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GIRSANOV'S THEOREM ON ABSTRACT WIENER SPACES**

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Abstract

Let (E, H, μ) be an abstract Wiener space in the sense of L. Gross. It is proved that if u is a measurable map from E to H such that $u \in W^{2,1}(H, \mu)$ and there exists a constant α, $0 < \alpha < 1$, such that either $\sum\limits_n \|D_n u(w)\|_H^2 \le \alpha^2$ a.s. or $\|u(w+h) - u(w)\|_H \le \alpha \|h\|_H$ a. s. for every $h \in H$ and $E\Big(\exp\Big(\frac{108}{(1-\alpha)^2}\Big(\sum \|D_n u\|_H\Big)\Big)\Big) < \infty$, then the measure $\mu \circ T^{-1}$ is equivalent to μ, where $T(w) = w + u(w)$ for $w \in E$. And the explicit expression of the Radon-Nikodym derivative (cf. Theorem 2.1) is given.

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§0. Introduction

Let (E, H, μ) be an abstract Wiener space in the sense of L. Gross, i.e., $(E, \|\cdot\|_E)$ is a separable Banach space, $(H, \|\cdot\|_H)$ is a separable Hilbert space with the inner product $(\cdot,\cdot)_H$, H is a dense subspace of E and the inclusion map is continuous, and μ is the probability measure on $(E, \mathfrak{B}_E)$ such that for $f \in E^*$,

$$\int_E \exp(i\, f(w))\ \mu(dw) = \exp(-\frac{1}{2}\|f\|_H^2),$$

where we have used the fact that E^* (the dual space of E) becomes a dense subspace of H when we make the natural identification between H^* and H itself.

Starting with the Cameron-Martin formula, a great effort has been devoted to the problem of finding the Radon-Nikodym derivative of the image of μ under good nonlinear maps with respect to μ (see [1–6]). In [6], O. Enchev and D. W. Stroock presented a very interesting Girsanov type theorem. However, they mainly focus on the standard Wiener space of $\mathbb{R}^d$-valued Brownian paths, and it is hard to apply their method to prove this Girsanov theorem in the case of abstract Wiener spaces. So it might be worth to make an effort of studying this problem on general abstract Wiener spaces. That is the motivation of this paper. Now let us describe our main ideas.

It is well known that there exist a sequence (G_n) in E^* and an orthonormal basis (a_n) of H with the property that for all n, m,

$$\int_E G_n(w)\ G_m(w)\ \mu(dw) = \delta_{n,m}, \quad G_n(a_m) = \delta_{n,m}.$$

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The system $\{(G_n);(a_n)\}$ is called an orthogonal decomposition of $(E,\ H,\ \mu)$. Now let us fix an orthogonal decomposition $\{(G_n);(a_n)\}$. A functional ϕ on E is called a smooth cylinder functional if it is of the form

$$\phi(w)=\varphi(G_1(w),G_2(w),\cdots,G_N(w)),\quad w\in E$$

for some N and $\varphi\in C_c^\infty(\mathbb{R}^N)$ (the space of infinitely differentiable functions with compact support in $\mathbb{R}^N$). We define the directional derivative of a_n, for any smooth cylinder functional ϕ and for all $w\in E$, as

$$D_{a_n}\phi(w)=\lim_{t\to 0}\frac{\phi(w+ta_n)-\phi(w)}{t}.$$

The smallest closed extension of $D_{a_n}$is denoted by D_n. As in [7], we can construct the Ornstein-Uhlenbeck operator $\mathcal{L}$ and its semi-group (P_t) on $L^2(E,\mu)$.

We first consider a cylinder map $Z:E\to E$ such that for all $w\in E$,

$$Z(w)=w+K(G_1(w),G_2(w),\cdots,G_N(w)),$$

where K is a smooth map from $\mathbb{R}^N$ to H. Then it is easy to check that for every smooth cylinder functional ϕ,

$$\int_E \phi(Z(w))\mathfrak{E}^Z(w)\ \mu(dw)=\int_E\phi(w)\ \mu(dw),$$

where

$$\begin{aligned}\mathfrak{E}^Z(w)=&\det\left(I_d+\|D_iG_j(K(w))\|_{1\le i,j\le N}\right)\\ &\cdot\exp\left(-\sum_n G_n(w)(a_n,K(w))_H-\frac{1}{2}\|K(w)\|_H^2\right).\end{aligned}$$

If a map $T:E\to E$ can be approximated in μ-measure by a sequence of cylinder maps (T_n) such that $\mathfrak{E}^{T_n}$ converges to $\mathfrak{E}^T$ in $L^1(E,\mu)$, then for every smooth cylinder functional ϕ,

$$\int_E \phi(T(w))\mathfrak{E}^T(w)\ \mu(dw)=\int_E\phi(w)\ \mu(dw).$$

In this paper we prove that if for all $w\in E$, $T(w)=w+u(w)$, where u is a measurable map from E into H and satisfies some conditions, which is similar to those given by O. Enchev and D. W. Stroock in [6], then the preceding procedure can be applied to T. In this way we get the generalization of Girsanov's theorem on general abstract Wiener spaces, particularly on the Brownian sheet sample space and the Brownian bridge path space. Finally, we present the concept of Gaussian operators and describe the Girsanov's theorem on the Gaussian measure spaces corresponding to some Gaussian operator.

§1. Some Lemmas

Let (E,H,μ) be an abstract Wiener space, $\{(G_n);(a_n)\}$ be an orthogonal decomposition of (E,H,μ), and (P_t) be the Ornstein-Uhlenbeck semigroup on $L^2(E,\mu)$.

We first introduce some notions which will be used constantly in the sequel.

Let $L^2(H,\mu)$ denote the space of H-valued square integrable functions on E with the norm

$$\|u\|_2^2=\int_E\|u(w)\|_H^2\mu(dw),\quad u\in L^2(H,\mu).$$

For $u \in L^2(H, \mu)$, define

$$P_t u = \sum_n P_t(u, a_n)_H a_n, \quad t \geq 0.$$

From [8], we have

$$P_t u(w) = \int_E u(\mathrm{e}^{-t} w + \sqrt{1 - \mathrm{e}^{-2t}} v) \ \mu(dv).$$

If $W^{2,k}(\mathbb{R}^1, \mu)$ is the completion of the space of smooth cylinder functional on E with respect to the norm

$$\|u\|_{2,k}^2 = E(|\,u\,|^2) + \sum_{i_1, i_2, \cdots, i_k} E(|D_{i_1} D_{i_2} \cdots D_{i_k} u|^2),$$

where $E(X) = \int X(w) \, \mu(dw)$, we set

$$\begin{aligned} W^{2,k}(H, \mu) = \Big\{ & u \in L^2(H, \mu) | u = \sum_n (u, a_n)_H a_n \ \text{ such that for all } \ n, \\ & (u, a_n)_H \in W^{2,k}(\mathbb{R}^1, \mu) \text{ and} \\ & \|u\|_{2,k}^2 = \sum_n \|(u, a_n)_H\|_{2,k}^2 < \infty \Big\}. \end{aligned}$$

When $u \in W^{2,k}(H, \mu)$, we write

$$D_{i_1} D_{i_2} \cdots D_{i_k} u = \sum_n D_{i_1} D_{i_2} \cdots D_{i_k} (u, a_n)_H a_n.$$

We are now going to prove some lemmas which will play a fundamental role in this paper.

Lemma 1.1. *If $u \in W^{2,1}(H, \mu)$ and $\epsilon > 0$, then*

$$\sum_n \|D_n P_\epsilon u\|_H^2 \leq P_\epsilon \Big(\sum_n \|D_n u\|_H^2 \Big) \quad \text{a.s.}$$

Proof. We observe that for all n,

$$D_n P_\epsilon u = \mathrm{e}^{-\epsilon} P_\epsilon D_n u.$$

Since P_ϵ is a Markov operator (see [8]), we have

$$\sum_n \|D_n P_\epsilon u\|_H^2 \leq \mathrm{e}^{-2\epsilon} P_\epsilon \Big(\sum_n \|D_n u\|_H^2 \Big) \quad \text{a.s.,}$$

which proves our claim.

Lemma 1.2. *Assume that $u \in W^{2,1}(H, \mu)$ and $u_\epsilon = P_\epsilon u$, $\epsilon > 0$. Let $u_{\epsilon,N} = E(u_\epsilon | \Omega_N)$, where Ω_N is the σ-field generated by $\{G_1, G_2, \cdots, G_N\}$ and $E(\cdot | \Omega_N)$ denote the condition expectation with respect to Ω_N. Then*

$$u_{\epsilon,N}(w) = K_{\epsilon,N}(G_1(w), G_2(w), \cdots, G_N(w)),$$

where $y \mapsto K_{\epsilon,N}(y)$ is a C^∞-map from $\mathbb{R}^N$ to H, and

$$\sum_n \|D_n u_{\epsilon,N}(w)\|_H^2 \leq P_\epsilon \Big(E \Big(\sum_n \|D_n u\|_H^2 \Big| \Omega_N \Big) \Big)(w) \ \textit{a.s.}$$

Proof. Straightforward.

Before studying our approximation lemmas, we introduce an operator $A(u; w)$ corresponding to $u \in W^{2,1}(H, \mu)$.

Let $u \in W^{2,1}(H,\mu)$, define for $a \in H$,

$$A(u;w)(a) = \sum_n (D_n u(w), a)_H a_n.$$

Because

$$\|A(u;w)\|^2_{\mathrm{H.S}} = \sum_m \|A(u;w)a_m\|^2_H$$
$$= \sum_n \sum_m (D_n u(w), a_m)^2_H = \sum_n \|D_n u(w)\|^2_H,$$

where we denote the Hilbert-Schmidt norm of an operator by $\|\cdot\|_{\mathrm{H.S}}$, for almost surely w, $A(u;w)$ is a Hilbert-Schmidt operator from H to H.

Lemma 1.3.[7] *If $u \in W^{2,1}(H,\mu)$ and $u_{\epsilon,N} = E(P_\epsilon u|\Omega_N)$, then*

$$E(\|A(u;w) - A(u_{\epsilon,N};w)\|^2_{\mathrm{H.S}}) \longrightarrow 0 \quad \text{as} \quad N \to +\infty \text{ and } \epsilon \downarrow 0.$$

and if there exists a constant α such that $\|A(u;w)\|^2_{H.S} \le \alpha^2$ a.s. , then for all ϵ and N, $\|A(u_{\epsilon,N};w)\|^2_{\mathrm{H.S}} \le \alpha^2$ a.s.

Lemma 1.4. *Suppose that $u \in W^{2,1}(H,\mu)$ and there exists a constant α, $0 < \alpha < 1$, such that $\sum\limits_n \|D_n u(w)\|^2_H \le \alpha^2$ a.s. . Assume that $u_{\epsilon,N} = E(u_\epsilon|\Omega_N)$ (see Lemma 1.2). Then u (resp. $u_{\epsilon,N}$) has μ-a.e. modification $\tilde u$ (resp. $\tilde u_{\epsilon,N}$) with the property that for all $w \in E$ and $h \in H$,*

$$\|\tilde u(w+h) - \tilde u(w)\|_H \le \alpha\|h\|_H,$$
$$\|\tilde u_{\epsilon,N}(w+h) - \tilde u_{\epsilon,N}(w)\|_H \le \alpha\|h\|_H,$$

and we can choose $\epsilon_k \downarrow 0$ and $N_k \uparrow \infty$ as $k \to \infty$ such that for all $w \in E$

$$\lim_{k\to\infty} \|\tilde u_{\epsilon_k,N_k}(w) - \tilde u(w)\|_H = 0.$$

Proof. We have noticed that

$$u_{\epsilon,N}(w) = K_{\epsilon,N}(G_1(w), G_2(w), \cdots, G_N(w)), \qquad \text{a.s.},$$

where $K_{\epsilon,N}$ is a C^∞-map from $\mathbb{R}^N$ to H. Since

$$D_n u_{\epsilon,N}(w) = \Big(\frac{\partial}{\partial y_n} K_{\epsilon,N}\Big)(G_1(w), G_2(w), \cdots, G_N(w)) \quad \text{a.s. },$$

we know that if $\mu(\Omega) = 1$, then the set $\{(G_1(w), G_2(w), \cdots, G_N(w)) \,|w \in \Omega\}$ is a dense subset of $\mathbb{R}^N$. Hence for all $y \in \mathbb{R}^N$,

$$\sum_{n=1}^N \|\frac{\partial}{\partial y_n} K_{\epsilon,N}(y)\|^2_H \le \alpha^2$$

and for all $w \in E$ and $h \in H$

$$\|u_{\epsilon,N}(w+h) - u_{\epsilon,N}(w)\| \le \alpha\|h\|_H.$$

Recall that $\lim\limits_{\epsilon\downarrow 0, N\uparrow\infty} E(\|u_{\epsilon,N} - u\|^2_H) = 0$. We can choose $\epsilon_k \downarrow 0$ and $N_k \uparrow \infty$ as $k \to \infty$ such that

$$u_k = u_{\epsilon_k,N_k}(w) \to u(w) \quad \text{in } H \text{ a.s. as } \ k \to \infty.$$

Let H_0 be a countable dense subset of H and

$$\Omega = \{w| \text{ for all } h \in H_0, \lim_{k\to\infty} u_k(w+h) \text{ exists in } H\}.$$

Obviously, Ω is a measurable set of E and $\mu(\Omega) = 1$ and if $a \in H$ and $w \in \Omega$, then we obtain that for all $h \in H_0$

$$\|u_{k_1}(w+a) - u_{k_2}(w+a)\| \le 2\alpha\|a-h\|_H + \|u_{k_1}(w+h) - u_{k_2}(w+h)\|_H.$$

Thus, $\lim u_k(w+a)$ exists in H, which means that for all $a \in H$

$$\Omega + a = \{w + a | w \in \Omega\} \subseteq \Omega.$$

Denote the indicator of set Ω by I_Ω and define $\tilde{u}_k = u_k I_\Omega$ and $\tilde{u}(w) = \lim \tilde{u}_k(w)$ for $w \in E$. Since $(E \setminus \Omega) + a \subseteq E \setminus \Omega$ for all $a \in H$, $\tilde{u}(w+a) = \lim\limits_{k\to\infty} \tilde{u}_k(w+a)$ in H and for all $w \in E$ and $a \in H$,

$$\begin{aligned}
&\|\tilde{u}(w+a) - \tilde{u}(w)\|_H \le \alpha\|a\|_H,\\
&\|\tilde{u}_k(w+a) - \tilde{u}_k(w)\|_H \le \alpha\|a\|_H,\\
&\lim_{k\to\infty} \|\tilde{u}(w) - \tilde{u}_k(w)\|_H = 0.
\end{aligned}$$

Lemma 1.5. *If u and u_k, $k = 1, 2, \cdots$, are measurable maps from E to H with the following properties*

(1) *there exists a constant α, $0 < \alpha < 1$, such that for $h \in H$ and $w \in E$ and all k,*

$$\|u(w+h) - u(w)\|_H \le \alpha\|h\|_H, \quad \|u_k(w+h) - u_k(w)\| \le \alpha\|h\|_H.$$

(2) *for every $w \in E$, $\lim\limits_{k\to\infty} \|u_k(w) - u(w)\|_H = 0$.*

Let $T(w) = w + u(w)$ and $T_k(w) = w + u_k(w)$ for $w \in E$. Then $T(E) = T_k(E) = E$, and T^{-1} is also a measurable map. If $w_k + u_k(w_k) = w + u(w)$, we have

$$\lim_{k\to\infty} \|w_k - w\|_H = 0.$$

Proof. For $w \in E$, we define a metric space $X = \{w + h | h \in H\}$ with the metric $\rho(x, y) = \|x - y\|_H$ for $x, y \in X$. It is clear that X is a complete metric space. Let $K(x) = w - u(x)$ for $x \in X$. Then

$$\rho(K(x), K(y)) \le \alpha\|x - y\|_H = \alpha\rho(x, y).$$

Thus $x = \lim K^n(w)$ is the fixed point of K, this fact ensures that $T(E) = E$ and T^{-1} is also a measurable map from E to E. Similarly, the same conclusion holds for T_k. If $w \in E$ and $w_k + u_k(w_k) = w + u(w)$, we get

$$\begin{aligned}
\|w_k - w\|_H &\le \|u(w) - u_k(w)\|_H + \|u_k(w) - u_k(w_k)\|_H\\
&\le \|u(w) - u_k(w)\|_H + \alpha\|w - w_k\|_H.
\end{aligned}$$

Consequently,

$$\|w_k - w\|_H \le \frac{1}{1-\alpha}\|u(w) - u_k(w)\|_H \longrightarrow 0 \quad \text{as } k \to \infty.$$

Corollary 1.1. *Let $\lambda > 0$ and $A_{\lambda,k} = \{w : \|u_k(w)\| < \lambda\}$ and $A_\lambda = \{w : \|u(w)\|_H < \lambda\}$. Assume that $T(w) = w + u(w)$ and $T_k(w) = w + u_k(w)$ for $w \in E$. Then*

$$\varliminf_{k\to\infty} T_k(A_{\lambda,k}) \supseteq T(A_\lambda).$$

Proof. Suppose that $w \in T(A_\lambda)$. Then $w = w_0 + u(w_0)$, $\|u(w_0)\|_H < \lambda$. If $w_k + u_k(w_k) = w_0 + u(w_0)$, by Lemma 1.5, $\lim\limits_{k\to\infty} \|w_k - w_0\|_H = 0$. Thus

$$\begin{aligned}
\|u_k(w_k)\|_H &\le \|u_k(w_k) - u_k(w_0)\|_H + \|u_k(w_0) - u(w_0)\|_H + \|u(w_0)\|_H\\
&\le \alpha\|w_k - w_0\|_H + \|u_k(w_0) - u(w_0)\|_H + \|u(w_0)\|_H,
\end{aligned}$$

so that there exists an integer K such that for $k \geq K$, $\|u_k(w_k)\|_H < \lambda$, i.e, $w \in \bigcap_{k \geq K} T_k(A_{\lambda,k})$.

Lemma 1.6. *Suppose that $u \in W^{2,1}(H,\mu)$ and $u_\epsilon = P_\epsilon u$ with $\epsilon > 0$. Let K be a finite subset of natural numbers $\mathbb{N}$. Then*

$$E\Big(\Big|\sum_{n \in K} (D_n^* u_\epsilon, a_n)_H\Big|^2\Big) = \sum_{n \in K} E((u, a_n)_H^2) + \sum_{n,m \in K} E((D_m u, a_n)_H (D_n u, a_m)_H).$$

Proof. See [7].

Corollary 1.2. *$\sum_n (D_n^* u_\epsilon, a_n)_H$ converges to $\delta(u_\epsilon)$ in $L^2(E,\mu)$ and $\delta(u_\epsilon)$ converges to $\delta(u)$ in $L^2(E,\mu)$ as $\epsilon \downarrow 0$.*

Remark 1.1. $\delta(u)$ is called the Skorohod integral of u on (E, H, μ).

§2. Main Theorem

Theorem 2.1. *Suppose that $w \mapsto u(w)$ is a measurable map from E to H and $u \in W^{2,1}(H,\mu)$ and there exists a constant α, $0 < \alpha < 1$, such that*

$$\sum_n \|D_n u(w)\|_H^2 \leq \alpha^2 \quad \text{almost surely.}$$

Let $T(w) = w + u(w)$ for $w \in E$. Then for every bounded measurable function ϕ on E, we have

$$\int \phi(T(w)) \mathfrak{E}^T(w) \mu(dw) = \int \phi(w) \mu(dw)$$

where

$$\mathfrak{E}^T(w) = \exp(-\delta(u) - \frac{1}{2}\|u(w)\|_H^2 - \mathfrak{F}(u;w)),$$
$$\mathfrak{F}(u;w) = \mathrm{Trace}(A(u;w)^2 B(u;w)),$$
$$B(u;w) = \sum_{n \geq 2} \frac{(-1)^{n-2}}{n} A(u;w)^{n-2}.$$

Here $\delta(u)$ is the Skorohod integral of u on (E,H,μ) (see Lemma 1.6) and $A(u;w)$ is a linear operator from H to H defined as $A(u;w)(a) = \sum_n (D_n u(w), a)_H a_n$ for all $a \in H$ (see Lemma 1.3).

Proof. By Lemma 1.4, we may assume that $\|u(w+h) - u(w)\|_H \leq \alpha \|h\|_H$ for $w \in E$ and $h \in H$, and suppose that there exists a sequence (u_k), $u_k = \tilde{u}_{\epsilon_k, N_k}$ for $k \in \mathbb{N}$ such that $\|u_k(w+h) - u_k(w)\|_H \leq \alpha \|h\|_H$ for $w \in E$ and $h \in H$. Moreover, we can assume that for every $w \in E$,

$$\lim_{k \to \infty} \|u(w) - u_k(w)\|_H = 0,$$
$$\sum_n \|D_n u_k(w)\|_H^2 = \|A(u_k;w)\|_{\mathrm{H.S}}^2 \leq \alpha^2 \quad \text{a.s. for all } k,$$
$$\|A(u;w) - A(u_k;w)\|_{\mathrm{H.S}} \to 0 \quad \text{a.s. as } k \to \infty.$$

Write $T_k(w) = w + u_k(w)$ for $w \in E$. It is easily seen that if ϕ is a smooth cylinder functional which is of the form

$$\phi(w) = \varphi(G_1(w), G_2(w), \cdots, G_N(w)),$$

where $\varphi \in C_c^\infty(\mathbb{R}^N)$, then we deduce that

$$\int \phi(T_k(w))\mathfrak{E}^{T_k}(w)\mu(dw) = \int \phi(w)\mu(dw),$$

where

$$\begin{aligned}\mathfrak{E}^{T_k}(w) =& \exp\Big(-\sum_n G_n(w)(u_k(w), a_n)_H - \frac{1}{2}\|u_k(w)\|_H^2\Big)\\ &\times \det(I_d + \|D_i(a_j, u_k(w))_H\|_{1\le i,j\le N_k}).\end{aligned}$$

Here we denote the identity by I_d and we have used the fact that

$$G_j(u_k(w)) = (a_j, u_k(w))_H \text{ and } D_j u_k = 0 \text{ a.s. for } j > N_k.$$

Let P be the orthogonal projection from H to the subspace generated by $\{a_1, a_2, \cdots, a_{N_k}\}$. Thus the matrix $\|D_i(a_j, u_k(w))_H\|_{1\le i,j\le N_k}$ is corresponding to the operator $A(u_k; w)P$ from H to H. Because $\|A(u_k; w)\|_{\text{H.S}} \le \alpha$ a.s., we know that the series $\sum_n \frac{(-1)^{n-1}}{n}(A(u_k; w)P)^n$ converges almost surely to $\log(I_d + A(u_k; w)P)$ with respect to the Hilbert-Schmidt norm. Thus, we deduce that

$$\det(I_d + A(u_k; w)) = \exp\Big(\sum_n (D_n u_k, a_n)_H\Big) \exp\Big(\sum_{n\ge 2} \frac{(-1)^{n-1}}{n}\text{Trace}((A(u_k; w)P)^n)\Big).$$

Observe that

$$\text{Trace}((A(u_k; w)P)^n) = \text{Trace}(PA(u_k; w)^n) = \text{Trace}(A(u_k; w)^n).$$

Consequently

$$\det(I_d + \|D_i(a_j, u_k(w)\|_{1\le i,j\le N_k}) = \exp\Big(\sum_n (D_n u_k(w), a_n)_H\Big) \exp(-\mathfrak{F}(u_k; w)) \text{ a.s.}$$

It follows that

$$\mathfrak{E}^{T_k}(w) = \exp\Big(-\delta(u_k) - \frac{1}{2}\|u_k(w)\|_H^2 - \mathfrak{F}(u_k; w)\Big) \quad \text{a.s.}$$

since

$$\begin{aligned}&|\text{Trace}(A(u_k; w)^2 B(u_k; w)) - \text{Trace}(A(u; w)^2 B(u; w))|\\ &\le 2\|A(u_k; w) - A(u; w)\|_{\text{H.S}}\|B(u_k; w)\|_{\text{H.S}}\\ &\quad + \|B(u_k; w) - B(u; w)\|_{\text{H.S}}\|A(u; w)\|_{\text{H.S}}^2\end{aligned}$$

and

$$\|B(u_k; w)\|_{\text{H.S}} \le \sum_{n\ge 2} \|A(u_k; w)\|_{\text{H.S}}^{n-2} \le \frac{1}{1-\alpha} \quad \text{a.s. },$$

$$\lim_{k\to\infty} \|B(u_k; w) - B(u; w)\|_{\text{H.S}} = \lim_{k\to\infty} \|A(u_k; w) - A(u; w)\|_{\text{H.S}} = 0 \text{ a.s. .}$$

Summarizing, we have obtained that $\mathfrak{E}^{T_k} \to \mathfrak{E}^T$ in μ-measure as $k \to \infty$.

The crucial point of our proof is to show that $\mathfrak{E}^{T_k}$ converges to $\mathfrak{E}^T$ in $L^1(E, \mu)$ as $k \to \infty$. Instead of (u_k), we consider (σu_k), where constant $\sigma > 1$ and $\sigma\alpha < 1$. Obviously, for all k,

$$\int \exp\Big(-\sigma\delta(u_k) - \frac{\sigma^2}{2}\|u_k(w)\|_H^2 - \mathfrak{F}(\sigma u_k; w)\Big)\ \mu(dw) = 1.$$

Let $\lambda > 0$ and $A_{\lambda,k} = \{w : \|u_k(w)\|_H < \lambda\}$. Then for all $k \in \mathbb{N}$,

$$\int I_{A_{\lambda,k}} \exp\Big(-\sigma\delta(u_k)\Big)\mu(dw) \le \exp\Big(\frac{\sigma^2\lambda^2}{2} + \frac{\sigma^2\alpha^2}{1-\sigma\alpha}\Big).$$

Consequently,

$$\int (I_{A_{\lambda,k}} \cdot \mathfrak{E}^{T_k})^\sigma \mu(dw) \le \exp\Big(\frac{\sigma^2\lambda^2}{2} + \frac{\sigma^2\alpha^2}{1-\sigma\alpha} + \frac{\sigma\alpha^2}{1-\alpha}\Big).$$

We know that for any $\delta > 0$ there exists a measurable subset Ω of E such that $\mu(E\setminus\Omega) < \delta$ and $\lim\limits_{k\to\infty} \int\limits_\Omega \big|\mathfrak{E}^{T_k} - \mathfrak{E}^T\big| \mu(dw) = 0$. Then we yield that

$$\int I_{A_{\lambda,k}} \mathfrak{E}^{T_k}\mu(dw) \le c\mu(E\setminus\Omega)^{\frac{\sigma-1}{\sigma}} + \int_\Omega |\mathfrak{E}^{T_k} - \mathfrak{E}^T|\,\mu(dw) + \int_\Omega \mathfrak{E}^T\mu(dw),$$

where

$$c = \exp\Big(\frac{\sigma\lambda^2}{2} + \frac{\sigma\alpha^2}{1-\sigma\alpha} + \frac{\sigma\alpha^2}{1-\sigma\alpha}\Big).$$

Since for all k

$$\int I_{A_{\lambda,k}} \mathfrak{E}^{T_k}\mu(dw) = \int I_{T_k(A_{\lambda,k})}\mu(dw),$$

we deduce, by Fatou's Lemma, that for any $\epsilon > 0$,

$$\epsilon + \int_E \mathfrak{E}^T\mu(dw) \ge \varliminf_{k\to\infty} \int I_{T_k(A_{\lambda,k})}\mu(dw) \ge \int \varliminf_{k\to\infty} I_{T_k(A_{\lambda,k})}\mu(dw).$$

By Corollary 1.1,

$$\varliminf_{k\to\infty} I_{T_k(A_{\lambda,k})} \ge I_{T(A_\lambda)},$$

thus for any $\lambda > 0$, we have

$$\int \mathfrak{E}^T\mu(dw) \ge \mu(T(A_\lambda)) \longrightarrow 1 \quad \text{as } \lambda\to\infty,$$

which means that $\int \mathfrak{E}^T\mu(dw) \ge 1$. On the other hand, $\int \mathfrak{E}^T\mu(dw) \le 1$, so we conclude that $\int \mathfrak{E}^T\mu(dw) = 1$. Now we are ready to prove that $\mathfrak{E}^{T_k}$ converges to $\mathfrak{E}^T$ in $L^1(E,\mu)$. Actually,

$$\int |\mathfrak{E}^{T_k} - \mathfrak{E}|\,\mu(dw) = 2 - 2\int \mathfrak{E}^{T_k}\wedge \mathfrak{E}^T\mu(dw) \longrightarrow 0 \quad \text{as } k\to\infty,$$

by the dominated theorem.

We can now claim that for any bounded measurable function ϕ we have

$$\int \phi(T(w))\mathfrak{E}^T(w)\mu(dw) = \int \phi(w)\mu(dw).$$

Theorem 2.2. *Suppose that $w \mapsto u(w)$ is a measurable map from E to H and $u \in W^{2,1}(H,\mu)$ and there exists a constant α, $0<\alpha<1$, such that for every $h\in H$*

$$\|u(w+h) - u(w)\|_H \le \alpha\|h\|_H \quad a.s.,$$

and

$$E\Big(\exp\Big(\frac{108}{(1-\alpha)^2}\Big(\sum_n \|D_n u\|_H^2\Big)\Big)\Big) < \infty.$$

Then the conclussion of Theorem 2.1 holds.

Proof. Under the assumptions

$$|\mathfrak{F}(u_k;w)| \le \frac{\|A(u;w)\|^2_{\text{H.S.}}}{(1-\alpha)},$$

and we can find constants σ and σ_1, $1<\sigma_1<\sigma<1/\alpha$, such that

$$\int_{A_k,\lambda} \exp\{-\sigma\delta(u_k)\}\exp\Big\{-\frac{\sigma^2}{1-\sigma\alpha}\|A(u_k,w)\|^2_{\text{H.S.}}\Big\}\mu(dw) \le e^{\frac{\sigma^2\lambda^2}{2}},$$

and

$$\int_{A_k,\lambda} (\mathfrak{E}^{T_k})^{\sigma_1} \mu(dw) \leq c(\lambda, \alpha, \sigma.\sigma_1) \quad \text{for all } k,$$

where $c(\lambda, \alpha, \sigma.\sigma_1)$ is a constant indepent of k. Thus we may now complete the proof easily.

§3. Gaussian Operators and Girsanov's Theorem

In this section we introduce the concept of Gaussian operators and give the Girsanov's theorem on Gaussian measure spaces related to some Gaussian operator.

Let $\mathbb{R}^d$ be the Euclidean space and T a bounded closed subset of $\mathbb{R}^d$ and $m(dx)$ the Lebesque measure on $\mathbb{R}^d$ and $L^2(T)$ the space of the integrable functions on T with the usual inner product denoted by $(\cdot, \cdot)_2$.

Definition 3.1. *A densely defined linear closed operator L on $L^2(T)$ is called a Gaussian operator on T if there exists a strictly positive constant c such that for all $u \in H(L)$*

$$\|Lu\|_2 \geq c\|u\|_2,$$

where $H(L)$ is the domain of L and $\|\cdot\|_2$ is the norm of $L^2(T)$.

In that case if we define an inner product on $H(L)$ such that $[u, v] = (Lu, Lv)_2$ for $u, v \in H(L)$, then $H(L)$ becomes a Hilbert space, and we can find a sequence $(a_n) \subseteq H(L) \cap C(T)$ (the space of continuous functions on T with the maximum norm) such that (a_n) is an orthonormal basis of $H(L)$ and $\sum a_n e_n(w)$ converges a.s. in $C(T)$, where $(e_n(w))$ is a sequence of i.i.d. Gaussian random variables with mean 0 and variance 1 on some probability space $(\Omega, \mathfrak{F}, P)$.

We set

$$X(w) = \sum_n a_n e_n(w), \quad \mu(L) = P \circ X^{-1};$$

hence we get a Gaussian measure space ($C(T), \mathfrak{B}(C(T)), \mu(L)$), and the Gaussian random field corresponding to L, $W(t, w) = w(t)$ for $t \in T$ and $w \in C(T)$, and $\mu(L)$, the Gaussian measure of L.

In this section we will state without proofs some facts about Gaussian operators.

Proposition 3.1. *Let L be a Gaussian operator on T and $\mu(L)$ the Gaussian measure of L. Then $H(L) \subseteq C(T)$ and $(C(T), H(L), \mu(L))$ becomes an abstract Wiener space in the sense of L. Gross.*

Now let us turn to a comparison theorem about Gaussian operators.

Theorem 3.1. *Suppose that L is a Gaussian operator on T and L_1 be another densely defined linear operator on $L^2(T)$ such that the domain of L_1, $H(L_1)$, is contained in $H(L)$ and for all $u \in H(L_1)$*

$$\|L_1 u\|_2 \geq \|Lu\|_2.$$

Then L_1 is also a Gaussian operator on T.

Remark 3.1. If $T = [0, 1]$ and $L = \dfrac{d}{dt}$ and

$$H(L) = \{u | u(0) = 0 \text{ and } u \text{ is absolutely continuous on } T \text{ such that } u' \in L^2(T)\},$$

then L is a Gaussian operator on T and the stochastic process corresponding to L is the Wiener process on T.

When $T=[0,1]^d\subset\mathbb{R}^d$ and $L_0=\dfrac{\partial^d}{\partial t_1\cdots\partial t_d}$ and

$$H(L_0)=\{u|u(t)=0 \text{ if } t=(t_1,\cdots,t_d)\in T \text{ such that } t_1\cdots t_d=0 \text{ and } u\in W^{2.d}(T)\},$$

where $W^{2,d}(T)$ is the Sobolev space, then the smallest closed extension L of L_0 is a Gaussian operator and the Gaussian field corresponding to L is the Brownian Sheet on T with the covariance function

$$\Gamma(s,t)=(t_1\wedge s_1)(t_2\wedge s_2)\cdots(t_d\wedge s_d)\text{for } t=(t_i),\quad s=(s_i)\quad\text{in } T.$$

Corollary 3.1. *Let $T=[0,1]$, and $L_1=\dfrac{d}{dt}$ and*

$$H(L_1)=\{u|\,u\ \ \textit{is absolutely continuous on } T\ \ \textit{such that}\ u'\in L^2(T)\ \textit{and}\ \ u(0)=u(1)=0\}.$$

Then L_1 is a Gaussian operator on T and the Gaussian process corresponding to L_1 is the Brownian bridge on T.

Corollary 3.2. *Suppose that U is a bounded open subset of $\mathbb{R}^2$ with C^2-class boundary ∂U, and $a_{i,j}(x),b_j(x)\in C^2(\overline{U})$ such that*

$$\sum_{1\le i,j\le 2}a_{i,j}(x)t_it_j\ge\theta(t_1^2+t_2^2),$$

where θ is a strictly positive constant. Set

$$L_0u(x)=\sum_{1\le i,j\le 2}\frac{\partial}{\partial x_i}\Big(a_{i,j}(x)\frac{\partial}{\partial x_j}u(x)\Big)+\sum_{j=1}^{2}b_j(x)\frac{\partial}{\partial x_j}u(x),$$

and

$$H(L_0)=\{u:u\in W^{2,2}(U)\ \textit{and}\ \ u|_{\partial U}=0\}.$$

Then the smallest closed extension of L_0 is a Gaussian operator on $\overline{U}$.

Proposition 3.2. *Let L be a Gaussian operator on T. Then there exists a function $G(t,s)$, $t,s\in T$, such that for each $t\in T$, $G(t,\cdot)\in L^2(T)$ and for each $\varphi\in R(L)$ (the range of L), we have*

$$\int G(t,s)\varphi(s)ds\in H(L)$$

and

$$L\Big(\int G(\cdot,s)\varphi(s)ds\Big)(t)=\varphi(t).$$

This function G is called the Green function of L.

Definition 3.2. *If L is a Gaussian operator on T and G is its Green function, A map T from $C(T)$ to $C(T)$ is called a drift map if it is of the form*

$$T(w)(t)=w(t)+\int G(t,s)g(s,w)ds,\quad w\in C(T),$$

where $g(s,w)\in L^2(T\times C(T),m\times\mu(L))$ such that for each w, $g(\cdot,w)\in R(L)$.

Remark 3.2. When $T=[0,1]^d\subset\mathbb{R}^d$ and $L=\dfrac{\partial^d}{\partial t_1\cdots\partial t_d}$, if L is a Gaussian operator with suitable domain, then the drift map is of the form

$$T(w)(t) = w(t) + \int_0^{t_1} \cdots \int_0^{t_d} g(s, w) ds,$$

where $t = (t_1, \cdots, t_d)$.

When L is the second order elliptic differential operator mentioned in Corollary 3.2, if G is the Green function of L with the Dirichlet boundary condition, then the drift map is as

$$T(w)(t) = w(t) + \iint_T G(t, s) g(s, w) ds \quad \text{for } w \in C(T).$$

Definition 3.3. *Suppose $g(s, w) \in L^2(T \times C(T), m \times \mu(L))$ and for each $s \in T$, $g_s(w) = g(s, w) \in W^{2,1}(\mathbb{R}, \mu(L))$ and*

$$\sum_n E\Big(\int |D_n g_s(w)|^2 ds\Big) < \infty.$$

By Lemma 1.6, *we know that $\sum\limits_n \int_T D_n^* P_\epsilon g_s(w)\, La_n(s)\, ds$ converges to $\delta(g)$ as $\epsilon \to 0$ in $L^2(C(T), \mu(L))$. $\delta(g)$ is called the Skorohod integral of g.*

Theorem 3.2. *Suppose that L is a Gaussian operator on T and G is its Green function. Suppose that g is measurable on $(T \times C(T), \mathfrak{B}(T) \times \mathfrak{B}(C(T)))$ such that $g \in L^2(T \times C(T), m \times \mu(L))$ and for each $w \in C(T)$, $g(\cdot, w) \in R(L)$ and for every $s \in T$, $g_s(w) = g(s, w) \in W^{2,1}(\mathbb{R}, \mu(L))$. Furthermore, we assume that there exists a constant α, $0 < \alpha < 1$, such that*

$$\sum_n \int |D_n g(s, w)|^2 ds \le \alpha^2 \ \ a.s.$$

Let

$$T(w)(t) = w(t) + \int_T G(t, s) g(s, w)\ ds,\ \ w \in C(T).$$

Then, for all bounded measurable function ϕ, we have

$$\int \phi(T(w)) \mathfrak{E}^T(w) \mu(dw) = \int \phi(w) \mu(dw),$$

where

$$\mathfrak{E}^T(w) = \exp\left(-\delta(g) - \frac{1}{2}\int_T |g(s, w)|^2 ds - \mathrm{Trace}(A(w)^2 B(w))\right).$$

Here $A(w)$, $w \in C(T)$, is a linear operator from $R(L)$ to $R(L)$ which is

$$A(w)(\varphi)(t) = \int K(t, s)\varphi(s)\ ds, \quad \varphi \in R(L),$$

$$K(t, s) = \sum_n D_n g(s, w) La_n(t),$$

$$B(w) = \sum_{n \ge 2} \frac{(-A(w))^{n-2}}{n}.$$

Proof. By Proposition 3.2, we have

$$(u(w), a_n)_{H(L)} = \int g(s, w) La_n(s) ds,$$

where $u(w) = \int G(t, s) g_s ds$. Hence $u : C(T) \to H(L)$ is measurable and it is easy to check that

$$\sum_n \|D_n u(w)\|_{H(L)}^2 = \sum_n \int |D_n g(s, w)|^2 ds \le \alpha^2 \quad \text{a.s.}$$

Now we deduce this theorem by Theorem 2.1 immediately.

Remark 3.3. If $T=[0,1]$ and $L=\dfrac{d}{dt}$ and

$$H(L)=\{u|u(0)=0,\ \ u \text{ is an absolutely continuous path from } T \text{ to } \mathbb{R}^d \text{ and } u'\in L^2(T)\},$$

then we get the result due to O. Enchev and D. W. Stroock (see [6]).

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