

$$p_m(r)=\frac{2+\sin r+3r\cos r-r^2\sin r}{r\ln r}$$

is an alternating function. So Theorem 1 in [16] does not ensure the oscillation of equation (4.2) in an exterior domain Ω. Since

$$\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\frac{1}{r}\int_{t_0}^{r}\tau\ln\tau p_m(\tau)d\tau\ dr=1>0,$$

it follows from Theorem 3.2 that equation (4.2) is oscillatory in Ω. For $n\ge 3$, to the best of our knowledge, there is no corresponding result, and thus it remains an open problem.

References

[1] Allegretto, W., Oscillation criteria for quasilinear equations, *Canad. J. Math.,* **26**(1974), 931-947.

[2] Allegretto, W., Oscillation criteria for semilinear equations in general domains, *Canad. Math. Bull.,* **19**(1976), 137-144.

[3] Bugir, M. K., A remark on the conditions for the oscillation of solutions of nonlinear differential equations, *Ukrain. Math. Zh.,* **42:** 4(1990), 458-464.

[4] Elbert, A., Generalized Riccati equation for half linear second order differential equations, Differential equations, Qualitative theory, Szeged (Hungary),**47**, (1984), 247-249.

[5] Fiedler, F., Oscillation criteria of Nehari type for Sturm Liouville operators and elliptic differential operators of second order and the lower spectrum, *Proc. Roy. Soc. Edinburgh, Sect.,* **A 109:** 1-2 (1988), 127-144.

[6] Gilbarg, D. & Trudinger, N. S., Elliptic partial differential equation of second order, Springer-Verlag, Berlin Heidelburg, New York, Tokyo, 1983.

[7] Kitamura, Y. & Kusano, T., An oscillation theorem for a sublinear Schrödinger equation, *Utilitas Math.,* **14**(1978), 171-175.

[8] Kusano, T. & Naito, M., Oscillation criteria for a class of perturbed Schrödinger equations, *Canad. Math. Bull.,* **25:**1 (1982), 71-77.

[9] Kura, T., Oscillation criteria for a class of sublinear elliptic equations of the second order, *Utilitas Math.,* **22** (1983), 709-718.

[10] Kusano, T. & Naito, M., Oscillation theory of entire solutions of second order superlinear elliptic equations, *Funkcial Ekvac,* **30** (1987), 269-282.

[11] Leighton, W., The detection of the oscillation of solutions of second order linear differential equations, *Duke J. Math.,* **17** (1950), 57-62.

[12] Naito, M. & Yashida, N., Oscillation criteria for a class of higher order elliptic equations, *Math Rep. Toyama Univ.,* **12** (1989), 29-40.

[13] Noussair, E. S. & Swanson, C. A., Oscillation of semilinear elliptic inequalities by Riccati Transformation, *Canad. J. Math.,* **32** (1980), 908-923.

[14] Noussair, E. S. & Swanson, C. A., Oscillation theory for semilinear elliptic Schrödinger equations and inequalities, *Proc. Roy. Soc. Edinburg,* **75A** (1975/76), 67-81.

[15] Swanson, C. A., Criteria for oscillatory sublinear Schrödinger equations, *Pacific J. Math.,* **104** (1983), 483-493.

[16] Swanson, C. A., Positive solutions of $-\Delta u = f(x, u)$, *Nonlinear Analysis, Theory, Methods and Applications,* **9:**12 (1985), 1319-1323.

[17] Swanson, C. A., Comparison and oscillation theory of linear differential equations, Academic Press, N. Y., 1968.

[18] Swanson, C. A., Semilinear second order elliptic oscillation, *Canad. Math. Bull.,* **22:**2 (1979), 139-157.

[19] Yan Jurang, Oscillation properties for second order nonlinear differential equation with "integrally small" coefficient, *Acta Math. Sinica,* **30** (1987), 206-215.

[20] Yan Jurang, Oscillation theorem for second order linear differential equations with damping, *Proc. Amer. Math. Soc.,* **98:**2 (1986), 276-282.

[21] Yashida, N., Oscillation properties of solutions of second order elliptic equations, *SIAM. J. Math. Anal.,* **14:**4 (1983), 709-718.