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A REMARK ON EXTINCTION OF A CLASS OF SUPERPROCESSES**

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Abstract

The extinction of a class of superprocesses associated with general branching characteristics and underlying Markov processes is investigated. The extinction is closely associated with the branching characteristics and the recurrence and transience of underlying processes.

Keywords Superprocess, Extinction, Nonlinear evolution equation, Branching characteristic, Recurrence and transience
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§1. Introduction

Superprocesses, i.e., measure-valued Markov processes, have been an attractive topic in the recent years, which can be defined as follows. Let $(\xi, P_x)_{x\in \mathbb{R}^d}$ be a Markov process with strong continuous Markov semigroup S_t (denote by A its infinitesimal operator, and by $p_t(x,y)$ the associated transition function).

$$\psi : \mathbb{R}^d \times \mathbb{R}_+ \longmapsto \mathbb{R}_+, (x,\lambda) \to \gamma(x)\lambda^{1+\beta}, \quad 0<\beta\le 1.$$

Denote by M_p the set of all Radon measures μ satisfying

$$\int_{\mathbb{R}^d} \frac{1}{1+\|x\|^p}\mu(dx) < \infty, \quad p\ge 0,$$

and by $pC(\mathbb{R}^d)$ (resp. $pC_c(\mathbb{R}^d)$) the set of positively continuous functions (resp. with compact support) in $\mathbb{R}^d$. The so-called superprocess $(X, P^\mu)_{\mu\in M_p}$ with parameters ξ and ψ and taking value in M_p is determined by the following Laplacian functional:

$$P^\mu \exp\{-\langle X_t, f\rangle\} = \exp\{-\langle V_t f, \mu\rangle\}, \ f\in pC_c(\mathbb{R}^d), \ \mu\in M_p, \tag{1.1}$$

where $V_t f$ satisfies the following integral equation:

$$V_t f(x) + \int_0^t S_s\psi(x, V_{t-s}f(x))ds = S_t f(x), \ x\in \mathbb{R}^d \tag{1.2}$$

and $\langle f,\mu\rangle$ means the integral of f with respect to μ. See [4,7] for more general definitions. From [10] it is not difficult to prove that if the initial measure $\mu\in M_p$ for some $p\ge 0$, then, for any $t>0$, $X_t\in M_p$ almost surely.

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In this paper we are interested in the extinction of a class of superprocesses. As one of the most fundamental problems for stochastic processes, the extinction of superprocesses has been investigated by a number of authors. Dynkin[3] and Dawson[2] had considered such kinds of questions for different cases. More precisely, for $\gamma \equiv C$, a positive constant, Dynkin has proved that for $p = 0$ (M_p is thereby the set of all finite measures) X_t is extinct in the sense of

$$\langle X_t, 1\rangle \longrightarrow 0 \text{ (a.s. as } t \to \infty), \ \mu \in M_0, \tag{1.3}$$

i.e., the total mass of X_t vanishes as time tends to infinite. Indeed, this can be easily verified from the following simple observation. For $f \equiv 1$, the unique solution of equation (1.2) is

$$V_t = (1 + C\beta t)^{-1/\beta} \tag{1.4}$$

which is independent of $x \in \mathbb{R}^d$. Clearly, $V_t \to 0$ as $t \to \infty$, so from our following argument we know that (1.3) holds.

On the other hand, for $p > d$ the Lebesgue measure (denoted by L) belongs to M_p. In this case, Dawson (1977) demonstrated for α-stable processes and $\alpha < d$ that X_t with the Lebesgue measure as its initial measure is vaguely extinct in probability, i.e., for any compact set K and $\epsilon > 0$,

$$P^\mu\{X_t(K) > \epsilon\} \longrightarrow 0 \text{ as } t \longrightarrow \infty$$

if ξ is recurrent. Some more delicate results concerning extinction are due to [5,6,9,11] and so on. However, the above results as well as most previous others were obtained under the condition that γ is a constant. The spatial homogeneity has played an essential role. When γ is a general non-negative function, the corresponding superprocesses are generally not spatially homogeneous. This destroys much of the simplicity, and we have to seek a new way to approach our goals.

We now assume that the semigroup S_t satisfies:

(a) $S_t 1 = 1$, i.e., ξ is conservative;

(b) for any $x \in \mathbb{R}^d$ and any Borel set U whose Lebesgue measure is positive, then $S_t 1_U(x) > 0$ for some $t > 0$.

Clearly, these assumptions make sense for very wide situations.

For $p = 0$, [13] studied the absolute continuity of X_t and showed that, for any fixed $t > 0$, X_t on the "non-branching" set $\{x \in \mathbb{R}^d; \gamma(x) = 0\}$ is almost surely absolutely continuous with respect to Lebesgue measure. Now we turn to the investigation of the extinction of X_t in the sense of (1.3), and show that the extinction of X_t closely depends on the recurrence of underlying process ξ, as well as the branching characteristic γ. At first, we have

Theorem 1.1. *Under the assumptions* (a) *and* (b)*, suppose* supp(γ) (*the support of* γ) *is of positive Lebesgue measure. If ξ is a recurrent Hunt process then X_t is extinct in the sense of* (1.3).

On the other hand, we shall investigate the case that ξ is transient. We have

Theorem 1.2. *Suppose* supp(γ) *is of finite Lebesgue measure. If ξ is transient then for any $\mu \in M_p$, X_t is not a.s. extinct.*

To prove the above theorems, we first have to investigate a class of non-linear equations about which very few results can be found in existing literature. In the next section we will find that probabilistic approach to some non-linear equations is proved to be powerful. From above results we have the following

Corollary 1.1. *Suppose the Lebesgue measure of* supp(γ) *is finite and positive and* ξ *is* α*-stable process in* $\mathbb{R}^d$ $(0 < \alpha \leq 2)$*, then the corresponding superprocess* X_t *is extinct if* $\alpha \geq d$*, but not if* $\alpha < d$.

The proof of the theorems will be carried out in Section 2. We shall give some remarks in Section 3.

§2. Proof of Theorems

We prove the theorems through several lemmas. Consider

$$V_t(x) + \int_0^t P_x \gamma(\xi_s) V_{t-s}^{1+\beta}(\xi_s) ds = 1. \tag{2.1}$$

The following lemma is about the stability of solution of integral equation (2.1).

Lemma 2.1. *The convergence* $V_t \downarrow V_\infty$ $(t \longrightarrow \infty)$ *exists pointwise. Moreover,* V_∞ *is an invariant function of nonlinear semigroup* V_t *and* $X_t(\mathbb{R}^d) \longrightarrow$*some random variable* $X_\infty(\mathbb{R}^d)$ *almost surely as* $t \to \infty$.

Proof. Since $V_t(x) \leq 1$, we have

$$P^{X_s} e^{-\langle 1, X_t \rangle} = e^{-\langle V_{t-s}, X_s \rangle} \geq e^{-\langle 1, X_s \rangle}$$

and therefore $e^{-\langle 1, X_s \rangle}$ is a bounded submartingale. So

$$V_t(x) = -\log P^{\delta_x} e^{-\langle 1, X_t \rangle} \downarrow V_\infty(x)$$

as $t \to \infty$.

From this and martingale convergence theorem it is easy to prove the rest part of this lemma.

We here approach the stability of solution of nonlinear equation (2.1) probabilistically, which appears rather simple. In fact, such kind of questions are of special interesting for many people who are devoted to nonlinear problems. However, because their methods involve some abstruse knowledge in modern analysis, they are difficult for us to understand.

Lemma 2.2. *If condition* (b) *in Section* 1 *holds, then either* $V_\infty \equiv 0$ *or* $V_\infty > 0$ *for all* $x \in \mathbb{R}^d$.

Proof. At first, to prove that "if $V_\infty(x_0) = 0$ for some $x_0 \in \mathbb{R}^d$, then for a.e-L $x \in \mathbb{R}^d$, $V_\infty(x) = 0$." If not, by Assumption (b), we have

$$P^{\delta_{x_0}} \langle X_t, V_\infty \rangle = S_t V_\infty(x_0) > 0. \tag{2.2}$$

Therefore

$$\begin{aligned} e^{-V_\infty(x_0)} &= e^{-V_t(V_\infty)(x_0)} \quad \text{(by Lemma 2.1)} \\ &= P^{\delta_{x_0}} e^{-\langle X_t, V_\infty \rangle} > 0, \end{aligned} \tag{2.3}$$

i.e., $V_\infty(x_0) > 0$. This is absurd.

Then to prove "if $V_\infty(x) = 0$ for a.e.-L, then $V_\infty \equiv 0$." In fact, for each $x \in \mathbb{R}^d$,

$$P^{\delta_x} \langle X_t, V_\infty \rangle = S_t(V_\infty)(x) = 0.$$

This means $\langle X_t, V_\infty \rangle = 0$, a.s.-$P^{\delta_x}$. Therefore

$$V_\infty(x) = -\log P^{\delta_x} e^{-\langle X_t, V_\infty \rangle} = 0.$$

The proof is complete.

Proof of Theorem 1.1. Since $L(\text{supp}(\gamma))> 0$, then $\text{supp}(\gamma)$ is not polar set of ξ. Because of the recurrence of ξ we know that either $\int_0^\infty P_x\gamma(\xi_s)V_\infty^{1+\beta}(\xi_s)ds = 0$ or ∞ (cf. [1]). If $V_\infty > 0$, then

$$\int_0^t P_x\gamma(\xi_s)V_\infty^{1+\beta}(\xi_s)ds > 0,$$

therefore

$$\int_0^\infty P_x\gamma(\xi_s)V_\infty^{1+\beta}(\xi_s)ds \equiv \infty.$$

By Lemma 2.1,

$$\int_0^t P_x\gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds \geq \int_0^\infty P_x\gamma(\xi_s)V_\infty^{1+\beta}(\xi_s)ds \equiv \infty.$$

Consequently

$$\lim_{t\to\infty}\int_0^t P_x\gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds = \infty.$$

On the contrary, we have that for any $x \in \mathbb{R}^d$ and any $t > 0$,

$$\int_0^t P_x\gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds \leq 1.$$

So we get a contradiction. This means $V_\infty \equiv 0$, and thereby $X_\infty = 0$ a.s.

Proof of Theorem 1.2. Similarly, we only need to prove $V_\infty \not\equiv 0$. If $V_\infty \equiv 0$, then for any $\epsilon > 0$, we can find t_1 large enough such that

$$P_x\int_{t_1}^\infty 1_{\text{supp}(\gamma)}(\xi_s)ds < \epsilon/c\|\gamma\|,$$

i.e., for any $t > t_1$,

$$P_x\int_{t_1}^t \gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds < \epsilon.$$

Note that, for $t > t_1$,

$$P_x\int_0^t \gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds = P_x\int_0^{t_1} \gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds + P_x\int_{t_1}^t \gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds.$$

Because of $V_t \downarrow 0$, the monotonous convergence theorem implies that

$$\lim_{t\to\infty} P_x\int_0^{t_1} \gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds = 0.$$

Hence

$$\lim_{t\to\infty} P_x\int_0^t \gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds < \epsilon.$$

By the arbitrariness of ϵ, we have that for $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} P_x\int_0^t \gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds = 0.$$

On the other hand, from (1.2) and our assumptions we have

$$\lim_{t\to\infty} P_x\int_0^t \gamma(\xi_s)V_{t-s}^{1+\beta}(\xi_s)ds = 1.$$

So we have a contradiction. This means that $V_\infty > 0$ in $\mathbb{R}^d$, and the proof is complete.

§3. Remarks

In this section we shall discuss the possibilities of generalizing our results. For any non-negative bounded measurable function γ, the following Comparison Lemma make us possible to extend our results.

Proposition 3.1 (**Comparison Lemma**). *Suppose that* $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *and* $\bar{u}(t,x)$ *and* $\underline{u}(t,x)$ *satisfy the condition that* $\bar{u}$ *is a supersolution and* $\underline{u}$ *is a subsolution, i.e.,*

$$\dot{\bar{u}}(t,x) - A\bar{u}(t,x) + f(t,x,\bar{u}) \geq 0$$

and

$$\begin{aligned}
&\dot{\underline{u}}(t,x) - A\underline{u}(t,x) + f(t,x,\underline{u}) \leq 0;\\
&\sup_{t\leq s,\ x\in\mathbb{R}^d} |\bar{u}(t.x)| + |\underline{u}(t,x)| < \infty \quad \textit{for every} \quad s < T;\\
&\bar{u}(0,x) \geq \underline{u}(0,x) \quad \textit{for } x \in \mathbb{R}^d.
\end{aligned}$$

Then

$$\bar{u}(t,x) \geq \underline{u}(t,x) \quad \textit{in } (0,T) \times \mathbb{R}^d.$$

The theorem was stated in [8] when A is a strongly elliptic operator. If the Comparison Lemma is true for some A, an immediate result is the following proposition.

Proposition 3.2. *Suppose that* X_t^1 *and* X_t^2 *are superprocesses associated with* γ_1 *and* γ_2 *respectively, and* $\gamma_1 \leq \gamma_2$. *If* X_t^1 *is extinct in the sense of* (1.3), *so is* X_t^2. *On the other hand, if* X_t^2 *is not a.s. extinct, neither is* X_t^1.

There are still some unsolved problems. An interesting question is that under the condition that ξ is transient (e.g. Brownian motion for $d > 2$), for which kind of γ, X_t is still extinct? We know from Section 1 that it is true when γ is a constant. However, for a general γ this question may be not easy to answer. Anyway, we have the following

Conjecture 3.1. *If for some* $c > 0$ *the set* $\{x \in \mathbb{R}^d, \gamma(x) > c\}$ *is of infinite potential with respective to* ξ, *i.e.,*

$$\int_0^\infty S_t 1_{\{x\in\mathbb{R}^d,\gamma(x)>c\}} dt = \infty,$$

then the corresponding X_t *is extinct.*

A further question is whether $\mathrm{supp}(X_t)$ almost surely equals to $\emptyset$ for t large enough when X_t is extinct in the sense of (1.3). Unlike the case which γ is a positive constant, the above question has a negative answer for a general branching characteristic γ. This will be proved in a later paper.

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