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STABILITY OF GLOBAL GEVREY SOLUTION TO WEAKLY HYPERBOLIC EQUATIONS

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Abstract

This work is concerned with the proof of stability of global Gevrey solution to the following quasilinear weakly hyperbolic equation: $u_{tt} - a(x,t)u_{xx} = f(x,t,u,u_x)$ in $P \times [0,T]$ with initial data $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$. Here weak hyperbolicity means that $a(x,t) \geq 0$, that is, there exist, in general, characteristic roots of variable multiplicity. One has to distinguish between the case of spatial degeneracy and that of time degeneracy. The connection to the life span of solutions is given.

Keywords Quasilinear weakly hyperbolic equations, Gevrey functions, Global solvability, Stability, Life span of solutions

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§1. Introduction

In the last years the theory of weakly hyperbolic equations was developed in an astonishing way. A lot of papers have led to a general view about linear weakly hyperbolic equations (even) of higher order. Using Gevrey analysis first success was obtained for nonlinear weakly hyperbolic equations. In this context we remind of the general local existence theorem for nonlinear hyperbolic equations of order m, $m \geq 2$, which has been proved in [5] for local Gevrey classes of order $s \leq j/(j-1)$, where j denotes the highest multiplicity of characteristic roots.

To include Gevrey classes of order $s > j/(j-1)$ one has to formulate Levi conditions. We call Levi condition any algebraic condition between lower order terms and the principal part.

For example, if one wants to study well-posedness for the weakly hyperbolic Cauchy problem $(a(x) \geq 0)$

$$u_{tt} - (a(x)u_x)_x - b(x)u_x = f(x,t), \quad u(x,0) = u_0(x), u(x,0) = u_1(x)$$

in all Gevrey classes of order $s > 2$, the Levi condition $|b(x)|^2 \leq Ca(x)$ (C denotes here and in the following an universal constant) is sufficient[4,15] and necessary[3] if $a(x)$ has a zero of finite order.

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The authors studied in two papers[11,12] the influence of Levi conditions to some typical problems of hyperbolic theory for quasilinear weakly hyperbolic equations of the form

$$u_{tt} - a(x,t)u_{xx} = f(x,t,u,u_x). \tag{1.1}$$

The reason why we had to represent the results in two different papers is that the goals, methods and especially Levi conditions differ in the cases of time $(a(x,t) = \lambda^2(t)b(x,t),\ b(x,t) \geq C > 0,\ \lambda(0) = \lambda'(0) = 0,\ \lambda'(t) > 0 \text{ for } t > 0)$ and spatial $(a(x,t) \geq 0,\ Aa(x,t) - a_t(x,t) \geq 0)$ degeneracy.

In the case of time degeneracy[11] local existence of solutions and existence of cone of dependence could be proved. In opposite to this type of degeneracy (if $t > 0$ the equation (1.1) becomes strictly hyperbolic) in the case of spatial degeneracy not only the question for local existence and cone of dependence, but also for global regularity of solutions[12,14] is reasonable (if $t > 0$ the equation (1.1) still remains weakly hyperbolic).

From the strictly hyperbolic theory $(a(x,t) \geq C > 0$ in (1.1)) it is known that growth conditions of f with respect to u and u_x are necessary for the global existence of solutions[6,7,10]. The situation is unchanged in the degenerate case (1.1) because this case contains $a(x,t) = 0$. In this case the Levi condition implies $\partial_p f(x,t,u,p) \equiv 0$. Consequently, the degenerate case includes ordinary differential equations for which the blow-up behaviour of solutions is well studied.

But one has to take into consideration the Levi conditions, too. If the Levi condition is not satisfied, then the global existence of solutions cannot be expected. The following example shows that for quasilinear weakly hyperbolic equations without fulfillment of Levi condition which was proposed in [11], even stability of local existence is not valid in all Gevrey spaces. Hence, it is necessary to seek for sharp Levi conditions depending on the Gevrey order. This will be done in a forthcoming paper. For the linear case see [4,15,16].

Example 1.1. Let us consider the equation (b is real)

$$u_{tt} - t^{2j}u_{xx} - bt^k u_x - t^{2j}(u_x)^2 + (u_t)^2 = 0. \tag{1.2}$$

If $k < j-1$, then the Levi condition from [11] is not satisfied. Using an idea of [6] it is easy to see that if $u = u(x,t)$ is a solution, then $v(x,t) = \exp u(x,t)$ is a solution of the following linear equation

$$v_{tt} - t^{2j}v_{xx} - bt^k v_x = 0. \tag{1.3}$$

We seek for real-valued solutions $v^{(n)}$ of this equation of the form

$$v^{(n)}(x,t) = a^{(n)}(t)\cos(nx) + b^{(n)}(t)\sin(nx).$$

Consequently, $(a^{(n)}(t), b^{(n)}(t))$ is a solution of the system

$$\begin{cases} a_{tt}^{(n)}(t) + t^{2j}n^2 a^{(n)}(t) - bt^k n b^{(n)}(t) = 0, \\ b_{tt}^{(n)}(t) + t^{2j}n^2 b^{(n)}(t) + bt^k n a^{(n)}(t) = 0. \end{cases}$$

Setting $c^{(n)}(t) = a^{(n)}(t) + ib^{(n)}(t)$ this function is a solution of the equation

$$c_{tt}^{(n)}(t) + t^{2j}n^2 c^{(n)}(t) + ibt^k n c^{(n)}(t) = 0.$$

Adding the initial conditions $c^{(n)}(0) = 0, c_t^{(n)}(0) = \rho^{(n)}$ ($\rho^{(n)}$ is real) which are equivalent to $a^{(n)}(0) = b^{(n)}(0) = 0,\ a_t^{(n)}(0) = \rho^{(n)}, b_t^{(n)}(0) = 0$, where the sequence $\{\rho^{(n)}\}$ will be chosen later, then according to [13] the following representation holds for $c^{(n)}$:

$$c^{(n)}(t) = \rho^{(n)} \sum_{m=1}^{2} a_m(t) n^{-1} \exp[C_m n^\sigma + in(-1)^m t^{j+1}/(j+1)](1+o(1)), \tag{1.4}$$

where $\sigma = (j-k-1)/(2j-k), a_m(t) \neq 0$ and the real part of at least one C_m is positive.

Now, the function $\tilde{u}(x,t) \equiv 1$ is a globally defined solution of (1.2) with data $\tilde{u}(x,0) = 1$, $\tilde{u}_t(x,0) = 0$. Let us suppose that, for some n which will be chosen later, $u(x,t)$ is a solution of (1.2) with the data $u(x,0) = 1$ and $u_t(x,0) = e^{-1}\rho^{(n)}\cos(nx)$. Then $v(x,t) = \exp u(x,t)$ is a solution of (1.3) with initial data $v(x,0) = e$ and $v_t(x,0) = \rho^{(n)}\cos(nx)$. On the other hand the function $e + v^{(n)}(x,t)$ is a solution of (1.3) with the same initial values. Uniqueness of the Cauchy problem (Holmgren's theorem) for (1.3) implies that $v(x,t) = e + v^{(n)}(x,t)$.

Furthermore, if $\rho^{(n)} = \exp(-n^{\sigma_1})$, $\sigma_1 < \sigma$, then we have for every $\eta > 0$ and $s > 1/\sigma_1$

$$\rho^{(n)} \sup_{k \in N, x \in P} \eta^k \mid d_x^k \cos(nx) \mid / k^{sk} \leq \exp(-n^{\sigma_1} + s(\eta n)^{1/s}/e).$$

The right hand side of the last inequality tends to 0 when n tends to infinity. At the same time $n^{-1}\rho^{(n)}\exp(C_m n^\sigma)$ tends to infinity when $\operatorname{Re} C_m > 0$.

Thus, by (1.4) for every time interval $[0,\delta]$ and for every ε-neighbourhood of the initial data $(1,0)$ in the Gevrey space $G^{(s)}(s > 1/\sigma)$ there exist initial data from this neighbourhood such that the solution of (1.2) does not exist in $C^2([0,\delta]; Y^s_{+0}(P))$. For the definition of the space $Y^s_{+0}(P)$ see below.

The same example can be constructed in the spaces C^k using asymptotic solutions which have been constructed, for example, in [3,17].

If the global existence of solutions cannot be expected, as usual, after the question for local existence that for life span is studied. The analytic case for more general quasilinear equations than (1.1) was studied in [2]. In the analytic case Levi conditions do not appear. But in the present paper we shall consider the stability of global Gevrey solvability to equations of type (1.1).

Problem. "Let $u(x,0) = u_0(x), u_t(x,0) = u_1(x)$, respectively, $f(x,t,u,u_x)$ be the data, the right hand side of (1.1), where f is defined on some interval $[0,T]$ with respect to t. From the results of [11,12] the local existence and uniqueness of Gevrey solutions $u = u(x,t)$ is known. Let us, additionally, suppose that this solution exists globally on $[0,T]$. The problem, roughly speaking, is that if other initials U_0, U_1 and F are sufficiently close in suitable function spaces (see (1.6), (1.8), (1.9)) to the reference initials u_0, u_1 and f, then the corresponding Cauchy problem

$$u_{tt} - a(x,t)u_{xx} = F(x,t,u,u_x), \quad u(x,0) = U_0(x), u_t(x,0) = U_1(x) \tag{1.5}$$

has a global solution $U(x,t)$ with respect to t, too?"

Here we shall give a positive answer to this question. We have to remark that among other things the same answer for the three dimensional Navier-Stokes system was given in [9].

To be precise we have to explain perturbations. We restrict ourselves to the case of periodical solutions with respect to x. In the following we discuss only the case of spatial degeneracy. The case of time degeneracy will be considered in Section 5.

Let P be a bounded interval in R. Let us define the space of P-periodical functions

$$Y^s_{+0}(P) = \Big\{ u(x) \in C^\infty(P) : \|\partial_x^k u(x)\| \frac{\rho_0^k}{k!^s} \le C_u \quad \text{for all} \quad k \in N_0$$
$$\text{and a suitable positive } \rho_0 \Big\}.$$

These are periodical Gevrey functions of order s, while $\|\cdot\|$ denotes the $L_2(P)$-norm. Then an ε-perturbation of data u_0, u_1 is contained in the set

$$\big\{ U_0, U_1 \in Y^s_{+0}(P) : \|U_0 - u_0\|_{\rho_0,s} + \|U_1 - u_1\|_{\rho_0,s} < \varepsilon \big\}, \tag{1.6}$$

where $\|\cdot\|_{\rho_0,s}$ denotes one semi-norm of the space $Y^s_{+0}(P)$ endowed with the inductive limit topology.

Perturbations of f are only reasonable to study in sets of functions satisfying Levi conditions which depend on the coefficient $a = a(x,t)$.

Case of Spatial Degeneracy (see [12]):

(C1) The function $a = a(x,t)$ is P-periodic in x and belongs to $C^1([0,T]; Y^s_{+0}(P))$. Moreover, $a(x,t) \ge 0$ and $Aa(x,t) - a_t(x,t) \ge 0$ with a suitable positive constant A.

(C2) The function f is P-periodic in x and belongs to

$$C([0,T]; X^{(s,s',s')}_{\mathrm{loc}}(P \times R_u \times R_p)), \quad s' < s.$$

Here

$$\begin{aligned} &X^{(s,s',s')}_{loc}(P \times R_u \times R_p)) \\ &= \Big\{ f(x,u,p) : \text{ for every compact set } \quad K \subset R_u \times R_p \text{ there exist} \\ &\quad \text{constants} \quad C_K \quad \text{and} \quad M_K \quad \text{such that} \\ &\quad |\partial_x^i \partial_u^{\nu_1} \partial_p^{\nu_2} f(x,u,p)| \le C_K M_K^{s(i+\nu_1+\nu_2)} (\nu_1! \nu_2!)^{s'} i!^s \\ &\quad \text{for all} \quad (x,u,p) \in P \times K \Big\}. \end{aligned}$$

Additionally f has to satisfy the Levi conditions ($l \ge 1$)

$$|\partial_p^l f(x,t,u,p)| \le C_K M_K^{ls} l!^{s'} \sqrt{a(x,t)} \quad \text{for all} \quad (x,t,u,p) \in P \times [0,T] \times K. \tag{1.7}$$

The set of functions satisfying conditions (C1) and (C2) we denote by

$$C([0,T]; X^{(s,s',s')}_{\mathrm{loc}}(P \times R_u \times R_p))_{LC_x},$$

where LC_x stands for Levi condition with respect to x.

Now we are able to explain perturbations for $f = f(x,t,u,p)$. Let f and F be two functions from $C([0,T]; X^{(s,s',s')}_{loc}(P \times R_u \times R_p))_{LC_x}$. Then F belongs to an ε-neighbourhood of f if there exists a compact set $K \subset R_u \times R_p$ such that

$$|\partial_x^i \partial_u^{\nu_1} \partial_p^{\nu_2} (f(x,t,u,p) - F(x,t,u,p))| \le \varepsilon C_K M_K^{(i+\nu_1+\nu_2)s} (\nu_1! \nu_2!)^{s'} i!^s, \tag{1.8}$$

$$|\partial_p^l (f(x,t,u,p) - F(x,t,u,p))| \le \varepsilon C_K M_K^{ls} (l!)^{s'} \sqrt{a(x,t)} \tag{1.9}$$

for all $(x,t,u,p) \in P \times [0,T] \times K, l \ge 1$.

We can now state our stability result.

Theorem 1.1. *Let us consider the periodical with respect to x Cauchy problem*

$$u_{tt} - (a(x,t)u_x)_x = f(x,t,u,u_x),\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x). \tag{1.10}$$

The function $a(x,t)$ satisfies (C1), *the data belong to $Y^s_{+0}(P)$. Moreover, the function f belongs to $C([0,T];X^{(s,s',s')}_{loc})_{LC_x}$. Let $u \in C^2([0,T];Y^s_{+0}(P))$ be a global solution. Then there is a positive constant ε depending on u and f such that the Cauchy problem*

$$u_{tt} - (a(x,t)u_x)_x = F(x,t,u,u_x),\ u(x,0) = U_0(x),\ u_t(x,0) = U_1(x) \tag{1.11}$$

has a global solution $U \in C^2([0,T];Y^s_{+0}(P))$ for all U_0, U_1, F from an ε-neighbourhood of u_0, u_1, f (see (1.6),(1.8), (1.9)).

The paper is organized as follows. In Section 2 we shall describe the philosophy of approach to get a better understanding of the main ideas to consider the case of spatial degeneracy. Section 3 is devoted to summarizing the main energy estimates. Finally, the proof of Theorem 1.1 is given in Section 4. The case of time degeneracy is sketched in Section 5.

§2. Philosophy of Approach

Let us restrict ourselves to the case of spatial degeneracy. We suppose that $u \in C^2([0,T];Y^s_{+0}(P))$ is the global solution of the Cauchy problem (1.10).

For further considerations we need the energies of finite order $E_N(u)(t)$ and partial energies $e_j(u)(t)$, $j,N \ge 1$, which are defined by

$$E_N(u)(t) = \sum_{j=1}^{N} e_j(u)(t)\rho(t)^{j-k}\frac{j^{ks}}{j!^s}, \tag{2.1}$$

where k is a fixed natural number ($k=3$ in one-dimensional case), $\rho(t) = \rho_0 \exp(-Ct)$, $\rho_0 \le 1$, and

$$e_j(u)(t) = \left(\int_P e^{-At}(a(x,t)|\partial_x^j u|^2 + |\partial_x^{j-1}u_t|^2 + j^2|\partial_x^{j-1}u|^2)\,dx\right)^{1/2}. \tag{2.2}$$

The assumption $u \in C^2([0,T];Y^s_{+0}(P))$ implies $E_N(u)(t) \le H$ for all $N \ge 1$ with a suitable positive constant H. To make small data arguments possible we consider instead of (1.10)

$$u_{tt} - (a(x,t)u_x)_x = f(x,t,u,u_x), \quad u(x,0) = U_0(x),\ u_t(x,0) = U_1(x), \tag{2.3}$$

where (U_0, U_1) is an ε-perturbation of (u_0, u_1) (see (1.6)). If $U \in C^2([0,T];Y^s_{+0}(P))$ is a solution, then $w = U - u$ solves

$$\begin{gathered} w_{tt} - (a(x,t)w_x)_x = g_1(x,t,w,w_x)w + g_2(x,t,w_x)w_x, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \end{gathered}$$

where the data w_0, w_1 are an ε-perturbation of the homogeneous data. Using formula of Hadamard gives the representations

$$\begin{aligned} g_1(x,t,w,w_x) &= \int_0^1 f_u(x,t,u+w\tau,(u+w)_x)\,d\tau, \\ g_2(x,t,w_x) &= \int_0^1 f_p(x,t,u,u_x+w_x\tau)\,d\tau. \end{aligned}$$

We consider the linearized equation at an arbitrary point $w \in C^1([0,T];Y^s_{+0}(P))$ which is given by

$$W_{tt} - (a(x,t)W_x)_x = g_1(x,t,w,w_x)W + g_2(x,t,w_x)W_x,$$
$$W(x,0) = w_0(x), \quad W_t(x,0) = w_1(x). \tag{2.5}$$

The condition (C1) together with results from linear theory[12] guarantees that to each $w \in C^1([0,T]; Y^s_{+0}(P))$ there exists a uniquely determined solution $W \in C^2([0,T]; Y^s_{+0}(P))$. Especially, the uniqueness is important for the definition of a mapping $Q : w \mapsto W$. Using energies of finite order E_N one can refine this statement. Let w be taken from the set

$$X_D = \left\{ w \in C^1([0,T]; \bigcap_{N=1}^{\infty} W_2^{N-1}(P)) : \quad E_N(w)(t) \le D \quad \text{for all } N \right\}. \tag{2.6}$$

Here $W_2^{N-1}(P)$ denotes the Sobolev space of periodical functions with exponent $N-1$. Some estimates (see Section 3) imply for the solution W of (2.5)

$$E'_N(W)(t) \le CE_N(W)(t) \tag{2.7}$$

with a constant C depending on D, T but independent of N. By Lemma of Gronwall from (2.7) it follows that

$$E_N(W)(t) \le E_N(W)(0)\exp(Ct). \tag{2.8}$$

Consequently, we find the small ε-perturbation of data in $E_N(W)(0)$. This results from the special Cauchy problem (2.5) and enables us to study the operator Q . With a suitable $\varepsilon = \varepsilon(D,T)$ the continuous operator Q maps X_D into itself and is even a compact one. By Tychonoff's fixed point theorem there exists a fixed point w. This fixed point is a global solution of (2.5), and $U = u + w$ is a global solution of (2.3).

An additional perturbation of f needs no other essential ideas. We have only to add to the right-hand side of (2.4) the term

$$g(x,t,u,u_x) = F(x,t,u,u_x) - f(x,t,u,u_x). \tag{2.9}$$

Taking account of (1.8), (1.9) we will derive instead of (2.8)

$$E_N(W)(t) \le (E_N(W)(0) + CT\varepsilon)\exp(Ct). \tag{2.10}$$

The term in the parenthesis can be chosen sufficiently small.

§3. Some Energy Estimates

Let us consider instead of (2.5) the linear weakly hyperbolic Cauchy problem

$$W_{tt} - (a(x,t)W_x)_x = h_1(x,t)W + h_2(x,t)W_x + h(x,t),$$
$$W(x,0) = w_0(x), \quad W_t(x,0) = w_1(x). \tag{3.1}$$

For the derivation of energy estimates we suppose that h_1, h_2 and h belong to $C([0,T]; Y^s_{+0}(P))$. The function $a = a(x,t)$ satisfies condition (C1). Consequently, one can find positive constants D and M such that

$$|\partial_x^j h_1| + |\partial_x^j h_2| + |\partial_x^j h| + |\partial_x^j a| \le DM^{js} j!^s (j+1)^{-ks} \quad \text{for all} \quad (x,t) \in P \times [0,T]. \tag{3.2}$$

Lemma 3.1. *If h_2 satisfies the Levi condition*

$$|h_2(x,t)| \le C\sqrt{a(x,t)} \quad \text{for all } (x,t) \in P \times [0,T]$$

and $W \in C^2([0,T]; Y^s_{+0}(P))$ *is a solution of* (3.1), *then there exist a weight function* $\rho = \rho(t) = \rho_0 \exp(-Ct)$ *on* $[0,T]$ *and a suitable positive constant* $C_1 = C_1(D, M, \rho_0)$ *such that for all* $N \geq 1$

$$E'_N(W)(t) \leq C_1\ E_N(W)(t) + E_N(h)(t). \tag{3.3}$$

Proof. For the proof see [12].

The following lemma explains how the condition (C2) is transferred to conditions for g_1 and g_2 from (2.5).

Lemma 3.2. *Let* $u \in C^2([0,T]; Y^s_{+0}(P))$ *be the global solution of* (1.10). *The function* $w \in X_D$ *(see* (2.6)) *is chosen arbitrarily. Then the function* g_1 *which is defined by*

$$g_1(x,t,w,w_x) = \int_0^1 f_u(x,t,u+w\tau,(u+w)_x)d\tau$$

satisfies condition (C2) *without Levi condition* (1.9). *The function* g_2 *which is defined by* $g_2(x,t,w_x) = \int_0^1 f_p(x,t,u,(u+w\tau)_x)d\tau$ *satisfies the full condition* (C2), *the Levi condition even for* g_2 *itself.*

Proof. Using assumptions concerning u and w there exists a positive constant

$$C = C(D,M)\,(|\partial_x^j u| \leq CM^{js} j!^s (j+1)^{-ks} \text{ for all } (x,t) \in P \times [0,T])$$

such that $(u + w\tau_1, (u+w\tau_2)_x) \in K = [-C,C] \times [-C,C]$ uniformly for all $w \in X_D, \tau_1, \tau_2 \in [0,1]$. To this compact set K there exist corresponding constants $\widetilde{C}_K$ and $\widetilde{M}_K$ such that (C2) is satisfied for f. Consequently, we obtain for g_2

$$\begin{aligned} |\partial_x^i \partial_u^{\nu_1} \partial_p^{\nu_2} g_2(x,t,w_x)| &\leq \int_0^1 |\partial_x^i \partial_u^{\nu_1} \partial_p^{\nu_2+1} f(x,t,u,(u+w\tau)_x)| d\tau \\ &\leq \widetilde{C}_K \widetilde{M}_K^{(i+\nu_1+\nu_2+1)s} (\nu_1!(\nu_2+1)!)^{s'} i!^s \\ &\leq C_K M_K^{(i+\nu_1+\nu_2)s} (\nu_1!\nu_2!)^{s'} i!^s \end{aligned}$$

for all $(x,t) \in P \times [0,T]$ and all $w \in X_D$. The new constants C_K and M_K are independent of $w \in X_D$ and $\tau \in [0,1]$. By the same reasoning the Levi conditions ($l \geq 0$)

$$\begin{aligned} |\partial_p^l g_2(x,t,w_x)| &\leq \int_0^1 |\partial_p^{l+1} f(x,t,u,(u+w\tau)_x)| d\tau \\ &\leq \widetilde{C}_K \widetilde{M}_K^{(l+1)s} (l+1)!^{s'} \sqrt{a(x,t)} \leq C_K M_K^{ls} l!^{s'} \sqrt{a(x,t)} \end{aligned}$$

can be shown. Analogously, we get condition (C2) (without Levi condition (1.7)) for g_1.

Now let us turn to the linearization (2.5). The main difficulty consists in estimating the nonlinear terms $g_1 = g_1(x,t,w,w_x)$ and $g_2 = g_2(x,t,w_x)$.

Lemma 3.3. *Let* $u \in C^2([0,T]; Y^s_{+0}(P))$ *be the global solution of* (1.10). *Then there exists a weight function* $\rho = \rho(t)$ *on* $[0,T]$ *such that for* $N \geq 6$

$$\begin{aligned} B_N(w,W)(t) &= \sum_{j=1}^N \|\partial_x^{j-1}(g_1(x,t,w,w_x)W + g_2(x,t,w_x)W_x)\| \frac{\rho(t)^{j-k} j^{ks}}{j!^s} \\ &\leq C_1 E_N(W)(t) \end{aligned} \tag{3.4}$$

for all $w \in X_D$, *where* C_1 *depends on* C_K, M_K, M, ρ_0, D *but not on* $w \in X_D$.

Proof. For the proof see [12].

Corollary 3.1. *Let us consider the linearized problem* (2.5)*, where we odd* (2.9) *to the right-hand side. Here* $u \in C^2([0,T];Y^s_{+0}(P))$ *is the global solution of* (1.10)*. The function* w *is an arbitrary function from* X_D*, where* $\rho = \rho(t)$ *is a sufficiently small weight function which is taken from Lemma* 3.1*. Then the linearized problem has a uniquely determined solution* $W \in C^2([0,T],Y^s_{+0}(P))$*. There exists a positive constant* C_1 *independent of* $w \in X_D$ *and* $N \geq 6$ *such that for all* $t \in [0,T]$

$$E'_N(W)(t) \leq C_1 E_N(W)(t) + E_N(g(x,t,u,u_x)). \tag{3.5}$$

Proof. It is clear that for given functions u and w the nonlinear terms g_1, g_2 and g (see (2.9),(3.4)) belong to $C([0,T];Y^s_{+0}(P))$. The data belong to $Y^s_{+0}(P)$. From results of the linear theory we get the global existence and uniqueness of solution $W \in C^2([0,T];Y^s_{+0}(P))$. The energy inequalities which are given in Lemmas 3.1 and 3.3 imply (3.5). The corollary is proved.

The next considerations serve to estimate the energies

$$E_N(g(x,t,u,u_x))(t) = E_N(F(x,t,u,u_x) - f(x,t,u,u_x)).$$

Due to Leibniz formula

$$\begin{aligned}\partial_x^{j-1} g(x,t,u,u_x) = &\sum_{i+l=j-1} \frac{(j-1)!}{i!} \sum_{l_1+l_2=l} \sum_{\nu_1=0}^{l_1} \sum_{\nu_2=0}^{l_2} \frac{1}{\nu_1!\nu_2!} \\ &\times (F-f)^{(i,\nu_1,\nu_2)}(x,t,u,p) \Big(\sum_{|h|=l_1, 1\leq h_i} \frac{\partial_x^{h_1} u \cdots \partial_x^{h_{\nu_1}} u}{h_1! \cdots h_{\nu_1}!} \Big) \\ &\times \Big(\sum_{|m|=l_2, 1\leq m_i} \frac{\partial_x^{m_1+1} u \cdots \partial_x^{m_{\nu_2}+1} u}{m_1! \cdots m_{\nu_2}!} \Big).\end{aligned}$$

Let $u \in C^2([0,T];Y^s_{+0}(P))$ be the given solution of (1.10). Then there exists a constant $C(u)$ such that (x,t,u,u_x) belongs to

$$P \times [0,T] \times [-C(u), C(u)] \times [-C(u), C(u)].$$

Hence, only the behaviour of f on the compact set $P \times [0,T] \times [-C(u),C(u)] \times [-C(u),C(u)]$ is essential, and condition (C2) is satisfied with suitable constants C_K and M_K. If F is now a function from

$$C([0,T]; X^{(s,s',s')}_{loc}(P \times R_u \times R_p))_{LC_x}$$

which satisfies condition (1.7) for the above compact set, then by the same reasoning to prove Lemma 3.3 (see [12]) one obtains

Lemma 3.4. *The energies of finite order* $E_N(g(t,x,u,u_x))$ *can be estimated by*

$$E_N(g(x,t,u,u_x)) \leq C_K \varepsilon \sum_{j=1}^{\infty} (M_K^s C_0(P) E_N(u)(t))^j j!^{s'-s} \leq C_1 \varepsilon,$$

where C_1 *is independent of* N*.*

§4. Proof of Theorem 1.1

Now let $u \in C^2([0,T];Y^s_{+0}(P))$ be a global solution of (1.10). Let us consider the Cauchy problem

$$u_{tt} - (a(x,t)u_x)_x = F(x,t,u,u_x), u(x,0) = U_0(x), u_t(x,0) = U_1(x). \tag{4.1}$$

If $U \in C^2([0,T]; Y^s_{+0}(P))$ is a solution, then $w = U - u$ satisfies (see (2.9) and (2.4) with F instead of f)

$$w_{tt} - (a(x,t)w_x)_x = g_1(x,t,w,w_x)w + g_2(x,t,w_x)w_x + g(x,t,u,u_x),$$
$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \tag{4.2}$$

Let us consider the linearized equation

$$W_{tt} - (a(x,t)W_x)_x = g_1(x,t,w,w_x)W + g_2(x,t,w_x)W_x + g(x,t,u,u_x),$$
$$W(x,0) = w_0(x), \quad W_t(x,0) = w_1(x). \tag{4.3}$$

Now the function $w = w(x,t)$ is taken arbitrarily from the set (2.6) with a fixed positive constant D. Then there exists a positive constant $C = C(D)$ independent of $\rho = \rho(t)$ on $[0,T]$, $\rho_0 \le 1$, such that all points $(u + w\tau_1, (u + w\tau_2)_x)$ belong to the compact set $\widetilde{K} = [-C,C] \times [-C,C]$. Consequently, there exist constants $C_{\widetilde{K}}$ and $M_{\widetilde{K}}$ such that F satisfies (C2) for all $(x,t) \in P \times [0,T]$ and $w \in X_D$.

According to $M_{\widetilde{K}}$ and M from (3.2) for the estimation of $a = a(x,t)$ we can determine a weight function $\rho = \rho(t)$ on $[0,T]$ satisfying

$$\rho'(t)/2 + (C_2+1)\rho(t) = 0, \quad \rho(0) = \rho_0, \quad M_{\widetilde{K}}\rho_0 < 1 \quad \text{and } M\rho_0 < 1,$$

where the constant C_2 depends on the properties of $a = a(x,t)$ and the initial value ρ_0 depends on the data w_0 and w_1, too. In general we have to choose ρ_0 smaller to guarantee the energy estimates on $[0,T]$ from Lemmas 3.3 and 3.4. Due to Corollary 3.1 it is possible to construct the operator $Q : w \to W$, here the well-posedness of (3.1) with respect to x in $Y^s_{+0}(P)$ plays an important rule. Now let F be from an ε-neighbourhood of f in $C([0,T]; X^{(s,s',s')}_{loc}(P \times R_u \times R_p))_{LC_x}$ (see (1.8),(1.9)), where the neighbourhood is generated by a compact set K containing all points $(u, u_x), (x,t) \in P \times [0,T]$. Then due to (3.5) and Lemma 3.4 for $N \ge 6$

$$E'_N(W)(t) \le C_1 E_N(W)(t) + C_1\varepsilon \tag{4.4}$$

for all $t \in [0,T]$ and $w \in X_D$, where C_1 depends only on $M, M_{\widetilde{K}}, M_K, D$ and ρ_0 but not on N and $w \in X_D$. Using Lemma of Gronwall it follows that

$$E_N(W)(t) \le E_N(W)(0)\exp(C_1 t) + C_1\varepsilon\exp(C_1 t). \tag{4.5}$$

Choosing the data (U_0, U_1) from an ε-neighbourhood of (u_0, u_1) (see (1.6)) then with the fixed $\rho(0) = \rho_0$ the energies of finite order can be estimated in $t = 0$ by $E_N(W)(0) \le C_3\varepsilon$ for all $N \ge 1$, where C_3 depends on M and ρ_0 but not on N. Consequently,

$$E_N(W)(t) \le C_3\varepsilon\exp(C_1 t) + C_1\varepsilon\exp(C_1 t).$$

If ε is small enough, then $E_N(W)(t) \le D$ for all $N \ge 1$ and $t \in [0,T]$. But this means that Q maps X_D into itself. Moreover, by (3.1) W belongs to $C^2\Big([0,T]; \bigcap\limits_{N=1}^{\infty} W_2^{N-1}(P)\Big)$. Thus Q maps X_D into

$$X_D \cap C^2\Big([0,T]; \bigcap_{N=1}^{\infty} W_2^{N-1}(P)\Big),$$

that is, Q is even compact. But this operator depends continuously on $w \in X_D$. If a sequence $\{w_k\}$ tends to w in X_D, then the coefficients $g_1(x,t,w_k,w_{k,x}), g_2(x,t,w_{k,x})$ of the linearized equation tend to $g_1(x,t,w,w_x), g_2(x,t,w_x)$ respectively. Pay attention that g_1 and g_2 are of Gevrey order $s' < s$ with respect to w_k and $w_{k,x}$, respectively, w and w_x. The solution W depends continuously on the coefficients. Consequently, Qw depends continuously on $w \in X_D$. By using Tychonoff's fixed point theorem there exists a fixed point $\tilde{w} \in X_D$ belonging even to $C^2([0,T]; \bigcap_{N=1}^{\infty} W_2^{N-1}(P))$. This fixed point is a solution of

$$w_{tt} - (a(x,t)w_x)_x = F(x,t,u+w,(u+w)_x) - f(x,t,u,u_x),$$
$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x).$$

Consequently, $U = u + \tilde{w}$ is a solution of (4.1). But this solution is uniquely determined with the uniqueness result from [12]. Obviously, U belongs to $C^2([0,T]; Y^s_{+0}(P))$. Thus, the theorem is completely proved.

Connection to the Life Span of Solutions

Theorem 1.1 gives us the following result under the assumption that all the conditions with respect to t are globally satisfied, especially $a = a(x,t)$ is uniformly bounded on $[0,\infty) \times P$:

Corollary 4.1. *Let* $u \in C^2([0,\infty); Y^s_{+0}(P))$ *be a solution of* (1.10). *In connection to* (1.10) *we consider perturbation of data of the form*

$$U_0(x) = u_0(x) + \varepsilon w_0(x), \quad U_1(x) = u_1(x) + \varepsilon w_1(x),$$

where w_o *and* w_1 *are arbitrary functions belonging to* $Y^s_{+0}(P)$. *Then there exists to each small* ε *a uniquely determined solution* $U_\varepsilon \in C^2([0,T(\varepsilon)), Y^s_{+0}(P))$, *where* $T(\varepsilon) \to +\infty$ *for* $\varepsilon \to 0$.

Proof. The given solution u belongs to $C^2([0,T]; Y^s_{+0}(P))$ for all $T > 0$. By Theorem 1.1, there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(T)$ such that (1.10) has a global solution $U_\varepsilon \in C^2([0,T], Y^s_{+0}(P))$ for all $\varepsilon \le \tilde{\varepsilon}(T)$. Hence, to a given small ε there exists a solution $U_\varepsilon \in C^2([0,T(\varepsilon)); Y^s_{+0}(P))$, where $T(\varepsilon)$ tends to infinity if ε tends to 0. Otherwise, we could find a constant T_0 such that (1.10) has no solution $U_\varepsilon \in C^2([0,T_0]; Y^s_{+0}(P))$ with $\varepsilon \to 0$ for data U_0 and U_1. This is impossible by Theorem 1.1.

Remark 4.1. The life span T_ε can be estimated by $T_\varepsilon \ge \mu \log \frac{1}{\varepsilon}$ for small ε. This follows from the fact that the constant C_1 from (4.5) depends only on $M, M_K, M_{\widetilde{K}}, D$ and ρ_0. Consequently,

$$E_N(W)(0)\exp(C_1 t) \le C_3 \varepsilon \exp(C_1 T) \le D$$

implies the above estimate.

§5. The Case of Time Degeneracy

In this section we only want to sketch the considerations which lead to a corresponding result to Theorem 1.1 in the case of time degeneracy. Let us consider

$$u_{tt} - \lambda^2(t)a(x,t)u_{xx} = f(x,t,u,u_x), \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{5.1}$$

Instead of (C1) and (C2) we have to assume:

(C3) The coefficient $b(x,t) = \lambda^2(t)a(x,t)$ belongs to $C^\infty([0,T]; Y^s_{+0}(P))$, where $a(x,t) \ge C > 0$ for all $(x,t) \in P \times [0,T]$ and

$$|\partial_t^k \partial_x^i a(x,t)| \le C_k M_k^{is} i!^s (\lambda(t)/\Lambda(t))^k \text{ for all } t > 0 \text{ and } i,k \ge 0.$$

Here $\Lambda(t) = \int_0^t \lambda(\tau)d\tau$. For $\lambda = \lambda(t)$ we need the additional conditions

$$\lambda(0) = \lambda'(0) = 0, \quad \lambda'(t) > 0 \ \text{ for } t > 0, \quad \lambda(t) \in C^\infty([0,T]), \quad \lambda^2\Lambda^{-1} \in C^\infty([0,T]),$$

$c\lambda(t)/\Lambda(t) \le \lambda'(t)/\lambda(t) \le c_0\lambda(t)/\Lambda(t)$ for all $t > 0$, $c > 1/2$ and c_0 are positive constants,

$|\lambda^{(k)}(t)| \le c_k(\lambda'(t)/\lambda(t))^{k-1}\lambda'(t)$ for all $t > 0$ and $k \ge 1$.

(C4) The function $f = f(x,t,u,p)$ is P-periodic in x and belongs to

$$C^\infty([0,T]; X_{\mathrm{loc}}^{(s,s',s')}(P \times R_u \times R_p)).$$

Additionally, f has to satisfy the Levi conditions $(i, \nu_1 \ge 0,\ \nu_2 \ge 1)$

$$|\partial_x^i \partial_u^{\nu_1} \partial_p^{\nu_2} f(x,t,u,p)| \le C_K M_K^{(i+\nu_1+\nu_2)s} i!^s (\nu_1!\nu_2!)^{s'} \lambda^2(t)/\Lambda(t)$$

for all $(x,t,u,p) \in P \times [0,T] \times K$.

The set of functions satisfying condition (C4) we denote by

$$C^\infty([0,T]; X_{\mathrm{loc}}^{(s,s',s')}(P \times R_u \times R_p))_{LC_t},$$

where LC_t stands for Levi condition with respect to t.

Remark 5.1. We underline that the conditions for λ and a and their derivatives exclude rapid oscillations with respect to t. Such oscillations can lead to non-uniqueness of solutions[1].

Let f and F be two functions from

$$C^\infty([0,T]; X_{\mathrm{loc}}^{(s,s',s')}(P \times R_u \times R_p))_{LC_t}.$$

Then F belongs to an ε-neighbourhood of f if there exist a compact set $K \subset R_u \times R_p$ and a nonnegative integer n_0 such that

$$|\partial_t^k \partial_x^i \partial_u^{\nu_1} \partial_p^{\nu_2}(f(x,t,u,p) - F(x,t,u,p))| \le \varepsilon\, C_{K,n_0} M_{K,n_0}^{(i+\nu_1+\nu_2)s} i!^s (\nu_1!\nu_2!)^{s'}, \tag{5.2}$$

$$|\partial_x^i \partial_u^{\nu_1} \partial_p^{\nu_2}(f(x,t,u,p) - F(x,t,u,p))| \le \varepsilon\, C_K M_K^{(i+\nu_1+\nu_2)s} i!^s (\nu_1!\nu_2!)^{s'} (\lambda^2/\Lambda)(t) \tag{5.3}$$

(in (5.3) $\nu_2 \ge 1$) for all $(x,t,u,p) \in P \times [0,T] \times K$, $0 \le k \le n_0$ and $i, \nu_1, \nu_2 \ge 0$.

Remark 5.2. Later it remains to choose F from an ε-neighbourhood of f which is generated by $n_0 = 0$ and a suitable compact set K.

Now we devote to the perturbed problem

$$u_{tt} - \lambda^2(t)a(x,t)u_{xx} = F(x,t,u,u_x), \quad u(x,0) = U_0(x), \quad u_t(x,0) = U_1(x) \tag{5.4}$$

and suppose that (5.1) has a global solution $u \in C^\infty([0,T]; Y_{+0}^s(P))$. Without loss of generality we can suppose in (5.1) homogeneous data, this implies U_0 and U_1 are sufficiently small.

In opposite to the case of spatial degeneracy it seems to be impossible to study (5.4) directly. If F would have an improved asymptotical behaviour for $t \to 0$, then we are able to find energy estimates leading to an existence result.

Let us consider the following Cauchy problem:

$$u_{tt}^{(0)} = F(x,t,u+u^{(0)},u_x) - u_{tt} + \lambda^2(t)a(x,t)u_{xx},$$
$$u^{(0)}(x,0) = U_0(x), \quad u_t^{(0)}(x,0) = U_1(x). \tag{5.5}$$

Lemma 5.1. *If (F, U_0, U_1) belongs to an $\tilde\varepsilon$-neighbourhood of $(f,0,0)$, $\tilde\varepsilon$ is sufficiently small, then there exists a global solution $u^{(0)} \in C^2([0,T]; Y_{+0}^s(P))$, where $u^{(0)} \to 0$ if $\tilde\varepsilon \to 0$ in $C^2([0,T]; Y_{+0}^s(P))$.*

Proof. If we replace F by f, then (5.5) has the global solution $u^{(0)} \equiv 0$. Now let us choose F from an $\tilde{\varepsilon}$-neighbourhood of f which is generated by $n_0 = 0$ and a compact set K containing all points $(u, u_x), (x,t) \in P \times [0,T]$ (see (5.2), (5.3)). We are able to interpret (5.5) as a weakly hyperbolic equation with spatial degeneracy ($a(x,t) = 0$, right hand side does not depend on p). But the conditions (5.2), (5.3) imply (1.8),(1.9), this gives the statement of this lemma. Moreover $u^{(0)}$ belongs to an ε-neighbourhood of 0 in $C^2([0,T]; Y^s_{+0}(P))$.

Now let us consider the following system of nonlinear ordinary differential equations which can be interpreted as a system of weakly hyperbolic equations with spatial degeneracy:

$$u^{(1)}_{tt} = F(x,t,u+u^{(0)}+u^{(1)}, u_x + u^{(0)}_x) - F(x,t,u+u^{(0)},u_x) + \lambda^2(t)a(x,t)u^{(0)}_{xx},$$

and in general for $i = 2, \cdots, n$

$$\begin{aligned} u^{(i)}_{tt} &= F(x,t,u+u^{(0)}+\cdots+u^{(i)}, u_x+u^{(0)}_x+\cdots+u^{(i-1)}_x) \\ &\quad - F(x,t,u+u^{(0)}+\cdots+u^{(i-1)}, u_x+u^{(0)}_x+\cdots+u^{(i-2)}_x) \\ &\quad + \lambda^2(t)a(x,t)u^{(i-1)}_{xx} \end{aligned} \tag{5.6}$$

with homogeneous initial conditions $u^{(i)}(x,0) = u^{(i)}_t(x,0) = 0, i = 1, \cdots, n$. Then one can prove the following

Lemma 5.2. *The above system of nonlinear ordinary differential equations possesses uniquely determined solutions $u^{(i)} \in C^2([0,T]; Y^s_{+0}(P))$, $i = 1, 2, \cdots, n$, for all F belonging to some sufficiently small $\tilde{\varepsilon}$-neighbourhood of f. Moreover,*

$$E_N(u^{(i)})(t) \le C_{i,\tilde{\varepsilon}}\lambda^i(t) \tag{5.7}$$

for all $t \in [0,T]$ and $N \ge 0$, where $C_{i,\tilde{\varepsilon}} \to 0$ if $\tilde{\varepsilon} \to 0$.

Keeping in mind these functions $u^{(i)}$ we seek for a function $v = v(x,t)$ as a solution of the problem

$$v_{tt} - \lambda^2(t)a(x,t)v_{xx} = G_n(x,t,v,v_x), \quad v(x,0) = v_t(x,0) = 0, \tag{5.8}$$

where

$$\begin{aligned} G_n(x,t,v,v_x) &= F(x,t,u+u^{(0)}+\cdots+u^{(n)}+v, u_x+u^{(0)}_x+\cdots+u^{(n)}_x+v_x) \\ &\quad - F(x,t,u+u^{(0)}+\cdots+u^{(n)}, u_x+u^{(0)}_x+\cdots+u^{(n-1)}_x) \\ &\quad + \lambda^2(t)a(x,t)u^{(n)}_{xx}, \end{aligned}$$

especially, $E_N(G_n(x,t,0,0)) \le C_{n,\tilde{\varepsilon}}\lambda^n(t)$ due to (5.7). We study the linearized equation

$$\begin{aligned} V_{tt} - \lambda^2(t)a(x,t)V_{xx} &= h_1(x,t,v,v_x)V + h_2(x,t,v,v_x)V_x + G_n(x,t,0,0), \\ V(x,0) &= V_t(x,0) = 0. \end{aligned} \tag{5.9}$$

Applying results of [8,16] to this equation gives instead of (4.4)

$$E'_N(V)(t) \le q\frac{\lambda'(t)}{\lambda(t)}\, E_N(V)(t) + E_N(G_n(x,t,0,0))$$

for all $t \in [0,T]$ and $v \in X_D$, where a sufficiently small $\tilde{\varepsilon}$ in Lemma 5.2 guarantees that q depends only on the given solution u and the coefficient $\lambda^2(t)a(x,t)$. Using Lemma of Nersesian[8] we arrive from this differential inequality with a singular coefficient at

$$E_N(V)(t) \le \lambda^q(t)\int_0^t \lambda^{n-q}(\tau)\lambda^{-n}(\tau)E_N(G_n(x,\tau,0,0))d\tau \le C_{n,\tilde{\varepsilon}}T\lambda^n(t),$$

where $C_{n,\tilde{\varepsilon}}$ is independent of N. Hence, one can find a sufficiently small $\tilde{\varepsilon} > 0$ such that the operator $Q : v \to V$ maps

$$\Big\{v \in C^1\Big([0,T]; \bigcap_{N=1}^{\infty} W_2^{N-1}(P)\Big) : E_N(v)(t) \le D\Big\} \quad \text{into}$$

$$\Big\{v \in C^2\Big([0,T]; \bigcap_{N=1}^{\infty} W_2^{N-1}(P)\Big) : E_N(v)(t) \le D \ \text{ and}$$

$$E_N(v)(t) = O(\lambda^n(t)) \ \text{ for } \ t \to 0\Big\}.$$

As in the case of spatial degeneracy this leads to a globally defined solution V of (5.9). But then $U = u + \sum_{i=0}^{n} u^{(i)} + V$ is a globally defined solution of (5.4), where U belongs to an ε-neighbourhood of u in $C^2([0,T]; Y^s_{+0}(P))$.

Thus, we have proved the following result.

Theorem 5.1. *Let us consider the periodical with respect to x Cauchy problem*

$$u_{tt} - \lambda^2(t)a(x,t)u_{xx} = f(x,t,u,u_x), \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x).$$

The functions $\lambda = \lambda(t)$ and $a = a(x,t)$ satisfy (C3), *the data belong to $Y^s_{+0}(P)$. Moreover, f belongs to $C^\infty([0,T]; X^{(s,s',s')}_{\text{loc}}(P \times R_u \times R_p))_{LC_t}$. Let $u \in C^\infty([0,T]; Y^s_{+0}(P))$ be a global solution. Then there is a positive constant ε depending on u, f, λ, and a such that*

$$u_{tt} - \lambda^2(t)a(x,t)u_{xx} = F(x,t,u,u_x), \quad u(x,0) = U_0(x), \quad u_t(x,0) = U_1(x)$$

has a global solution $U \in C^\infty([0,T]; Y^s_{+0}(P))$, too, for all U_0, U_1 and F from an ε-neighbourhood of u_0, u_1, f (see (1.6), (5.2),(5.3)).

The connection between Theorem 5.1 and the life span of solutions can be drawn in the same way as in the case of spatial degeneracy (see Corollary 4.1. and Remark 4.1).

Remark 5.3. In [11] we have proved a local existence result for

$$u_{tt} - \lambda^2(t)a(x,t)u_{xx} = f(x,t,u_x), \quad u(x,0) = u_t(x,0) = 0$$

under the Levi conditions

$$|\partial_x^i \partial_p^{\nu_2} f(x,t,p)| \le C_K \ M_K^{(i+\nu_2)s} i!^s \nu_2!^{s'} o(\lambda^2/\Lambda).$$

If we know the existence of a global solution u for (5.1), then the global existence of solutions for the perturbed problems (5.4) can be proved under the weaker Levi condition (C4).

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