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THE PERTURBATION THEORY FOR THE SUMMABILITY OF SELFADJOINT OPERATORS**

C. W. Leung*

Abstract

This paper introduces the new notion of $(p+0)$-summable operator. It is shown that this property is stable under small perturbation by selfadjoint operators.

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§0. Introduction

Let H be a complex Hilbert space and A be a complex unital Banach algebra acting on H. And let D be a densely defined selfadjoint operator H. A pair (H, D) is said to be an unbounded p-summable Fredholm module (ungraded) over A if it satisfies the following conditions:

(α) $\operatorname{tr}(1+D^2)^{-\frac{p}{2}} < \infty$;

(β) For any element a in A, the operator $[D, a]$ extends to a bounded operator on H.

We may replace the p-summability condition of (α) by the weaker, θ-summability condition;

(α') $\operatorname{tr} e^{-tD^2} < \infty, \quad \forall t > 0$.

Such a Fredholm module plays an important role in non-commutative geometry. In [2], E. Getzler and A. Szenes have shown that the θ-summability property is stable under perturbation by bounded selfadjoint operators. And in [1], A. Connes and H. Moscovici have obtained the similar result in finite summable case.

In this paper, we consider the pairing (H, D), where H is a complex Hilbert space and D is a densely defined selfadjoint operator on H. We introduce the new notion of $(p+0)$-summability. It is easy to see that each pair (H, D) which satisfies the condition (α) in above is $(p+0)$-summable. We show that every $(p+0)$-summable is stable under small perturbation by a selfadjoint operator which satisfies some certain conditions. For the θ-summable case, we also obtain a similar result.

Throughout this paper, let H be a complex Hilbert space and $\mathcal{L}^1$ be the family of all trace class operators on H.

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*Department of Mathematics, The Chinese University of Hong Kong, Shatin. N. T. Hong Kong, China.
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§1. Lemmas

The following lemma is slightly modified to the Connes and Moscovici's result[1].

Lemma 1.1. *Let S be a selfadjoint operator on H. Suppose that there exists $\lambda > 0$ such that* $\operatorname{tr}(\lambda + S^2)^{-\frac{r}{2}} < \infty$ *for some $r \geq 1$. Then we have*

$$\operatorname{tr} e^{-t^2S^2} \leq C \cdot e^{t^2\lambda^2} t^{-r} \quad \textit{for any } t > 0,$$

where the constant C does not depend on t. Thus we have $\operatorname{tr} e^{-t^2S^2} = O(t^{-r})$ *as $t \to 0^+$.*

Proof. By the Hölder's inequality for operators, we see that we have

$$\begin{aligned}\operatorname{tr} e^{-t^2S^2} &= \operatorname{tr}\{(\lambda + S^2)^{-\frac{r}{2}} e^{-t^2S^2} (\lambda + S^2)^{\frac{r}{2}}\} \\ &\leq \|e^{-t^2S^2}(\lambda + S^2)^{\frac{r}{2}}\| \operatorname{tr}(\lambda + S^2)^{-\frac{r}{2}}, \quad \text{for any} \quad t > 0.\end{aligned}$$

Note that we have

$$\begin{aligned}\|e^{-t^2S^2}(\lambda + S^2)^{\frac{r}{2}}\| &\leq \|e^{-t^2x^2}(\lambda + x^2)^{\frac{r}{2}}\|_\infty \\ &= \sup_{y \geq \sqrt{\lambda}} e^{-t^2(y^2-\lambda)}(y^2)^{\frac{r}{2}} \quad \text{where} \quad y^2 = \lambda + x^2 \\ &\leq \text{const.} e^{t^2\lambda^2} \sup_{y \geq \sqrt{\lambda}} \frac{y^r}{(t^2y^2)^{\frac{r}{2}}}.\end{aligned}$$

Then the lemma follows.

We are now going to prove the following critical lemma in this paper.

Lemma 1.2. *Let S be a selfadjoint operator on H. Let $1 \leq p < \infty$. Suppose that there exists $\lambda > 0$ such that* $\operatorname{tr}(\lambda + S^2)^{-\frac{p+\varepsilon}{2}} < \infty$, *for any $\varepsilon > 0$. Then we have* $\operatorname{tr}(\mu + S^2)^{-\frac{p+\varepsilon}{2}} < \infty$, *for any $\mu > 0$ and for any $\varepsilon > 0$.*

Proof. Let ε and μ be arbitrary positive numbers. Recall that the Mellin transform of $e^{-t^2(\mu+S^2)}$ is formally given by

$$\int_0^\infty e^{-t^2(\mu+S^2)} t^{p+\varepsilon-1} dt = \text{const.}(\mu + S^2)^{-\frac{p+\varepsilon}{2}},$$

where the constant depends on p and ε only.

We only need to show that the above integral is Bochner integrable in $\mathcal{L}^1$ (see [5]).

We see that we have

$$\int_1^\infty \|e^{-t^2(\mu+S^2)} t^{p+\varepsilon-1}\|_1 dt \leq \|e^{-S^2}\| \int_1^\infty e^{-t^2\mu} t^{p+\varepsilon-1} dt < \infty.$$

On the other hand, by Lemma 1.1 and the assumption, we have

$$\|e^{-t^2(\mu+S^2)}\|_1 \leq \text{const.} e^{-t^2(\mu-\lambda)} t^{-(p+\eta)}$$

for any $\eta > 0$ and any $t > 0$.

Then we have

$$\int_0^1 \|e^{-t^2(\mu+S^2)} t^{p+\varepsilon-1}\|_1 dt \leq \text{const.} \int_0^1 e^{-t^2(\mu-\lambda)} t^{-(p+\eta)} t^{p+\varepsilon-1} dt. \qquad (*)$$

Thus if we take $0 < \eta < \varepsilon$, then $(*)$ is integrable. Thus we have

$$\int_0^1 \|e^{-t^2(\mu+S^2)} t^{p+\varepsilon-1}\|_1 dt < \infty \quad \text{for any } \varepsilon > 0.$$

On the other hand, by the Hölder's inequality for operators, we see that the function $t \in (0, \infty) \to e^{-t^2(\mu+S^2)} \in \mathcal{L}^1$ is continuous. Thus the integral $\int_0^1 e^{-t^2(\mu+S^2)} t^{p+\varepsilon-1} dt$ is Bochner integrable in $\mathcal{L}^1$. Then the proof is completed.

By the above results, we are naturally led to have the following definition.

Definition 1.1. *A selfadjoint operator D on H is said to be $(p+0)$-summable, where $1 \le p < \infty$, if for any $\varepsilon > 0$, we have*

$$\operatorname{tr}(1+D^2)^{-\frac{p+\varepsilon}{2}} < \infty.$$

Remark. Obviously, every unbounded p-summable operator (see [1]) is $(p+0)$-summable.

§2. Main Results

We first recall the definition of relative boundedness between two selfadjoint operators on H (see [3]). Let A and B be two densely defined operators on H. B is said to be A-bounded if the following conditions hold:

(i) $\mathrm{Dom}(A) \subset \mathrm{Dom}(B)$ and

(ii) there exist a, b in $\mathbb{R}$ such that

$$\|B\xi\|^2 \le a^2\|A\xi\|^2 + b^2\|\xi\|^2, \quad \text{for any } \xi \in \mathrm{Dom}(A).$$

The greatest lower bound of $|a|$ of all possible constants a in the condition (ii) is called the relative bound of A.

On the other hand, we have to make use of the following fact: If A and B both are the positive selfadjoint operators on H, then we have $\operatorname{tr} e^{-A-B} \le \operatorname{tr} e^{-B}$ (see [4, Section 8]).

We are now in a position to prove our main result.

Theorem 2.1. *Let D be a $(p+0)$-summable operator on H. Let V be a D bounded selfadjoint operator on H with D-bound less than 1. Suppose that there exists $\delta > 0$ such that $(D+V)^2 - sD^2$ is selfadjoint, for any $0 < s < \delta$. Then $D+V$ is also $(p+0)$-summable.*

Proof. Since V has D-bound less than 1, by the Kato-Rellich Theorem (see [3, Theorem X. 12]), $D+V$ is also a selfadjoint operator. And there exists a constant a with $0 < a < 1$ and a constant b so that $\|V\xi\|^2 \le a^2\|D\xi\|^2 + b^2\|\xi\|^2$, for any $\xi \in \mathrm{Dom}D$. Now we choose $\beta > 0$ and $\gamma > 0$ such that $a^2 < \frac{\beta}{\gamma} < 1$ and $0 < 1 - \frac{\beta}{\gamma} + a^2(1-\frac{\gamma}{\beta}) < \delta$.

Let α be an arbitrary positive real number. Then we have

$$\begin{aligned}((1+\alpha+(D+V)^2)\xi,\xi) &\ge \Big(\Big(1+\alpha+D^2+V^2-\frac{\beta}{\gamma}D^2-\frac{\gamma}{\beta}V^2\Big)\xi,\xi\Big)\\ &= \Big(\Big(1+\alpha+\Big(1-\frac{\beta}{\gamma}\Big)D^2+\Big(1-\frac{\gamma}{\beta}\Big)V^2\Big)\xi,\xi\Big)\\ &\ge \Big(\Big(1+\alpha+\Big(1-\frac{\beta}{\gamma}\Big)D^2+\Big(1-\frac{\gamma}{\beta}\Big)(a^2D^2+b^2)\Big)\xi,\xi\Big)\\ &= \Big(\Big(1+\alpha+\Big(1-\frac{\gamma}{\beta}\Big)b^2+\Big(1-\frac{\beta}{\gamma}+a^2\Big(1-\frac{\gamma}{\beta}\Big)\Big)D^2\Big)\xi,\xi\Big)\end{aligned}$$

for any $\xi \in \mathrm{Dom}D^2 \cap \mathrm{Dom}(D+V)^2$.

Since $0 \le 1 - \frac{\beta}{\gamma} + a^2(1-\frac{\gamma}{\beta}) < \delta$, we have $0 \le \frac{\beta}{\gamma} - a^2(1-\frac{\gamma}{\beta}) < 1$. And then we choose α large enough so that we have $\alpha + (1-\frac{\gamma}{\beta})b^2 \ge 0$.

Now let $e = 1+\alpha+(1-\frac{\gamma}{\beta})b^2$ and $f = 1-\frac{\beta}{\gamma}+a^2(1-\frac{\gamma}{\beta})$.

Thus we have $1+\alpha+(D+V)^2 - e - fD^2$ is a positive selfadjoint operator.

Then by Lemma 1.1, for any $\eta > 0$, we have

$$\operatorname{tr} e^{-t^2(1+\alpha+(D+V)^2)} \le \operatorname{tr} e^{-t^2(e+fD^2)} = O(t^{-(p+\eta)}), \quad \text{as } \ t \to 0^+.$$

Thus the integral $\int_0^\infty e^{-t^2(1+\alpha+(D+V)^2)} t^{p+\varepsilon-1} dt$ is integrable in $\mathcal{L}^1$, for any $\varepsilon > 0$. Then by the proof of Lemma 1.2, we see that

$$\operatorname{tr}(1+\alpha+(D+V)^2)^{-\frac{p+\varepsilon}{2}} < \infty, \quad \text{for any } \ \varepsilon > 0.$$

Thus by Lemma 1.2 again, we have

$$\operatorname{tr}(1+(D+V)^2)^{-\frac{p+\varepsilon}{2}} < \infty, \quad \text{for any } \ \varepsilon > 0.$$

The proof is finished.

By the above theorem, we can now obtain the following result immediately.

Corollary 2.1. *Let D be a $(p+0)$-summable operator. Let V be a selfadjoint bounded operator. Suppose that there exists $\delta > 0$ such that $(D+V)^2 - sD^2$ is selfadjoint, for any $0 < s < \delta$. Then $D+V$ is also $(p+0)$-summable.*

§3. Remarks

In the study of entire cyclic cohomology[2], we are naturally led to have the following definition which is a generalization of the finite summability of an operator.

Definition 3.1. *A selfadjoint operator D on H is said to be θ-summable if for any $t > 0$, we have*

$$\operatorname{tr} e^{-t^2 D^2} < \infty.$$

By Lemma 1.1, we can obtain the following proposition immediately.

Proposition 3.1. *Every $(p+0)$-summable operator D is θ-summable, for any $1 \le p < \infty$.*

Analogous to the finite summable case, we can also obtain the following stability theorem for θ-summable operators.

Theorem 3.1. *Let D be a θ-summable operator on H. Let V be a D-bounded selfadjoint operator with D-bound less than 1. Suppose that there exists $\delta > 0$ such that $(D+V)^2 - sD^2$ is selfadjoint, for any $0 < s < \delta$. Then $D+V$ is also θ-summable.*

Proof. By the proof of (2.1), we have

$$1 + \alpha + (D+V)^2 \ge e + fD^2,$$

for some suitable positive numbers α, e, f. Then the result follows.

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