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A STRASSEN LAW OF THE ITERATED LOGARITHM FOR PROCESSES WITH INDEPENDENT INCREMENTS**

Wang Jiagang*

Abstract

Let $X = \{X(t),\, t \geq 0\}$ be a process with independent increments (PII) such that

$$\mathsf{E}[X(t)] = 0, \qquad D_X(t) \triangleq \mathsf{E}[X(t)]^2 < \infty, \qquad \lim_{t\to\infty} \frac{D_X(t)}{t} = 1,$$

and there exists a majoring measure G for the jump ΔX of X. Under these assumptions, using rather a direct method, a Strassen's law of the iterated logarithm (Strassen LIL) is established. As some special cases, the Strassen LIL for homogeneous PII and for partial sum process of i.i.d. random variables are comprised.

Keywords Strassen law of the iterated logarithm, process with independent increments, stochastic calculus

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§0. Introduction

Let $\{W(t),\, t \geq 0\}$ be a Brownian motion,

$$W_n(t) = \frac{W(nt)}{\sqrt{2n \operatorname{llg} n}}, \qquad t \in [0,1], \quad n \geq 1,$$

where $\operatorname{llg} x = \log(\log(x \vee e^e))$. Strassen[9] proved that with probability one ($W_n = \{W_n(t),\, t \geq 0\}$, $n \geq 1$) is relatively compact in $C([0,1])$ endowed with uniform norm and the set of its limit points coincides with the following $\mathcal{K}_1$:

$$\mathcal{K}_1 = \Big\{ f \colon f \text{ is absolutely continuous on } [0,1] \text{ and } f(0) = 0, \int_{[0,1]} [f'(t)]^2 dt \leq 1 \Big\}.$$

This is also called functional law of the iterated logarithm for Brownian motion. We shall abbreviate conclusions of this form by

$$\{W_n\} \to\to \mathcal{K}_1, \qquad \text{a.s.} \tag{0.1}$$

If $\{V_n,\, n \geq 1\}$ is a sequence of i.i.d. random variables with $\mathsf{E}[V_n] = 0$, $\mathsf{E}[V_n^2] = 1$, write

$$S_n = \sum_{j=1}^n V_j, \qquad \xi_n(t) = \frac{S_{[nt]} + (nt - [nt])(S_{[nt]+1} - S_{[nt]})}{\sqrt{2n \operatorname{llg} n}}, \qquad t \in [0,1], \quad n \geq 1.$$

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*Institute of Applied Mathematics, East China University of Science and Technology, Shanghai 200237, China.

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Strassen[9] also used Skorokhod embedding technique to prove the following similar result for $\{\xi_n\}$:

$$\{\xi_n,\, n \geq 1\} \to\to \mathcal{K}_1. \qquad \text{a.s.}$$

This is called the Strassen law of the iterated logarithm (in brief, Strassen LIL) or a.s. invariance principle (cf. [1]). As an immediate consequence, if φ is any continuous functional on $C[0,1]$, then with probability one $\{\varphi(\xi_n),\, n \geq 1\}$ is relatively compact and the set of its limit points is $\varphi(\mathcal{K}_1)$. In particular, if $\varphi(x) = x(1)$, then the Hartman-Wintner law of the iterated logarithm is just a corollary of the Strassen LIL. A vast array of other a.s. limit results follows immediately from the Strassen LIL by choosing suitable continuous functionals φ. Hence the Strassen LIL is one of the most important results in strong limit theory.

By using the large deviation theorem for Brownian motion Deuschel and Stroock[3] strengthened Strassen's result (0.1). Let $C(0,\infty)$ be the set of continuous functions on $[0,\infty)$ endowed with the following norm

$$\|x\| = \sup_t \frac{|x(t)|}{1+t}.$$

Then for $[0,\infty)$ instead of $[0,1]$, $\{W_n\} \to\to \mathcal{K}$, a.s.

Khoshnevisan[6] established a result to embedding compound Poisson processes into a Brownian motion. As a by-product, he also proved the Strassen LIL for compensated compound Poisson process on $D[0,1]$ endowed with uniform norm.

In this paper, we shall establish the Strassen LIL for more general (non-homogeneous or homogeneous) processes with independent increments (in brief, PII). As some special cases, this result also comprises the Strassen LIL for partial sum processes of i.i.d. random variables and for homogeneous PII. The method we used is a more direct method based on the stochastic calculus of PII, a similar procedure is also used to discuss the asymptotic behaviour of locally square integrable martingales[12]. The next section describes some notations and states the main theorem. The proof will be given in §2 and §3. In the last section, we will give some useful corollaries of main theorem.

§1. Notations and Main Theorem

In this paper we shall use the usual notations and symbols in stochastic calculus of PII according to [4] and [5], unless stated otherwise.

The general setting of this paper is a complete probability space $(\Omega,\, \mathcal{F},\, \mathsf{P})$. Let $X = \{X(t),\, t \geq 0\}$ be a PII with mean zero and $\mathsf{E}[X(t)]^2 < \infty$. Therefore X is a martingale and also locally square integrable. We always take its cadlag (right continuous and with finite left hand limits) version. Denote the continuous martingale part of X by X^c and the predictable quadratic variation of X^c by $\langle X^c \rangle = C$, where C is a deterministic increasing continuous function. Assume that the jump measure of X is μ, i.e., $\mu = \sum\limits_s \varepsilon_{(s,\Delta X(s))}$, where $\varepsilon_{\{a\}}$ is the unit measure concentrated on a. Then the dual predictable projection ν of μ is also deterministic and $\nu = \mathsf{E}[\mu]$, X^c and μ are independent mutually and X has the following integral representation (Lévy-Itô decomposition):

$$X = X^c + x * (\mu - \nu),$$

where $x * (\mu - \nu)$ denotes the stochastic integral of x with respect to martingale measure $\mu - \nu$. Meanwhile, the distribution of X is determined by (C, ν) uniquely. In particular,

$$D_X(t) \triangleq \mathsf{E}[X(t)]^2 = \langle X \rangle_t = C(t) + x^2 * \nu_t. \tag{1.1}$$

The ν has the following canonical predictable decomposition (see, e.g., [4, p. 381]):

$$\nu(dt, dx) = N_t(dx) D_X(dt), \tag{1.2}$$

where $N_t(dx)$ is a transition σ-finite measure from $(\boldsymbol{R}_+, \mathcal{B})$ to $(\boldsymbol{R}, \mathcal{B})$ with

$$\int_{\boldsymbol{R}} x^2 N_t(dx) = 1, \qquad \forall t \in \boldsymbol{R}_+. \tag{1.3}$$

Definition. *For a family of σ-finite measure $\{N_t,\, t \in I\}$ on $\boldsymbol{R}$, if there exists a finite measure G such that*

$$N_t(\{x : |x| \geq a\}) \leq G(\{x : |x| \geq a\}) < \infty \qquad \forall a \geq 1, \quad t \in I,$$

then we say that there exists a majoring measure G for the ΔX or for $\{N_t,\, t \in I\}$ and denote it by $(N_t) \prec N$.

The following Lemma is evident (cf. [11]).

Lemma 1.1. 1) *Suppose that for some $\delta > 0$ $\{N_t\}$ satisfies*

$$\sup_t \int_{\boldsymbol{R}} |x|^{2+\delta} N_t(dx) = C < \infty$$

and $G(dx) = 1_{|x| \geq 1} C(2(2+\delta)x^{3+\delta})^{-1} dx$. Then $(N_t) \prec G$ and $\int x^2\, G(dx) < \infty$.

2) *If $\{N_t\} \prec G$ and f is a nondecreasing non-negative function on $\boldsymbol{R}_+$ with $f(1) = 0$, then*

$$\int_{\boldsymbol{R}} f(|y|) N_t(dy) \leq \int_{\boldsymbol{R}} f(|y|) G(dy), \qquad \forall t. \tag{1.4}$$

In this paper we consider the PII $X = \{X(t), t \geq 0\}$ which satisfies the following more general assumption.

Assumption A. Let $X = \{X(t),\, t \geq 0\}$ be a PII with $\mathsf{E}X(t) = 0$, $D_X(t) = \mathsf{E}[X(t)]^2$,

$$\lim_{t \to \infty} \frac{D_X(t)}{t} = 1, \tag{1.5}$$

and there exists a majoring G for the jump ΔX of X with

$$\int x^2 G(dx) < \infty. \tag{1.6}$$

Since the trajectories of a PII with mean zero are cadlag, instead of the space of continuous functions we shall consider the set of all cadlag functions on $[0, \infty)$. Let

$$\mathcal{C} = \left\{ f \colon f \text{ is continuous on } [0, \infty), f(0) = 0, \lim_{t \to \infty} \frac{|f(t)|}{1+t} = 0 \right\}. \tag{1.7}$$

$$\mathcal{D} = \left\{ f \colon \begin{array}{c} f \text{ is right continuous and with finite left} \\ \text{limits on } [0, \infty), f(0) = 0, \displaystyle\lim_{t \to \infty} \frac{|f(t)|}{1+t} = 0 \end{array} \right\}. \qquad \|f\|_{\mathcal{D}} = \sup_{t > 0} \frac{|f(t)|}{1+t}. \tag{1.8}$$

Then both $\mathcal{C}$ and $\mathcal{D}$ endowed with norm $\|\cdot\|_{\mathcal{D}}$ are Banach spaces. $\mathcal{C}$ is a closed set of $\mathcal{D}$. Let

$$\mathcal{K} = \left\{ f \colon \begin{array}{c} f \text{ is absolutely continuous on } [0, \infty) \\ \text{and } f(0) = 0, \int_0^\infty [f'(t)]^2 dt \leq 1 \end{array} \right\}. \tag{1.9}$$

Similar to the case of $C[0, 1]$ (see, e.g., [2, Lemma 1.2]), $\mathcal{K}$ is a compact subset of $\mathcal{C}$.

The main result of this paper is the following theorem:

Main Theorem. *Suppose that* X *satisfies Assumption* A *and*

$$\xi_n(t) = \frac{X(nt)}{\sqrt{2n \operatorname{llg} n}}, \qquad t \geq 0, \quad n \geq 1. \tag{1.10}$$

Then $(\xi_n = \{\xi_n(t),\, t \geq 0\},\, n \geq 1)$ *is relatively compact in* $\mathcal{D}$ *with probability one and almost surely the set of its limit points is* $\mathcal{K}$, *i.e.,*

$$\{\xi_n,\, n \geq 1\} \to\to \mathcal{K}, \qquad a.s. \tag{1.11}$$

§2. The Case of Processes with Restricted Jumps

In this section we will consider the process with independent increments $Y = \{Y(t),\, t \geq 0\}$ which satisfies the following Assumption B.

Assumption B. *Let* $Y = \{Y(t),\, t \geq 0\}$ *be a process with independent increments such that* $\mathsf{E}[Y(t)] = 0$, $\mathsf{E}[Y(t)]^2 = D(t)$,

$$\lim_{t\to\infty} \frac{D(t)}{t} = 1 \tag{2.1}$$

and

$$\sup_{s\leq t} |\Delta Y(s)| \leq \varepsilon(t)\sqrt{t/\operatorname{llg} t}, \qquad \forall t > 0, \qquad a.s.$$

where $\varepsilon(t)$ *is a positive function with*

$$\lim_{t\to\infty} \varepsilon(t) = 0. \tag{2.2}$$

Obviously, if Y satisfies Assumption B, then Y is a locally square integrable martingale.

To begin with, we need the following inequality of probability of large deviation for martingale in [8], it will be one of important tools in this section.

Lemma 2.1.[8,p.899] *Let* T *be a positive number and* $M = \{M(t),\, t \geq 0\}$ *be a PII such that*

$$\mathsf{E}[M(t)] = 0, \qquad \mathsf{E}[M(T)]^2 \leq b(T), \qquad \sup_{t\leq T} |\Delta M(t)| \leq d(T).$$

Then

$$\mathsf{P}\Big(\sup_{t\leq T} |M(t)| \geq a\Big) \leq 2\exp\left[-\frac{a^2}{2b(T)}\psi\left(\frac{ad(T)}{b(T)}\right)\right], \qquad \forall a > 0, \tag{2.3}$$

where

$$\psi(x) = \frac{2}{x^2}\int_0^x \int_0^y \frac{dz}{1+z} dy = \frac{2(1+x)\log(1+x) - 2x}{x^2}, \qquad x > 0. \tag{2.4}$$

From (2.4) it is easy to know that ψ is a decreasing continuous function and

$$\psi(x) \leq 1, \qquad \lim_{x\to 0} \psi(x) = 1. \tag{2.5}$$

For a stochastic process X, write

$$\omega(\delta, t, X) = \sup_{\substack{0\leq u,v\leq t\\ |v-u|\leq\delta}} |X_v - X_u|.$$

From this definition it is easy to see that

$$\omega(\delta, t, X) \leq 3 \sup_{j:j\delta\leq t} \ \sup_{j\delta\leq s\leq (j+1)\delta\wedge t} |X(s) - X(j\delta)|. \tag{2.6}$$

We are going to establish the asymptotic behaviour of $\omega(\delta, t, Y)$ firstly.

Proposition 2.1. *Let Y be a PII satisfying Assumption* B. *Then for each $\alpha \in (0,1]$*

$$\overline{\lim_{t\to\infty}} \frac{\omega(\alpha t, t, Y)}{\sqrt{2t \operatorname{llg} t}} \le 3\sqrt{\alpha}. \tag{2.7}$$

Proof. Firstly, for $p > 1$ and positive integer n write

$$t_j = j\alpha p^n, \qquad j = 0, 1, 2, \cdots .$$

Due to (2.1) for given $\delta > 0$ there exists $n_0 > 0$ such that

$$\Big|\frac{D(t)}{t} - 1\Big| < \frac{\delta}{2([1/\alpha]+1)}, \qquad \forall t \ge \alpha p^{n_0}.$$

By direct calculating it deduces that

$$D(t_{j+1}) - D(t_j) < (1+\delta)(t_{j+1} - t_j), \qquad \forall j \le [1/\alpha], \quad n \ge n_0.$$

Consider $Z_j(t) = Y(t \vee t_j) - Y(t_j)$, $t \ge t_j$. Then $\{Z_j(t),\, t \ge t_j\}$ is a PII, and $\mathsf{E}[Z_j(t)] = 0$, $\mathsf{E}[Z_j(t)]^2 = D(t) - D(t_j)$,

$$\mathsf{E}[Z_j(t_{j+1})]^2 \le (1+\delta)(t_{j+1} - t_j) = (1+\delta)\alpha p^n, \qquad \forall j \le [1/\alpha], \quad n \ge n_0,$$
$$\sup_{t \le p^n} |\Delta Z_j(t)| \le \varepsilon(p^n)\sqrt{p^n/\operatorname{llg} p^n}, \qquad \text{a.s.}$$

By Lemma 2.1 for $\delta > 0$ we have

$$\begin{aligned}
&\mathsf{P}\Big(\sup_{t_j \le t \le t_{j+1}\wedge p^n} |Y(t) - Y(t_j)| \ge (1+\delta)\sqrt{2\alpha p^n \operatorname{llg} p^n}\,\Big) \\
&\le 2\exp\Big[-\frac{(1+\delta)^2 2\alpha p^n \operatorname{llg} p^n}{2(1+\delta)\alpha p^n}\psi\Big(\frac{\sqrt{2\alpha p^n \operatorname{llg} p^n}}{\alpha p^n}\varepsilon(p^n)\sqrt{\frac{p^n}{\operatorname{llg} p^n}}\,\Big)\Big] \\
&= 2\exp\Big[-(1+\delta)\operatorname{llg} p^n \psi\Big(\frac{\sqrt{2}\varepsilon(p^n)}{\sqrt{\alpha}}\Big)\Big], \qquad \forall j \le [1/\alpha].
\end{aligned} \tag{2.8}$$

Next, for $\delta > 0$ we have

$$\begin{aligned}
&\mathsf{P}\Big(\omega(\alpha p^n, p^n, Y) \ge 3(1+\delta)\sqrt{2\alpha p^n \operatorname{llg} p^n}\,\Big) \\
&\le \mathsf{P}\Big(\sup_{0 \le j \le [1/\alpha]} \sup_{t_j \le t \le t_{j+1}\wedge p^n} |Y_t - Y_{t_j}| \ge (1+\delta)\sqrt{2\alpha p^n \operatorname{llg} p^n}\,\Big) \qquad \text{(by (2.6))} \\
&\le \Big(\Big[\frac{1}{\alpha}\Big] + 1\Big) \sup_{0 \le j \le [1/\alpha]} \mathsf{P}\Big(\sup_{t_j \le t \le t_{j+1}\wedge p^n} |Y_t - Y_{t_j}| \ge (1+\delta)\sqrt{2\alpha p^n \operatorname{llg} p^n}\,\Big) \\
&\le \Big(\Big[\frac{1}{\alpha}\Big] + 1\Big) 2\exp\Big[-(1+\delta)\operatorname{llg} p^n \psi\Big(\frac{\sqrt{2}\varepsilon(p^n)}{\sqrt{\alpha}}\Big)\Big] \qquad \text{(from (2.8))} \\
&\le \frac{2([1/\alpha]+1)}{(n\log p)^{1+\delta/2}}, \qquad \forall n \ge n_0 \vee n_1.
\end{aligned} \tag{2.9}$$

The last inequality comes from the following facts: since $p > 1$, by (2.2) and (2.5)

$$\lim_{n\to\infty} \psi\Big(\frac{\sqrt{2}\varepsilon(p^n)}{\sqrt{\alpha}}\Big) = 1.$$

Hence there is an n_1 such that

$$\psi\Big(\frac{\sqrt{2}\varepsilon(p^n)}{\sqrt{\alpha}}\Big) > \frac{1+\delta/2}{1+\delta}, \qquad \forall n \ge n_1.$$

Now from (2.9) we have

$$\sum_n \mathsf{P}\big(\omega(\alpha p^n, p^n, Y) \geq 3(1+\delta)\sqrt{2\alpha p^n \operatorname{llg} p^n}\,\big) < \infty.$$

Hence from Borel-Cantelli lemma and the arbitrariness of $\delta > 0$ we get

$$\varlimsup_{n\to\infty} \frac{\omega(\alpha p^n, p^n, Y)}{\sqrt{2p^n \operatorname{llg} p^n}} \leq 3\sqrt{\alpha}, \qquad \text{a.s.}$$

At last, while $p^n \leq t \leq p^{n+1}$, from the definition of ω we have

$$\omega(\alpha t, t, Y) \leq \omega(\alpha p^{n+1}, p^{n+1}, Y), \qquad \frac{\omega(\alpha t, t, Y)}{\sqrt{2t \operatorname{llg} t}} \leq \frac{\omega(\alpha p^{n+1}, p^{n+1}, Y)}{\sqrt{2p^n \operatorname{llg} p^n}}.$$

Therefore

$$\varlimsup_{t\to\infty} \frac{\omega(\alpha t, t, Y)}{\sqrt{2t \operatorname{llg} t}} \leq 3\sqrt{\alpha p}, \qquad \text{a.s.}$$

Since p may be any number greater than 1, letting $p \downarrow 1$ (2.7) comes.

Now we are going to establish a law of the iterated logarithm for $\sum\limits_{j=1}^m \alpha_j[Y(jt) - Y((j-1)t)]$.

Proposition 2.2. *Let $Y = \{Y(t),\, t \geq 0\}$ be a PII satisfying Assumption B, $m \in \boldsymbol{N} = \{1, 2, \cdots\}$, $\alpha_1, \cdots \alpha_m \in \boldsymbol{R}$. Then*

$$\varlimsup_{t\to\infty} \frac{\sum\limits_{j=1}^m \alpha_j[Y(jt) - Y((j-1)t)]}{\sqrt{2t \operatorname{llg} t}} \leq \sqrt{\sum_{j=1}^m \alpha_i^2}, \qquad a.s. \tag{2.10}$$

Proof. No loss of generality we can assume $\sum\limits_{j=1}^m \alpha_j^2 = 1$. From (2.1) for given $\eta > 0$ there is a number t_0 such that

$$\Big|1 - \frac{D(jt) - D((j-1)t)}{t}\Big| < \frac{\eta}{2}, \qquad j \leq m, \quad \forall t \geq t_0.$$

For fixed $t \geq t_0$, put

$$H(s) = \sum_{j=1}^m \alpha_j 1_{](j-1)t, jt]}(s), \qquad U(s) = (H \cdot Y)_s,$$

where $H \cdot Y$ denotes the stochastic integral of H with respect to Y. Then U is a PII and also a square integrable martingale,

$$U(mt) = \sum_{j=1}^m \alpha_j[Y(jt) - Y((j-1)t)] \triangleq S(t), \tag{2.11}$$

$$\mathsf{E}[U(mt)]^2 = \sum_{j=1}^m \alpha_j^2[D(jt) - D((j-1)t)] \leq (1 + \eta/2)t,$$

$$\begin{aligned} \sup_s |\Delta U(s)| &\leq \max_{1\leq j\leq m} |\alpha_j| \varepsilon(mt)\sqrt{D(mt)/\operatorname{llg} D(mt)} \\ &\leq (1+\eta)\varepsilon(mt)\sqrt{mt/\operatorname{llg} t}, \qquad \text{for } t > t_0. \end{aligned}$$

Now for $\delta > 0$ from (2.11) and Lemma 2.1 we have

$$\begin{aligned} &\mathsf{P}\Big(|S(t)| \geq (1+\delta)\sqrt{(1+\eta)2t \operatorname{llg} t}\Big) \\ &\leq 2\exp\Big[-\frac{(1+\delta)^2(1+\eta)2t \operatorname{llg} t}{2(1+\eta)t}\psi\Big(\sqrt{2m(1+\eta)}(1+\delta)\varepsilon(mt)\Big)\Big]. \end{aligned}$$

Owing to (2.2) and (2.5) there exists t_1 such that

$$\psi\big(\sqrt{2m(1+\eta)}(1+\delta)\varepsilon(mt)\big) \geq \frac{1}{1+\delta}, \qquad \forall t \geq t_1.$$

Therefore

$$\mathsf{P}\big(|S(t)| \geq (1+\delta)\sqrt{(1+\eta)2t\operatorname{llg} t}\,\big) < \frac{2}{(\log t)^{1+\delta}}, \qquad \forall t \geq t_0 \vee t_1.$$

For $p > 1$, put $t = p^k$, we have

$$\sum_{k=1}^{\infty} \mathsf{P}\Big(|S(p^k)| \geq (1+\delta)\sqrt{(1+\eta)2p^k \operatorname{llg} p^k}\,\Big) < \infty.$$

By using Borel-Cantelli Lemma we have

$$\varlimsup_{t\to\infty} \frac{S(p^k)}{\sqrt{2p^k \operatorname{llg} p^k}} \leq (1+\delta)\sqrt{1+\eta}. \tag{2.12}$$

Meanwhile,

$$\begin{aligned}\sup_{p^k \leq t \leq p^{k+1}} |S(t) - S(p^k)| &\leq 2m(\max_j |\alpha_j|) \sup_{1\leq j\leq m} \sup_{p^k \leq t\leq p^{k+1}} |Y(jt) - Y(jp^k)| \\ &\leq 2m\omega(m(p-1)p^k, mp^{k+1}, Y).\end{aligned}$$

Therefore Proposition 2.1 gives

$$\varlimsup_{k\to\infty} \frac{\sup\limits_{p^k\leq t\leq p^{k+1}} |S(t) - S(p^k)|}{\sqrt{2p^k \operatorname{llg} p^k}} \leq 6m\sqrt{m(p-1)}. \tag{2.13}$$

Combining (2.12) and (2.13) we get

$$\varlimsup_{t\to\infty} \frac{|S(t)|}{\sqrt{2t\operatorname{llg} t}} \leq \sqrt{p}\big[(1+\delta)\sqrt{1+\eta} + 6m\sqrt{m(p-1)}\,\big], \qquad \text{a.s.} \tag{2.14}$$

Since δ, η may be arbitrary positive numbers and p is an arbitrary number greater than 1, letting $\delta \downarrow 0$, $\eta \downarrow 0$ and $p \downarrow 1$ in (2.14) yields

$$\varlimsup_{t\to\infty} \frac{|S(t)|}{\sqrt{2t\operatorname{llg} t}} \leq 1.$$

The proposition is proved.

In order to prove that (2.10) is an equality, we borrow a lemma from [7] (see, e.g., [7, p. 269] or [10, Lemma 1]).

Lemma 2.2. *Let S be a random variable. If for some $d > 0$*

$$\exp\Big(\frac{u^2}{2}(1-ud)\Big) \leq \mathsf{E}(e^{uS}) \leq \exp\Big(\frac{u^2}{2}\Big(1+\frac{ud}{2}\Big)\Big), \qquad \forall u \in [0, 1/d),$$

then for any $\gamma > 0$, there exists numbers $\varepsilon_0 > 0$ and $\eta_0 > 0$ (both depending on γ) such that

$$\mathsf{P}(S > x) > \exp\Big[-\frac{x^2}{2}(1+\gamma)\Big], \qquad \forall x \in (\varepsilon_0, \eta_0/d). \tag{2.15}$$

Proposition 2.3. *Let $U = \{U(t), t \geq 0\}$ be a PII such that $\mathsf{E}U(t) = 0$, $\mathsf{E}U(t)^2 = D(t)$ and $\sup\limits_{s\leq t} |\Delta U(s)| \leq d(t)$. Then*

$$\exp\Big[\frac{u^2}{2}(1-ud(t))D(t)\Big] \leq \mathsf{E}\big[e^{uU(t)}\big] \leq \exp\Big[\frac{u^2}{2}(1+ud(t))D(t)\Big], \quad \forall u \in \Big[0, \frac{1}{d(t)}\Big). \tag{2.16}$$

Proof. Let μ be the jump measure of U, $\nu = \mathsf{E}\mu$ be the dual predictable projection of μ and U^c be the continuous local martingale part of U. Then by the stochastic integral representation of PII we have

$$\mathsf{E}[e^{uU(t)}] = \prod_{0<s\le t}\Big(1+\int_{|x|\le d(t)}(e^{ux}-1-ux)\nu(\{s\},dx)\Big)$$
$$\times\exp\Big(\frac{u^2}{2}\mathsf{E}[U^c(t)]^2+\int_0^t\int_{|x|\le d(t)}(e^{ux}-1-ux)\nu^c(ds,dx)\Big), \tag{2.17}$$

where ν^c is the continuous part of ν (i.e., $\nu^c(\{s\},\boldsymbol{R})=0$, $\forall s>0$). Meanwhile,

$$D(t)=\mathsf{E}[U^c(t)]^2+(x^2*\nu)_t=\mathsf{E}[U^c(t)]^2+(x^21_{\{|x|\le d(t)\}}*\nu^c)_t+\sum_{s\le t}\Delta D(s), \tag{2.18}$$

$$\Delta D(s)=\int_{|x|\le d(t)}x^2\nu(\{s\},dx)\le d(t)^2,\qquad s\le t. \tag{2.19}$$

Note that for $|ux|<1$

$$\Big|\frac{e^{ux}-1-ux}{u^2x^2}-\frac12\Big|=\Big|\sum_{n=3}^{\infty}\frac{(ux)^{n-2}}{n!}\Big|\le\sum_{n=1}^{\infty}\frac{|ux|^n}{3^n}\le\frac{|ux|}{2}$$

and

$$\frac{u^2}{2}\Big(1-\frac{|ux|}{2}\Big)\le\frac{e^{ux}-1-ux}{x^2}\le\frac{u^2}{2}\Big(1+\frac{|ux|}{2}\Big).$$

Hence

$$\frac{u^2}{2}\Big(1-\frac{ud(t)}{2}\Big)(x^2*\nu^c)_t\le\int_0^t\int_{|x|\le d(t)}(e^{ux}-1-ux)\nu^c(ds,dx)$$
$$\le\frac{u^2}{2}\Big(1+\frac{ud(t)}{2}\Big)(x^2*\nu^c)_t,\quad\forall u\in[0,1/d(t)). \tag{2.20}$$

$$\frac{u^2}{2}\Big(1-\frac{ud(t)}{2}\Big)\Delta D(s)\le\int_{|x|\le d(t)}(e^{ux}-1-ux)\nu(\{s\},dx)$$
$$\le\frac{u^2}{2}\Big(1+\frac{ud(t)}{2}\Big)\Delta D(s),\quad s\le t,\quad\forall u\in[0,1/d(t)).$$

But for $u\in[0,1/d(t))$ and $s\le t$

$$\log\Big(1+\frac{u^2}{2}\Big(1-\frac{ud(t)}{2}\Big)\Delta D(s)\Big)\ge\frac{u^2}{2}\Big(1-\frac{ud(t)}{2}\Big)\Delta D(s)-\frac{u^4}{8}(\Delta D(s))^2$$
$$\ge\frac{u^2}{2}(1-ud(t))\Delta D(s)\quad(\text{by }(2.19)).$$

Therefore

$$\exp\Big(\frac{u^2}{2}(1-ud(t))\Delta D(s)\Big)\le1+\int_{|x|\le d(t)}(e^{ux}-1-ux)\nu(\{s\},dx)$$
$$\le\exp\Big(\frac{u^2}{2}\Big(1+\frac{ud(t)}{2}\Big)\Delta D(s)\Big). \tag{2.21}$$

Combining (2.20), (2.21) and referring (2.17), (2.18) yield (2.16).

Proposition 2.4. *Let Y be a PII satisfying Assumption B, $m\in\boldsymbol{N}$, $\alpha_1,\cdots,\alpha_m\in\boldsymbol{R}$. Then*

$$\varlimsup_{t\to\infty}\frac{\sum_{j=1}^{m}\alpha_j[Y(jt)-Y((j-1)t)]}{\sqrt{2t\operatorname{llg}t}}=\sqrt{\sum_{j=1}^{m}\alpha_j^2},\qquad a.s. \tag{2.22}$$

Proof. Without loss of generality we may suppose $\sum\limits_{j=1}^{m} \alpha_j^2 = 1$. For given $\vartheta > m$ put $t_k = \vartheta^k$, $k \geq 1$ and

$$H^{(k)}(s) = \alpha_1 I(mt_{k-1} < s \leq t_k) + \sum_{j=2}^{m} \alpha_j I((j-1)t_k < s \leq jt_k),$$
$$U^{(k)} = \{U^{(k)}(s) = (H^{(k)} \cdot Y)_s,\ s \geq 0\}.$$

Then $U^{(k)}$ is a PII, $k = 1, 2, \cdots$, and

$$U^{(k)}(mt_k) = \sum_{j=2}^{m} \alpha_j [Y(jt_k) - Y((j-1)t_k)] + \alpha_1 [Y(t_k) - Y(mt_{k-1})],$$
$$\mathsf{E}U^{(k)}(s) = 0,$$
$$L^{(k)} \triangleq \mathsf{E}[U^{(k)}(mt_k)]^2 = \sum_{j=1}^{m} \alpha_j^2 [D(jt_k) - D((j-1)t_k)] - \alpha_1^2 D(mt_{k-1}),$$
$$\lim_{k\to\infty} \frac{L^{(k)}}{t_k} = 1 - \frac{m}{\vartheta}\alpha_1^2. \tag{2.23}$$

Meanwhile, for given $\eta > 0$,

$$\sup_s |\Delta U^{(k)}(s)| \leq \max_{1\leq j\leq m} |\alpha_j| \varepsilon(mt_k) \sqrt{D(mt_k)/\operatorname{llg} D(mt_k)}$$
$$\leq \varepsilon(mt_k)(1+\eta)\sqrt{mt_k/\operatorname{llg} t_k} \triangleq d^{(k)}, \qquad \text{while } t_k \text{ is large enough.}$$

Now applying (2.16) to $U^{(k)}(mt_k)$ gives

$$\exp\left[\frac{u^2}{2}\left(1 - u\frac{d^{(k)}}{\sqrt{L^{(k)}}}\right)\right] \leq \mathsf{E}\left[\exp\left(u\frac{U^{(k)}(mt_k)}{\sqrt{L^{(k)}}}\right)\right]$$
$$\leq \exp\left[\frac{u^2}{2}\left(1 + \frac{u}{2}\frac{d^{(k)}}{\sqrt{L^{(k)}}}\right)\right], \quad \forall u \in \left[0, \frac{\sqrt{L^{(k)}}}{d^{(k)}}\right).$$

Hence using Lemma 2.2 for given $0 < \gamma < 1/2$ yields

$$\mathsf{P}\left(\frac{U^{(k)}(mt_k)}{\sqrt{L^{(k)}}} > x\right) > \exp\left(-\frac{x^2}{2}(1+\gamma)\right), \qquad \forall x \in \left(\varepsilon_0, \eta_0\frac{\sqrt{L^{(k)}}}{d^{(k)}}\right). \tag{2.24}$$

For given $\eta > 0$ if t_k is large enough from (2.23) we have

$$\frac{\sqrt{L^{(k)}}}{d^{(k)}} \geq \frac{(1-2\eta)\sqrt{1 - m\alpha_1^2/\vartheta}}{\varepsilon(mt_k)\sqrt{m}}\sqrt{\operatorname{llg} t_k}.$$

Hence $\sqrt{2(1-2\gamma)\operatorname{llg} t_k} \in (\varepsilon_0, \eta_0\sqrt{L^{(k)}}/d^{(k)})$ for k large enough and (2.24) yields

$$\mathsf{P}\left(\frac{U^{(k)}(mt_k)}{\sqrt{L^{(k)}}} > \sqrt{2(1-2\gamma)\operatorname{llg} t_k}\right) > \exp[-(1+\gamma)(1-2\gamma)\operatorname{llg} t_k)]$$
$$> \frac{C}{(\log t_k)^{1-\gamma}} = \frac{C}{k^{1-\gamma}},$$

where C is a constant, but it may vary in different expressions. Recall that $\{U^{(k)}(mt_k),\ k \geq 1\}$ is an independent sequence of random variables, by the Borel-Cantelli lemma we have

$$\mathsf{P}\left(\frac{U^{(k)}(mt_k)}{\sqrt{L^{(k)}}} > \sqrt{2(1-2\gamma)\operatorname{llg} t_k} \quad \text{i. o.} \right) = 1.$$

Therefore

$$\overline{\lim_{t\to\infty}} \frac{U^{(k)}(mt_k)}{\sqrt{2t_k \operatorname{llg} t_k}} \geq (1-2\gamma) \overline{\lim_{t\to\infty}} \sqrt{\frac{L^{(k)}}{t_k}} = (1-2\gamma)\sqrt{1-\frac{m\alpha_1^2}{\vartheta}}.$$

Furthermore,

$$\overline{\lim_{t\to\infty}} \frac{\sum\limits_{j=1}^{m} \alpha_j [Y(jt)-Y((j-1)t)] - \alpha_1 Y(mt/\vartheta)}{\sqrt{2t \operatorname{llg} t}} \geq (1-2\gamma)\sqrt{1-\frac{m\alpha_1^2}{\vartheta}}.$$

But from (2.10) we have

$$\overline{\lim_{t\to\infty}} \frac{|Y(mt/\vartheta)|}{\sqrt{2t\operatorname{llg} t}} \leq \sqrt{\frac{m}{\vartheta}}.$$

Hence

$$\overline{\lim_{t\to\infty}} \frac{\sum\limits_{j=1}^{m} \alpha_j [Y(jt)-Y((j-1)t)]}{\sqrt{2t \operatorname{llg} t}} \geq (1-2\gamma)\sqrt{1-\frac{m\alpha_1^2}{\vartheta}} - \sqrt{\frac{m}{\vartheta}}.$$

Recall that $\gamma > 0$, $\vartheta > 1$ may be any given numbers, now letting $\vartheta \uparrow \infty$ and $\gamma \downarrow 0$ yields

$$\overline{\lim_{t\to\infty}} \frac{\sum\limits_{j=1}^{m} \alpha_j [Y(jt)-Y((j-1)t)]}{\sqrt{2t \operatorname{llg} t}} \geq 1.$$

Combining this and Proposition 2.2 gives (2.22).

From Propositions 2.1 and 2.4 it is easy to deduce the following corollary.

Corollary 2.1. *Let Y be a PII satisfying Assumption* B, $d > 0$. *Then*

$$\overline{\lim_{n\to\infty}} \frac{\sum\limits_{j=1}^{m} \alpha_j [Y(jn/d)-Y((j-1)n/d)]}{\sqrt{2n \operatorname{llg} n}} = \sqrt{\frac{1}{d}\sum_{j=1}^{m} \alpha_j^2}$$

$$\forall m \in \boldsymbol{N}, \alpha_1, \cdots, \alpha_m \in \boldsymbol{R} \qquad a.s. \tag{2.25}$$

Now we shall state some results about the compactness in $\mathcal{D}$.

For $f \in \mathcal{D}$ define a mapping $T^{(d)}$ as follows:

$$T^{(d)}f(t) = f^{(d)}(t) \triangleq f\Big(\frac{[td]}{d}\Big) + \Big[f\Big(\frac{[td]+1}{d}\Big) - f\Big(\frac{[td]}{d}\Big)\Big](td - [td]),$$
$$\mathcal{C}^{(d)} = \{f^{(d)} \colon f \in \mathcal{D}\}.$$

Then $T^{(d)}$ is a continuous mapping from $\mathcal{D}$ onto $\mathcal{C}^{(d)}$ and $\mathcal{C}^{(d)}$ is a closed subset of $\mathcal{C}$.

Let

$$\mathcal{K}^{(d)} = \mathcal{K}\mathcal{C}^{(d)} = \{g^{(d)} \colon g \in \mathcal{K}\} = \Big\{g \colon g \in \mathcal{C}^{(d)}, \sum_{i=1}^{\infty} d[g((i+1)/d) - g(i/d)]^2 \leq 1\Big\}.$$

Similar to the case of $C[0,1]$ (see, e.g., [2, Lemma 2.1]), $f \in \mathcal{K}$ iff $f^{(d)} \in \mathcal{K}^{(d)}$ for $d = 1, 2, \cdots$.

Let

$$\mathcal{H} = \Big\{x = (x_i, i \geq 1) \in \boldsymbol{R}^\infty \colon \lim_{k\to\infty} \frac{\sum\limits_{i=1}^{k} x_i}{k+1} = 0\Big\}, \qquad \|x\|_{\mathcal{H}} = \sup_k \frac{|\sum\limits_{i=1}^{k} x_i|}{k+1}, \qquad x \in \mathcal{H}.$$

Then $\mathcal{H}$ endowed with norm $\|\cdot\|_{\mathcal{H}}$ is also a Banach space. It is also easy to verify that the following mapping Q_d is a homeomorphism from $\mathcal{C}^{(d)}$ onto $\mathcal{H}$:

$$Q_d\colon \mathcal{C}^{(d)} \ni f \mapsto \Big(f\Big(\frac{i}{d}\Big) - f\Big(\frac{i-1}{d}\Big), i \geq 1\Big) \in \mathcal{H}.$$

The following results are well-known.

Lemma 2.3. *A subset B in $\mathcal{H}$ is relatively compact iff for each $k \geq 1$ $\{x_k : x \in B\}$ is bounded and*

$$\lim_{k\to\infty} \frac{1}{k+1} \sup_{x\in B} \Big|\sum_{j=1}^{k} x_j\Big| = 0.$$

Proposition 2.5. *Let $\{\vartheta^{(n)} = (\vartheta_k^{(n)}, k \geq 1), n \geq 1\}$ be a sequence in $\mathcal{H}$. If for each $k \geq 1$ and $\alpha_1, \cdots, \alpha_k \in \boldsymbol{R}$*

$$\overline{\lim_{n\to\infty}} \sum_{j=1}^{k} \alpha_j \vartheta_j^{(n)} = a\sqrt{\sum_{j=1}^{k} \alpha_j^2}, \qquad \lim_{k\to\infty} \frac{1}{k+1} \sup_{n} \Big|\sum_{j=1}^{k} \vartheta_j^{(n)}\Big| = 0,$$

then $\{\vartheta^{(n)}, n \geq 1\}$ is relatively compact in $\mathcal{H}$ and the set L of its limit points is

$$B(a) \triangleq \Big\{x \in \boldsymbol{R}^\infty\colon \sum_{j=1}^{\infty} x_i^2 \leq a^2\Big\}.$$

For a process $Y = \{Y(t), t \geq 0\}$ satisfying Assumption B, define

$$\eta_n(t) = \frac{Y(nt)}{\sqrt{2n \operatorname{llg} n}}, \qquad t \geq 0, \quad n \geq 1, \tag{2.27}$$

$$\eta_n^{(d)}(t) = \frac{Y(\frac{n[td]}{d}) + \big[Y(\frac{n[td]+n}{d}) - Y(\frac{n[td]}{d})\big](td - [td])}{\sqrt{2n \operatorname{llg} n}}, \quad t \geq 0, \ n \geq 1. \tag{2.28}$$

Proposition 2.6. *Let Y be a PII satisfying Assumption* B *and $\eta_n^{(d)}$ be defined by* (2.28)*. Then $\{\eta_n^{(d)}, n \geq 1\} \to\to \mathcal{K}^{(d)}$, a.s.*

Proof. For fixed $d > 0$, put

$$\vartheta_i^{(n)} = \eta_n^{(d)}\Big(\frac{i}{d}\Big) - \eta_n^{(d)}\Big(\frac{i-1}{d}\Big) = \frac{Y(ni/d) - Y(n(i-1)/d)}{\sqrt{2n \operatorname{llg} n}}.$$

From (2.25) we have

$$\overline{\lim_{n\to\infty}} \sum_{j=1}^{k} \alpha_j \vartheta_j^{(n)} = \overline{\lim_{n\to\infty}} \frac{\sum\limits_{j=1}^{k} \alpha_j [Y(nj/d) - Y(n(j-1)/d)]}{\sqrt{2n \operatorname{llg} n}} = \sqrt{\frac{1}{d} \sum_{j=1}^{k} \alpha_j^2},$$
$$\forall \alpha_1, \cdots, \alpha_k \in \boldsymbol{R}, \quad k \geq 1, \quad \text{a.s.} \tag{2.29}$$

Also from (2.25) we have

$$\overline{\lim_{n\to\infty}} \frac{|Y(n/d)|}{\sqrt{2n \operatorname{llg} n}} = \frac{1}{\sqrt{d}}, \qquad \text{a.s.}$$

Hence

$$\lim_{k\to\infty} \frac{1}{k+1} \sup_{n\geq 1} \Big|\sum_{j=1}^{k} \vartheta_j^{(n)}\Big| = \lim_{k\to\infty} \frac{1}{k+1} \sup_{n\geq 1} \frac{|Y(nk/d)|}{\sqrt{2n \operatorname{llg} n}}$$
$$\leq \lim_{k\to\infty} (1+k)^{-1/4} \sup_{n\geq 1} \frac{|Y(nk/d)|}{\sqrt{2nk \operatorname{llg}(nk)}} \leq \lim_{k\to\infty} (1+k)^{-1/4} \sup_{n\geq k} \frac{|Y(n/d)|}{\sqrt{2n \operatorname{llg}(n)}} = 0, \qquad \text{a.s.} \tag{2.30}$$

Now from (2.29), (2.30) and Proposition 2.5 we get that $\{\vartheta^{(n)}, n \geq 1\}$ is relatively compact in $\mathcal{H}$ with probability 1 and the set of its limit points is $B(1/\sqrt{d})$ almost surely. This also means that $\{\eta_n^{(d)},\, n \geq 1\}$ is relatively compact in $\mathcal{C}^{(d)}$ with probability 1 and the set of its limit points is

$$\Big\{f \in \mathcal{C}^{(d)} : \sum_{i=1}^{\infty}\Big[f\Big(\frac{i}{d}\Big) - f\Big(\frac{i-1}{d}\Big)\Big]^2 \leq \frac{1}{d}\Big\} = \Big\{f \in \mathcal{C}^{(d)} : \int_0^{\infty} [f'(s)]^2 ds \leq 1\Big\} = \mathcal{K}^{(d)}.$$

Now we are ready to prove the Strassen law of the iterated logarithm for Y.

Proposition 2.7. *Let Y be a PII satisfying Assumption* B *and η_n be defined by* (2.27). *Then $\{\eta_n,\, n \geq 1\} \to\to \mathcal{K}$, a.s.*

Proof. We adhere to the notations of the last Proposition. Similar to (2.30) we also have

$$\lim_{t\to\infty} \frac{1}{1+t} \sup_{n\geq 1} \frac{|Y(nt)|}{\sqrt{2n \operatorname{llg} n}} = 0, \qquad \lim_{t\to\infty} \frac{1}{1+t} \sup_{n\geq 1} \frac{|Y(n[td]/d)|}{\sqrt{2n \operatorname{llg} n}} = 0.$$

Hence for $\delta > 0$ there exists a finite number $T > 0$ such that

$$\sup_{t\geq T} \frac{1}{1+t} \sup_{n\geq 1} \frac{|Y(nt)|}{\sqrt{2n \operatorname{llg} n}} < \frac{\delta}{3}, \qquad \sup_{t\geq T} \frac{1}{1+t} \sup_{n\geq 1} \frac{|Y(n[td]/d)|}{\sqrt{2n \operatorname{llg} n}} < \frac{\delta}{3}.$$

Now we have

$$\begin{aligned}\overline{\lim_{n\to\infty}} \|\eta_n - \eta_n^{(d)}\|_{\mathcal{D}} &\leq \frac{2}{3}\delta + \overline{\lim_{n\to\infty}} \sup_{k\leq Td} \sup_{k/d\leq t\leq (k+1)/d} \frac{|Y(nt) - Y(nk/d)|}{\sqrt{2n \operatorname{llg} n}} \\ &\leq \frac{2}{3}\delta + \overline{\lim_{n\to\infty}} \frac{\omega(n/d, n(T+1/d), Y)}{\sqrt{2n \operatorname{llg} n}} \leq \frac{2}{3}\delta + 3\sqrt{\frac{1}{d}}.\end{aligned}$$

Letting $\delta \downarrow 0$ yields

$$\overline{\lim_{n\to\infty}} \|\eta_n - \eta_n^{(d)}\|_{\mathcal{D}} \leq 3\sqrt{1/d}. \tag{2.31}$$

For $\delta > 0$, take $d > 9/\delta^2$. From Proposition 2.6 $\{\eta_n^{(d)},\, n \geq 1\}$ is relatively compact. Therefore there exists a finite δ-net for $\{\eta_n^{(d)},\, n \geq 1\}$ and from (2.31) there exists also a finite 2δ-net for $\{\eta_n,\, n \geq 1\}$. Since $\delta > 0$ may be any positive number, $\{\eta_n,\, n \geq 1\}$ is relatively compact in $\mathcal{D}$ almost surely.

Denote by $L(\omega)$ the set of all limit points of $\{\eta_n,\, n \geq 1\}$. Then $L(\omega)$ is a closed set. From the continuity of the mapping $T^{(d)}$, $L^{(d)} = T^{(d)}L$ is just the set of limit points of $\{\eta_n^{(d)} = T^{(d)}\eta_n,\, n \geq 1\}$. Therefore Proposition 2.6 shows $L^{(d)} = \mathcal{K}^{(d)}$, for all $d = 1, 2, \cdots$. This means that if $f \in L$, then $f^{(d)} \in \mathcal{K}^{(d)}$, i.e.,

$$f(0) = 0, \quad \sum_{i=1}^{\infty} d\Big[f\Big(\frac{i}{d}\Big) - f\Big(\frac{i-1}{d}\Big)\Big]^2 \leq 1, \qquad d = 1, 2, \cdots.$$

Hence $f \in \mathcal{K}$ and $L \subset \mathcal{K}$ a.s.

On the other hand, (2.31) means $\mathcal{K}^{(d)} \subset L^{3\sqrt{1/d}} = \Big\{y : \inf\limits_{x\in L} \|y - x\| \leq 3\sqrt{1/d}\Big\}$. Therefore $\mathcal{K} \subset (\mathcal{K}^{(d)})^{\sqrt{1/d}} \subset L^{4\sqrt{1/d}}$, a.s. and $\mathcal{K} \subset \bigcap\limits_{d=1}^{\infty} L^{4\sqrt{1/d}} = L$, a.s. Hence the conclusion $L = \mathcal{K}$ holds.

§3. The General Case

In this section we adhere to the notations of last section and consider the PII $X = \{X(t),\, t \geq 0\}$ satisfying Assumption A.

The following lemma (cf. [14]) is important in the use of truncation technique.

Lemma 3.1. *Let G be a finite measure on R with $\int y^2G(dy) < \infty$. Then there exists a function $\varepsilon(t)$ satisfying*

$$\varepsilon(t) \downarrow 0, \qquad 1 \le \varepsilon(t)\sqrt{t/\operatorname{llg} t} \to \infty, \qquad \text{as } t \to \infty \tag{3.1}$$

and

$$\int_0^\infty \frac{1}{\sqrt{2t \operatorname{llg} t}} \int_{\boldsymbol{R}} |y| I\Big(|y| > \varepsilon(t)\sqrt{\frac{t}{\operatorname{llg} t}}\Big) G(dy)dt < \infty. \tag{3.2}$$

Note that if X satisfies Assumption A, then X is also a locally square integrable martingale and has the following integral representation: $X = X^c + x * (\mu - \nu)$, where X^c is the continuous local martingale part of X, μ is the jump measure of X and $\nu = \mathsf{E}\mu$ is the dual predictable projection of μ. Meanwhile, X^c is also a continuous PII and

$$\mathsf{E}[X(t)]^2 = \mathsf{E}[X^c(t)]^2 + (x^2 * \nu)_t. \tag{3.3}$$

Now we shall use the truncation technique, i.e., use the following decomposition of X:

$$\begin{aligned} X &= X^c + x * (\mu - \nu) \\ &= [X^c + xI(|x| \le d(t)) * (\mu - \nu)] + xI(|x| > d(t)) * (\mu - \nu) \triangleq Y + Z, \end{aligned} \tag{3.4}$$

$$Y \triangleq X^c + xI(|x| \le d(t)) * (\mu - \nu), \tag{3.5}$$

$$Z \triangleq xI(|x| > d(t)) * (\mu - \nu), \tag{3.6}$$

where $d(t) = \varepsilon(t)\sqrt{t/\operatorname{llg} t}$ and $\varepsilon(t)$ is given by Lemma 3.1. Then both Y, Z are PII.

Proposition 3.1. *Suppose that $X = \{X(t),\, t \ge 0\}$ satisfies Assumption A and $Y = \{Y(t),\, t \ge 0\}$ is defined by* (3.4). *Then Y satisfies Assumption* B.

Proof. From the integral representation (3.5), the following facts are evident:

$$\mathsf{E}[Y(t)] = 0, \qquad \sup_{s \le t} |\Delta Y(s)| \le 2d(t) = 2\varepsilon(t)\sqrt{t/\operatorname{llg} t}, \qquad \forall t \quad \text{a.s.}$$

To this end it suffices to prove that $\lim\limits_{t\to\infty} \mathsf{E}[Y(t)]^2/t = 1$. From the definition of Y we have

$$\begin{aligned} \mathsf{E}[Y(t)]^2 &= \mathsf{E}[X^c(t)]^2 + \big(|x|^2 I(|x| \le d(\cdot))\big) * \nu_t - \sum_{s \le t} \big(\Delta[(xI(|x| \le d(\cdot))) * \nu](s)\big)^2 \\ &\le \mathsf{E}[X(t)]^2, \end{aligned}$$

where $\sum W$ denotes the summation process of a thin process W. Meanwhile,

$$\begin{aligned} 0 &\le \mathsf{E}[X(t)]^2 - \mathsf{E}[Y(t)]^2 \\ &= (|x|^2 I(|x| > d(\cdot))) * \nu_t + \sum_{s \le t} \big(\Delta[(xI(|x| \le d(\cdot))) * \nu](t)\big)^2 \\ &= (|x|^2 I(|x| > d(\cdot))) * \nu_t + \sum_{s \le t} \big(\Delta[(xI(|x| > d(\cdot))) * \nu](t)\big)^2 \quad (\text{by } \Delta[x * \nu] = 0) \\ &\le 2(|x|^2 I(|x| > d(\cdot))) * \nu_t \\ &= 2\int_0^t \int_{\boldsymbol{R}} |x|^2 I(|x| > d(s))) N_s(dx) D_X(ds) \qquad (\text{by } (1.2)) \\ &\le 2\int_0^t \int_{\boldsymbol{R}} |x|^2 I(|x| > d(s)) G(dx) D_X(ds) \qquad (\text{by } (1.4)). \end{aligned}$$

Owing to $\int y^2 G(dy) < \infty$ and (3.1), we have

$$\lim_{t\to\infty} \int_{\boldsymbol{R}} x^2 I(|x| > d(t)) G(dx) = \lim_{t\to\infty} \int_{\boldsymbol{R}} x^2 I(|x| > \varepsilon(t)\sqrt{t/\operatorname{llg} t}\,) G(dx) = 0.$$

Hence

$$\lim_{t\to\infty}\frac{|\mathsf{E}[X(t)]^2-\mathsf{E}[Y(t)]^2|}{D_X(t)}\le\lim_{t\to\infty}\frac{1}{D_X(t)}\int_0^t\int_{\boldsymbol{R}}x^2I\Big(|x|>\varepsilon(s)\sqrt{\frac{s}{\operatorname{llg}s}}\Big)G(dx)D_X(ds)=0.$$

Therefore according to (1.5) we get $\lim\limits_{t\to\infty}\mathsf{E}[Y(t)]^2/t=1$.

Proposition 3.2. *Suppose that X satisfies Assumption A, Z is defined by* (3.6) *and*

$$\zeta_n(t)=\frac{Z(nt)}{\sqrt{2n\operatorname{llg}n}},\qquad t\ge0,\quad n\ge1. \tag{3.7}$$

Then

$$\lim_{t\to\infty}\frac{Z(t)}{\sqrt{2t\operatorname{llg}t}}=0,\qquad\lim_{n\to\infty}\|\zeta_n\|_{\mathcal{D}}=0,\qquad\lim_{t\to\infty}\frac{\omega(t,t,Z)}{\sqrt{2t\operatorname{llg}t}}=0. \tag{3.8}$$

Proof. Firstly, note that Z is a locally square integrable martingale. For the $\varphi(t)=\sqrt{2t\operatorname{llg}t}$ from (3.2) we have

$$\begin{aligned}\Big(\frac{|x|}{\varphi(\cdot)}I(|x|>d(\cdot))*\nu\Big)_\infty&=\int_0^\infty\frac{1}{\varphi(t)}\int_{\boldsymbol{R}}|x|I(|x|>d(t))N_t(dx)D_X(dt)\\&\le\int_0^\infty\frac{1}{\varphi(t)}\int_{\boldsymbol{R}}|x|I(|x|>d(t))G(dx)D_X(dt)<\infty.\end{aligned}\tag{3.9}$$

But

$$\mathsf{E}\Big[\Big(\frac{|x|}{\varphi(\cdot)}I(|x|>d(\cdot))*\mu\Big)_t\Big]=\Big(\frac{|x|}{\varphi(\cdot)}I(|x|>d(\cdot))*\nu\Big)_t.$$

Hence from (3.9) the following limits exist and are finite:

$$\lim_{t\to\infty}\Big[\frac{|x|}{\varphi(\cdot)}I(|x|>d(\cdot))*\nu\Big]_t,\qquad\lim_{t\to\infty}\Big[\frac{|x|}{\varphi(\cdot)}I(|x|>d(\cdot))*\mu\Big]_t,$$

$$\lim_{t\to\infty}\Big[\frac{1}{\varphi(\cdot)}\cdot Z\Big]_t=\lim_{t\to\infty}\Big[\frac{x}{\varphi(\cdot)}I(|x|>d(\cdot))*\nu\Big]_t-\lim_{t\to\infty}\Big[\frac{x}{\varphi(\cdot)}I(|x|>d(\cdot))*\mu\Big]_t.$$

Now the Kronecker lemma implies

$$\lim_{t\to\infty}\frac{Z(t)}{\sqrt{2t\operatorname{llg}t}}=\lim_{t\to\infty}\frac{Z(t)}{\varphi(t)}=0,\qquad\text{a.s.}$$

i.e., the first equality of (3.8) holds and it contains that for each fixed $a>0$

$$\lim_{n\to\infty}\frac{\sup\limits_{0\le t\le a}|Z(nt)|}{\sqrt{2n\operatorname{llg}n}}=0,\qquad\text{a.s.} \tag{3.10}$$

Meanwhile,

$$\|\zeta_n\|_{\mathcal{D}}\le\frac{1}{\sqrt{2n\operatorname{llg}n}}\sup_{0\le t\le a}|Z(nt)|+\frac{1}{\sqrt{2n\operatorname{llg}n}}\sup_{t>a}\frac{|Z(nt)|}{1+t}.$$

This and (3.10) contain that

$$\varlimsup_{n\to\infty}\|\zeta_n\|_{\mathcal{D}}\le\varlimsup_{a\to\infty}\varlimsup_{n\to\infty}\frac{1}{\sqrt{2n\operatorname{llg}n}}\sup_{t\ge a}\frac{|Z(nt)|}{1+t}. \tag{3.11}$$

But for each $\varepsilon>0$ there is an integer N such that

$$|Z(nt)|\le\varepsilon\sqrt{2nt\operatorname{llg}(nt)}\le\varepsilon\sqrt{2n\operatorname{llg}n}\,t^{3/4},\qquad\forall n\ge N,\quad t\ge a\vee e^e.$$

This and (3.11) imply the second equality in (3.8).

At last, the third equality of (3.8) comes from $\omega(t,t,Z)\le2\sup\limits_{s\le t}|Z(s)|$.

From Proposition 3.2 we also have the following Corollary.

Corollary 3.1. *If X satisfies Assumption* A, *then for $\alpha \in [0,1]$*

$$\overline{\lim_{n\to\infty}} \frac{\omega(\alpha t, t, X)}{\sqrt{2t \operatorname{llg} t}} \le 3\sqrt{\alpha}, \qquad a.s. \tag{3.12}$$

Proof. Note that $\omega(\delta,t,X) \le \omega(\delta,t,Y) + \omega(\delta,t,Z)$, hence (3.12) comes from (2.7), (3.8).

Now we are ready to prove Main Theorem.

Proof of Main Theorem. According to (3.4),

$$X(t) = Y(t) + Z(t), \quad \xi_n(t) = \eta_n(t) + \zeta_n(t),$$

where $\eta_n(t)$, $\zeta_n(t)$ are given by (2.28), (3.7) respectively. Now the conclusion comes from Proposition 2.7 and (3.8).

§4. Some Corollaries

The Main Theorem is a rather general result, it comprises some important special cases, here we shall list some corollaries of it.

Let $\mathcal{C}_1 = \{f\colon f \text{ is continuous on } [0,1],\ f(0)=0\}$,

$$\mathcal{D}_1 = \{f\colon f \text{ is right continuous and with finite left limit on } [0,1],\ f(0)=0,\},$$

$$\mathcal{K}_1 = \Big\{f\colon f \text{ is absolutely continuous on } [0,1] \text{ and } f(0)=0, \int_0^1 [f'(t)]^2 dt \le 1\Big\}.$$

Then $\mathcal{C}_1$, $\mathcal{D}_1$ endowed with the uniform norm are Banach spaces, $\mathcal{K}_1$ is a compact subset of $\mathcal{C}_1$.

For $f \in \mathcal{D}$ define $(Pf)(t) = f(t)$, $0 \le t \le 1$. Then P is a continuous mapping from $\mathcal{D}$ onto $\mathcal{D}_1$ and $P\mathcal{K} \triangleq \{Pf\colon f \in \mathcal{K}\} = \mathcal{K}_1$. Therefore we have the following corollary.

Corollary 4.1. *let $X = \{X(t),\, t \ge 0\}$ be a PII satisfying Assumption* A *and*

$$\xi_n(t) = \frac{X(nt)}{\sqrt{2n \operatorname{llg} n}}, \qquad 0 \le t \le 1, \quad n \ge 1. \tag{4.1}$$

Then $\xi_n \to\to \mathcal{K}_1$ in $\mathcal{D}_1$, a.s.

Corollary 4.2. *Let $X = \{X(t),\, t \ge 0\}$ be a temporally homogeneous PII with $\mathsf{E}[X(1)] = 0$, $\mathsf{E}[X(1)]^2 = \sigma^2$, and ξ_n is defined by* (1.10) (*or* (4.1) *resp.*). *Then*

$$\frac{\xi_n}{\sigma} \to\to \mathcal{K} \qquad \text{in } \mathcal{D} \qquad \Big(\frac{\xi_n}{\sigma} \to\to \mathcal{K}_1 \qquad \text{in } \mathcal{D}_1 \text{ resp.}\Big) \qquad a.s. \tag{4.2}$$

Proof. Note that $\mathsf{E}[X(t)] = 0$, $D_X(t) = \mathsf{E}[X(t)]^2 = \sigma^2 t$, $\nu(dt,dx) = N(dx)dt$ and $\int x^2 N(dx) = \sigma^2 < \infty$. Therefore it is evident that $\{X(t)/\sigma, t \ge 0\}$ satisfies Assumption A and (4.2) comes from (1.11) and Corollary 4.1.

Corollary 4.3. *Let $\{V_n,\, n \ge 1\}$ be a sequence of independent random variables with $\mathsf{E}[V_n] = 0$, $\sigma_n^2 = \mathsf{E}[V_n]^2 < \infty$ and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \sigma_j^2 = 1. \tag{4.3}$$

Let F_n be the distribution of V_n. Suppose $\{\sigma_n^{-2} F_n\} \prec G$ with $\int x^2 G(dx) < \infty$ or

$$\sup_n \sigma_n^{-2} \mathsf{E}[V_n^{2+\delta}] < \infty, \qquad \text{for some } \delta > 0. \tag{4.4}$$

Let

$$\xi_n(t) = \frac{\sum_{j=1}^{[nt]} V_j}{\sqrt{2n \operatorname{llg} n}}, \qquad \overline{\xi}_n(t) = \frac{\sum_{j=1}^{[nt]} V_j + (nt - [nt]) V_{[nt]+1}}{\sqrt{2n \operatorname{llg} n}}, \qquad t \geq 0, \quad n \geq 1. \tag{4.5}$$

Then

$$\xi_n \to\to \mathcal{K} \quad in\ \mathcal{D}, \qquad \overline{\xi}_n \to\to \mathcal{K} \quad in\ \mathcal{C} \qquad a.s. \tag{4.6}$$

Proof. Write $X(t) = \sum_{j=1}^{[t]} V_j$. Then $\{X(t),\, t \geq 0\}$ is a PII with $\mathsf{E}[X(t)] = 0$, $D_X(t) = \mathsf{E}[X(t)]^2 = \sum_{j=1}^{[t]} \sigma_j^2$. From (4.3) $\lim\limits_{t\to\infty} \frac{D_X(t)}{t} = \lim\limits_{t\to\infty} \frac{1}{t} \sum_{j=1}^{[t]} \sigma_j^2 = 1$. In this case, the jump measure μ of X is $\mu(dt, dx) = \sum_{j=1}^{\infty} \varepsilon_{\{j\}}(dt) \varepsilon_{\{V_j\}}(dx)$, where $\varepsilon_{\{a\}}$ is a unit measure concentrate in $\{a\}$. The dual predictable projection ν of μ is

$$\nu(dt, dx) = \sum_{j=1}^{\infty} \varepsilon_j(dt) F_{V_j}(dx) = \sigma_{[t]}^{-2} F_{[t]}(dx) dD_X(t). \tag{4.7}$$

Thus $\{N_t(\,\cdot\,)\} = \{\sigma_{[t]}^{-2} F_{[t]}(\,\cdot\,)\} \prec G$. If (4.4) holds, then from Lemma 1.1 $\{N_t\} \prec G$ holds too. Therefore X satisfies Assumption A and (4.6) comes from Main Theorem.

Meanwhile, similar to (2.38) it is easy to show $\lim\limits_{n\to\infty} \|\xi_n - \overline{\xi}_n\|_{\mathcal{D}} = 0$. Hence $\overline{\xi}_n \to\to \mathcal{K}$ in $\mathcal{D}$ a.s. Note that $\overline{\xi}_n \in \mathcal{C}$, therefore $\overline{\xi}_n \to\to \mathcal{K}$ in $\mathcal{C}$ a.s. too.

In particulr, from Corollary 4.3 we have

Corollary 4.4. *Let $\{V_n,\, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathsf{E}[V_n] = 0$, $\mathsf{E}[V_n^2] = \sigma^2 < \infty$. Suppose that $\xi_n(t)$, $\overline{\xi}_n(t)$ are defined by (4.5). Then*

$$\frac{\xi_n}{\sigma} \to\to \mathcal{K}, \quad in\ \mathcal{D}, \qquad \frac{\overline{\xi}_n}{\sigma} \to\to \mathcal{K}, \quad in\ \mathcal{C}, \qquad a.s.$$

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