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ON TAYLOR'S CONJECTURE ABOUT THE PACKING MEASURES OF CARTESIAN PRODUCT SETS**

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Abstract

It is proved that if $E \subset \mathbf{R}, F \subset \mathbf{R}^n$, then $\mathcal{P}(E \times F, \varphi_1\varphi_2) \leq c \cdot \mathcal{P}(E, \varphi_1)\mathcal{P}(E, \varphi_2)$, where $\mathcal{P}(\cdot, \varphi)$ denotes the φ-packing measure, φ belongs to a class of Hausdorff functions, the positive constant c deponds only on φ_1, φ_2 and n.

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§1. Introduction

In the geometry of fractals, Hausdorff measure and dimension play a very important role. On the other hand, the recent introduction of packing measures has led to a greater understanding of the geometric theory of fractals, as packing measures behave in a way that is 'dual' to Hausdorff measures in many respects[2]. For example, denoting Hausdorff dimension and packing dimension by dim and Dim respectively, we have $\dim(E \times F) \geq \dim E + \dim F$, while $\mathrm{Dim}(E \times F) \leq \mathrm{Dim}E + \mathrm{Dim}F$. It is well-knowen that if $E \subset \mathbf{R}^m$, $F \subset \mathbf{R}^n$, then

$$\mathcal{H}(E \times F, \varphi_1\varphi_2) \geq b \cdot \mathcal{H}((E, \varphi_1)H(F, \varphi_2)$$

for some Hausdorff functions and constant b, where $\mathcal{H}(\cdot, \varphi)$ denotes the φ-Hausdorff measure. Taylor conjectures that we should have

$$\mathcal{P}(E \times F, \varphi_1\varphi_2) \leq c \cdot \mathcal{P}(E, \varphi_1)\mathcal{P}(F, \varphi_2).$$

In this paper, it is shown that if E or F is a subset of $\mathbf{R}$, then the conjecture is correct.

§2. Packing Premeasure

We restrict our attention to subsets of Euclidean space $\mathbf{R}^d (d \geq 1)$. The Cartesian product of sets $E \subset \mathbf{R}^m$ and $F \subset \mathbf{R}^n$ is denoted by $E \times F$. We use $|E|$ to denote the diameter of E and $\|x\|$ to denote the distance from 0 to $x \in \mathbf{R}^n$. The open ball with center at x and radius $r > 0$ is denoted by

$$B_r(x) = \{y \in \mathbf{R}^n : \|x - y\| < r\}.$$

Ω stands for the class of balls:

$$\Omega(E) = \{B_r(x) : r > 0, x \in E\}.$$

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Γ^* stands for the class of dyadic cubes in $\mathbf{R}^d$, $C \in \Gamma^*$ if it has side length $2^{-n}, n \in \mathbf{N}$, and each of its projections $\text{proj}_i C$ on the ith axis is a half-open interval of the form $[k_i 2^{-n}, (k_i+1)2^{-n})$ with $k_i \in \mathbf{Z}$. $u_n(x)$ denotes the unique cube which is in Γ^* and contains x with side legnth 2^{-n}.

$$\Gamma^*(E) = \{u_n(x) : x \in E, n \in \mathbf{N}\}.$$

Γ^{**} stands for the class of semidyadic cubes in $\mathbf{R}^d$, $C \in \Gamma^{**}$ if it has side legnth 2^{-n} and $\text{proj}_i C = [\frac{1}{2}k_i 2^{-n}, (\frac{1}{2}k_i + 1)2^{-n})$ with $k_i \in \mathbf{Z}$. $v_n(x)$ is the unique cube in Γ^{**} of side legnth 2^{-n} such that on the i-th axis the complement of $\text{proj}_i C$ is at distance 2^{-n-2} from $u_{n+2}(\text{proj}_i x) \subset \mathbf{R}$. It is not difficult to see that if $x \in \mathbf{R}^{m+n}$ and $n \in \mathbf{N}$, then $\text{proj}_{\mathbf{R}^n}(v_n(x)) = v_n(\text{proj}_{\mathbf{R}^n} x)$, where $v_n(\text{proj}_{\mathbf{R}^n} x)$ is in $\mathbf{R}^n$.

$$\Gamma^{**} = \{v_n(x) : x \in E, n \in \mathbf{N}\}.$$

Φ denotes the class of functions $\varphi : [0, +\infty) \to \mathbf{R}$ which are increasing, continous with $\varphi(0) = 0$ and

$$\varphi(2x) < c_0 \varphi(x) \quad \text{for some } c_0 > 0 \quad \text{and} \quad 0 < x < \frac{1}{2}. \tag{2.1}$$

We use $\mathcal{B}(\mathbf{R}^n)$ to denote the family of bounded subsets of $\mathbf{R}^n$. For $\mathcal{R} \subset \mathcal{B}(\mathbf{R}^n)$, put $\|\mathcal{R}\| = \sup\{|E| : E \in \mathcal{R}\}$ and

$$\varphi(R) = \sum_{R \in \mathcal{R}} \varphi(|E|). \tag{2.2}$$

We say $R \subset B(\mathbf{R}^n)$ is a packing of E if for all $F \in R, \overline{E} \cap \overline{F} \neq \emptyset$, and the sets in $\mathcal{R}$ are disjoint. Put

$$\tau(E, \varphi, \varepsilon) = \sup\{\varphi(\mathcal{R}) : \|\mathcal{R}\| \leq \varepsilon, \mathcal{R} \text{ is a packing of } E\}. \tag{2.3}$$

Particularly, if $\mathcal{R} \subset \Omega(E)$ or $\mathcal{R} \subset \Gamma^{**}(E)$, the corresponding $\tau(E, \varphi, \varepsilon)$ is denoted by $P(E, \varphi, \varepsilon)$ or $P^{**}(E, \varphi, \varepsilon)$.

Obviously $\tau(E, \varphi, \varepsilon)$ is an increasing function of ε. Let

$$\begin{aligned} \tau(E, \varphi) &= \lim_{\varepsilon \to 0} \tau(E, \varphi, \varepsilon), \\ P(E, \varphi) &= \lim_{\varepsilon \to 0} P(E, \varphi, \varepsilon), \\ P^{**}(E, \varphi) &= \lim_{\varepsilon \to 0} P^{**}(E, \varphi, \varepsilon). \end{aligned} \tag{2. 4}$$

§3. Packing Measure

For $E \subset \mathbf{R}^n$, let

$$\mathcal{P}(E, \phi) = \inf\Big\{\sum P(E_i, \varphi) : E_i \in \mathcal{B}(\mathbf{R}^n), E \subset \cup E_i\Big\}, \tag{3.1}$$

$$\mathcal{P}^{**}(E, \varphi) = \inf\Big\{\sum P^{**}(E_i, \varphi) : E_i \in \mathcal{B}(\mathbf{R}^n), E \subset \cup E_i\Big\}. \tag{3.2}$$

Then they are two outer measures. We call $\mathcal{P}(E, \varphi)$ the φ-packing measure of E.

§4. Packing Measures of Cartesian Product Sets

Lemma 4.1.[3] *Let $E \subset \mathbf{R}^n$. Then there exist $0 < c_1 \leq c_2 < +\infty$ such that*

$$c_1 P(E, \varphi) \leq P^{**}(E, \varphi) \leq c_2 P(E, \varphi). \tag{4.1}$$

c_1 *and* c_2 *depend only on* φ *and* n.

Proof. From the definition of $v_i(x)$, we can get $B_{2^{-i-2}}(x) \subset v_i(x) \subset B_{\rho \cdot 2^{-i}}(x)$, where $i \in N$, and $\rho = n^{\frac{1}{2}}$. So according to (2.3) and (2.4), the results is obvious.

Corollary 4.1.[3] Let $E \subset \mathbf{R}^n$. Then there exist $0 < c_1 \leq c_2 < +\infty$ such that

$$c_1 \mathcal{P}(E, \varphi) \leq \mathcal{P}^{**}(E, \varphi) \leq c_2 \mathcal{P}(E, \varphi).$$

c_1 and c_2 depend only on φ and n.

Proof. Use (3.1), (3.2) and Lemma 4.1.

Lemma 4.2. *Let* $E \subset [a, b], -\infty < a \leq b < +\infty, u = \{U_i, i = 1, 2, 3, \cdots\} \subset \Gamma^{**}(E)$. U_i *and* U_j *may be the same set when* $i \neq j, q > 0, \|u\| \leq q$. *For all* $x \in [a, b]$,

$$\sum_{U_i \in u} \chi_{U_i}(x) \leq n, \quad n \in \mathbf{N}, \tag{4.2}$$

where $\chi_{U_i}(x)$ *is the characteristic function of* U_i. *Then*

$$\sum_{U_i \in u} \varphi(|U_i|) \leq n \cdot P^{**}(E, \varphi, q). \tag{4.3}$$

Proof. Use mathematical induction.

If $n = 1$, then from (4.2) we know that u is a packing of E, so

$$\sum_{U_i \in u} \varphi(|U_i|) \leq P^{**}(E, \varphi, q).$$

Suppose that the lemma is true when $n = k - 1$. Let $n = k$. Let $u' = \{U_1, U_2, \cdots U_N\}$. Then

$$\sum_{U_i \in u'} \chi_{U_i}(x) \leq \sum_{U_i \in u} \chi_{U_i}(x) \leq k, \quad x \in [a, b]. \tag{4.4}$$

Let $U_i = [a_i, b_i), i = 1, 2, \cdots N$. We can assume that $a_1 \leq a_2 \leq \cdots \leq a_{N-1} \leq a_N$. Let $U_{r_1} = [a_1, b_1), U_{r_2} = [a_{r_2}, b_{r_2})$, where r_2 is the smallest number which satisfies $a_{r_2} \geq b_1$. Also we can get $U_{r_3} = [a_{r_3}, b_{r_3})$, where r_3 is the smallest number such that $a_{r_3} \geq b_{r_2}$. In such a way, we can get

$$U_{r_1}, U_{r_2}, \cdots U_{r_l}, \quad 1 = r_1 < r_2 < \cdots \leq r_l \leq N.$$

Let $\tilde{u} = \{U_{r_1}, U_{r_2}, \cdots, U_{r_l}\}, u'' = u' \backslash \tilde{u}$. Then $\tilde{u}$ is a packing of E, so

$$\sum_{i=1}^{l} \varphi(|U_{r_i}|) \leq P^{**}(E, \varphi, q). \tag{4.5}$$

We need to prove

$$\sum_{U \in u''} \chi_{U_i}(x) \leq k - 1, \quad x \in [a, b]. \tag{4.6}$$

If $x \in \bigcup_{i=1}^{l} U_{r_i}$, then (4.6) is obviously correct.

If $x \notin \bigcup_{i=1}^{l} U_{r_i}$, then there must exist r_i such that $x \in [b_{r_i}, a_{r_{i+1}})$ (If $i = l$, then let $a_{r_{i+1}} = b$). So if $x \in U_i \in u''$, then U_i must satisfy $U_i \bigcap [a_{r_i}, b_{r_i}) \neq \emptyset$; otherwise U_i should have been selected into $\tilde{u}$ before $[a_{r_{i+1}}, b_{r_{i+1}})$. So if there are more than $k-1$ sets containing

x in u'', we can find a point b'_{r_i} in the left neighborhood of b_{r_i} such that

$$\sum_{U_i\in u'}\chi_{U_i}(b'_{r_i})=\sum_{U_i\in\tilde{u}}\chi_{U_i}(b'_{r_i})+\sum_{U_i\in u''}\chi_{U_i}(b'_{r_i})>1+(k-1)=k, \tag{4.7}$$

which contradicts (4.4). So we have

$$\sum_{U_i\in u''}\chi_{U_i}(x)\le k-1$$

and (4.6) is correct. So

$$\sum_{U_i\in u''}\varphi(|U_i|)\le(k-1)\cdot P^{**}(E,\varphi,q),$$

and

$$\sum_{U_i\in u'}\varphi(|U_i|)=\sum_{U_i\in\tilde{u}}\varphi(|U_i|)+\sum_{U_i\in u''}\varphi(|U_i|)\le k\cdot P^{**}(E,\varphi,q).$$

Letting $N\to+\infty$. we complete the proof.

Lemma 4.3. *If $E\subset\mathbf{R},F\subset\mathbf{R}^n$, then*

$$P^{**}(E\times F,\varphi_1\varphi_2)\le c\cdot P^{**}(E,\varphi_1)\cdot P^{**}(F,\varphi_2), \tag{4.8}$$

where $0<c<+\infty$. c depends only on φ_1,φ_2 and n.

Proof. If E or F is an unbounded set, then $P^{**}(E,\varphi_1)=+\infty$ or $P^{**}(F,\varphi_2)=+\infty$ and (4.8) holds. So we need only to consider the case that both E and F are bounded sets.

Let $u=\{U_i\}\subset\Gamma^{**}(E\times F),\|u\|\le q$ and u be a packing of $E\times F$. Put

$$P_1(U_i)=\mathrm{proj}_{\mathbf{R}}(U_i),\quad P_2(U_i)=\mathrm{proj}_{\mathbf{R}^n}(U_i),\quad u_1=\{P_1(U_i):U_i\in u\}$$

and $u_2=\{P_2(U_i):U_i\in u\}$. Then

$$u_1\subset\Gamma^{**}(E),\quad u_2\subset\Gamma^{**}(F),\quad \|u_1\|\le q\ \text{ and }\ \|u_2\|\le q.$$

Suppose $E\subset[a,b],-\infty<a\le b<+\infty$. For any fixed $x\in[a,b],\{P_2(U_i):x\in P_1(U_i)\}$ is a packing of F. So

$$\sum_{U_i\in u}\varphi_2(|P_2(U_i)|)\cdot\chi_{P_1(U_i)}(x)\le P^{**}(F,\varphi_2,q),\quad x\in[a,b]. \tag{4.9}$$

For u we have

$$\begin{aligned}\sum_{U_i\in u}\varphi_1\varphi_2(|U_i|)&=\sum_{U_i\in u}\varphi_1(|U_i|)\cdot\varphi_2(|U_i|)\\&=\sum_{U_i\in u}\varphi_1(\sqrt{n+1}\ |P_1(U_i)|)\cdot\varphi_2(\frac{\sqrt{n+1}}{\sqrt{n}}\ |P_2(U_i)|).\end{aligned} \tag{4.10}$$

Let $u'=\{U_1,U_2,\cdots,U_N\}$. Then

$$\sum_{U_i\in u'}\varphi_2(|P_2(U_i)|)\cdot\chi_{P_1(U_i)}(x)\le P^{**}(F,\varphi_2,q). \tag{4.11}$$

Let $\varphi_2(|P_2(U_i)|)=f_i,i=1,2,\cdots,N$, and $P^{**}(F,\varphi_2,q)=g$. f_i and g can be approximated by rational numbers d_i and h so that

$$\frac{h}{1+\varepsilon}\le g\le h,\quad \frac{d_i}{1+\varepsilon}\le f_i\le d_i,\quad i=1,2,\cdots,N,$$

where ε is also a rational number. Then

$$\sum_{U_i\in u'}d_i\cdot\chi_{P_1(U_i)}(x)\le(1+\varepsilon)h.$$

Let M be the common demoninator of ε, h and $d_i, i = 1, 2, \cdots, N, d_i = \frac{k_i}{M}$. Then

$$\sum_{U_i \in u'} Mk_i \cdot \chi_{P_1(U_i)}(x) \leq (1+\varepsilon)hM^2.$$

Put $(1+\varepsilon)hM^2 = K$. Then $K \in N$. Using Lemma 4.2 we get

$$\sum_{U_i \in u'} Mk_i \cdot \varphi_1(|P_1(U_i)|) \leq K \cdot P^{**}(E, \varphi_1, q).$$

So

$$\sum_{U_i \in u'} d_i \cdot \varphi_1(|P_1(U_i)|) \leq (1+\varepsilon)h \cdot P^{**}(E, \varphi_1, q),$$

$$\sum_{U_i \in u'} f_i \cdot \varphi_1(|P_1(U_i)|) \leq (1+\varepsilon)^2 g \cdot P^{**}(E, \varphi_1, q).$$

$$\sum_{U_i \in u'} \varphi_2(|P_2(U_i)|) \cdot \varphi_1(|P_1(U_i)|) \leq (1+\varepsilon)^2 P^{**}(F, \varphi_2, q) \cdot P^{**}(E, \varphi_1, q).$$

Let $\varepsilon \to 0$ and then $N \to +\infty$. We get

$$\sum_{U_i \in u} \varphi_2(|P_2(U_i)|) \cdot \varphi_1(|P_1(U_i)|) \leq P^{**}(F, \varphi_2, q) \cdot P^{**}(E, \varphi_1, q). \tag{4.12}$$

From (4.12),(4.10) and (2.1) we get

$$\begin{aligned} \sum_{U_i \in u} \varphi_1\varphi_2(|U_i|) &\leq \sum_{U_i \in u} \varphi_1(2^n |P_1(U_i)|) \cdot \varphi_2(2|P_2(U_i)|) \\ &\leq \sum_{U_i \in u} {c_1}^n \varphi_1(|P_1(U_i)|) \cdot c_2\varphi_2(|P_2(U_i)|) \\ &\leq c \cdot P^{**}(E, \varphi_1, q) \cdot P^{**}(F, \varphi_2, q), \end{aligned} \tag{4.13}$$

where $c = {c_1}^n \cdot c_2$ depends only on φ_1, φ_2 and n. (4.13) is valid for any packing u of $E \times F$ on the condition that $u \in \Gamma^{**}(E \times F), \|u\| \leq q$ and q is small enough. So we have

$$P^{**}(E \times F, \varphi_1\varphi_2, q) \leq c \cdot P^{**}(E, \varphi_1, q) \cdot P^{**}(F, \varphi_2, q). \tag{4.14}$$

Let $q \to 0$. We get

$$P^{**}(E \times F, \varphi_1\varphi_2) \leq c \cdot P^{**}(E, \varphi_1) \cdot P^{**}(F, \varphi_2).$$

Corollary 4.2. *If $E \subset \mathbf{R}, F \subset \mathbf{R}^n$, then*

$$P(E \times F, \varphi_1\varphi_2) \leq c' \cdot P(E, \varphi_1) \cdot P(F, \varphi_2), \tag{4.15}$$

where $0 < c' < +\infty$, c' depends only on φ_1, φ_2 and n.

Proof. Use Lemma 4.1 and Lemma 4.3.

Now we can prove the main result.

Theorem 4.1. *If $E \subset \mathbf{R}, F \subset \mathbf{R}^n$, then*

$$\mathcal{P}^{**}(E \times F, \varphi_1\varphi_2) \leq c \cdot \mathcal{P}^{**}(E, \varphi_1) \cdot \mathcal{P}^{**}(F, \varphi_2), \tag{4.16}$$

where $0 < c < +\infty$. c depends only on φ_1, φ_2 and n.

Proof. According to (3.2), for any $\varepsilon > 0$ there exist $\{E_i, i = 1, 2, \cdots\}$ such that $E_i \subset \mathcal{B}(\mathbf{R}), E \subset \cup E_i$ and

$$\mathcal{P}^{**}(E, \varphi_1) \leq \sum P^{**}(E_i) \leq \mathcal{P}^{**}(E, \varphi_1) + \varepsilon.$$

We can also get $\{F_i, i=1,2,\cdots\}$ so that $F_i \subset \mathcal{B}(\mathbf{R}^n), F \subset \cup F_i$ and

$$\mathcal{P}^{**}(F,\varphi_2) \le \sum P^{**}(F_i) \le \mathcal{P}^{**}(F,\varphi_2)+\varepsilon.$$

Let $u=\{E_i \times F_j, i,j=1,2,\cdots\}$. Then $E_i \times F_j \subset \mathcal{B}(\mathbf{R}\times\mathbf{R}^n)$ and $E \times F \subset \bigcup_i \bigcup_j E_i \times F_j$. So

$$\mathcal{P}^{**}(E\times F,\varphi_1\varphi_2) \le \sum_i \sum_j P^{**}(E_i \times F_j,\varphi_1\varphi_2).$$

From Lemma 4.3, we have

$$P^{**}(E_i \times F_j,\varphi_1\varphi_2) \le c \cdot P^{**}(E_i,\varphi_1)\cdot P^{**}(F_j,\varphi_2).$$

So

$$\begin{aligned}\mathcal{P}^{**}(E\times F,\varphi_1\varphi_2) &\le \sum_i \sum_j c\cdot P^{**}(E_i,\varphi_1)\cdot P^{**}(F_j,\varphi_2)\\ &\le c\cdot(\mathcal{P}^{**}(E,\varphi_1)+\varepsilon)\cdot(\mathcal{P}^{**}(F,\varphi_2)+\varepsilon).\end{aligned}$$

Let $\varepsilon \to 0$. We get

$$\mathcal{P}^{**}(E\times F,\varphi_1\varphi_2) \le c\cdot P^{**}(E,\varphi_1)\cdot P^{**}(F,\varphi_2).$$

Theorem 4.2. *If $E \subset \mathbf{R}, F \subset \mathbf{R}^n$, then*

$$\mathcal{P}(E\times F,\varphi_1\varphi_2) \le c'\cdot \mathcal{P}(E,\varphi_1)\cdot\mathcal{P}(F,\varphi_2),$$

where $0<c'<+\infty$, c' depends only on φ_1,φ_2 and n.

Proof. Use Theorem 4.1 and Corollary 4.1.

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