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ABSOLUTE CONTINUITY FOR INTERACTING MEASURE-VALUED BRANCHING BROWNIAN MOTIONS**

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Abstract

The moments and absolute continuity of measure-valued branching Brownian motions with bounded interacting intensity are investigated. An estimate of higher order moments is obtained. The absolute continuity is verified in the one dimension case. This thereby verifies the conjecture of Méléard and Roelly in [5].

Keywords Interacting measure-valued branching process, Absolute continuity, Branching Brownian motion, Comparison lemma.

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§1. Introduction

Let $(E, \mathcal{E})$ be a Polish space, denote by $M(E)$ the family of finite measures on E, and equip $M(E)$ with the usual weak convergence topology. Let us first briefly introduce the approximating processes $(\mu_t.)_{t>0}$, called interacting branching diffusion processes. For each time t, μ_t is a random measure which modelizes branching and diffusing particles in the following way:

$$\mu_t = \sum_{i \in I_t} \delta_{X_t^i},$$

where I_t is the set of indexes of particles alive at time t. Their dynamics is the following: μ_0 is a finite measure on E, describing the initial configuration of the system. Each particle moves following a homogeneous Feller process whose generator is $(A,\ \mathcal{D}(A))$, and after a certain lifetime, it vanishes at the location x with the death rate $\lambda(x, \mu_t)$ and is replaced by a random number of children. The reproduction law depends on the state of the system by μ_t and the location x. This kind of interaction is well known in infinite particle systems. Under suitable hypothesis this interacting diffusion process μ_t approximates to the measure-valued branching diffusion processes $(X_t,\ P^\mu)_{\mu \in M(E)}$ which uniquely satisfies the following martingale problem: $\forall F \in C_b^2(R),\ \forall f \in \mathcal{D},\ F_f(\mu) = F(\langle \mu, f \rangle)$,

$$\begin{aligned} F_f(X_t) - F_f(X_0) - \int_0^t \Big[&\langle X_s, (A + b(X_s)) f \rangle F'(\langle X_s, f \rangle) \\ &+ \langle X_s, \frac{c(X_s)}{2} f^2 \rangle F''(\langle X_s, f \rangle) \Big] ds \end{aligned} \tag{1.1}$$

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is a P^μ-local martingale. In particular,

$$\forall f \in \mathcal{D}(A), \quad M_t(f) = \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \langle X_s, (A + b(X_s))f\rangle ds$$

is a P^μ-local martingale with variance process $\int_0^t \langle X_s, c(X_s)f^2\rangle ds$, where $c \geq 0$ and b : $M(E) \times E \longrightarrow R$ are measurable functions. When $b(\mu, x)$ and $c(\mu, x)$ are independent of μ (the global system), the corresponding process is the classical DW-superprocess. This situation has been extensively studied in the recent decades. However, in population genetic model, the interaction is somehow essential according to Darwin's population evolution theory. Therefore, the more interesting case is that b may depend on the global system. In this point of view we call b the interacting intensity, and further assume

$$\sup_{\mu\in M(E),\ x\in E} b(\mu, x) < C < \infty$$

and c is a non-negative and bounded function. For the details see [5], and [6]. We will directly work with this kind of processes.

Absolute continuity is always an interesting topic for random measures. As a typical problem for measure-valued processes, it is not surprising that many authors have worked on (see [2, 7, 8], etc.) and have completely solved (if we may say so) this kind of questions for the classical DW-superprocesses. This is still an open problem for MBPI (see the conjecture posed by Méléard and Roelly in [5]).

Konno and Shiga[2] provided us with a clue to prove the absolute continuity for measure-valued processes. An important prerequisite in their method is that we must know how to estimate the higher moments (at least, up to the second moment). However, it is difficult to do so for MBPI, because of lack of the so-called log-Laplace property (see Remark 4.2). Therefore, our first goal is to compute and bound the moments of X_t. The general cases turn out very complicated, for we need made many abstract assumptions. For simplicity, in the sequel we shall devote ourselves to the measure-valued branching Brownian motions with interaction, the simplest and most important case. That is, E is R^d and ξ is the d-dimension standard Brownian motion. Its transition function is denoted by

$$p_t(x, y)\ \Big(= \frac{1}{2\pi t^{d/2}} \exp\Big\{-\frac{|x-y|^2}{2t}\Big\}\Big).$$

Our main result in this paper is the following theorem (the conjecture posed by Méléard and Roelly in [5]).

Theorem 1.1. *X_t, the measure-valued Brownian motion with bounded interaction on the real line, is almost surely absolutely continuous with respect to the Lebesgue measure for $t > 0$.*

This paper is organized as following. Section 2 will present some results on the moments of X_t. The proof of Theorem 1.1 will be given in Section 3. In the last section we shall simply comment on the ideas to generalize our results.

§2. Moments

In this section we will work on the moments of MBPI. Because of lack of the log-Laplace property, we shall directly apply the martingale properties, and adopt the theory of partial differential equations.

Now let us define the nth-moment measure $M_n(t,\mu,dx_1\cdots dx_n)$ on $(R^d)^n$ by

$$\int_{(R^d)^n} f_1(x_1)\cdots f_n(x_n)M_n(t,\mu,dx_1\cdots dx_n)$$
$$= P^\mu\langle X_t,f_1\rangle\cdots\langle X_t,f_n\rangle,\quad f_i\in pB(R^d),\ i=1,\cdots,n.$$

Lemma 2.1. *The nth-moment measure $M_n(t,\mu,dx_1\cdots dx_n)$, $n=1,2,\cdots$, is well defined.*

Proof. In fact, it suffices to prove $P^\mu\langle X_t,1\rangle^n<\infty$. For this, noticing that $1\in\mathcal{D}(\triangle)$, we know from (1.1) that

$$\langle X_t,1\rangle^n-\langle X_0,1\rangle^n-\int_0^t[n\langle X_s,b(X_s)\rangle\langle X_s,1\rangle^{n-1}+\frac{1}{2}n(n-1)\langle X_s,c(X_s)\rangle\langle X_s,1\rangle^{n-2}]ds$$

is a P^μ-local martingale. Therefore, there exists a sequence of stopping time $T_k,\ k\geq 1,\ T_k\to\infty$ as $k\to\infty$, and for any fixed k

$$P^\mu\langle X_{t\wedge T_k},1\rangle^n=\langle\mu,1\rangle^n+P^\mu\int_0^{t\wedge T_k}[n\langle X_s,b(X_s)\rangle\langle X_s,1\rangle^{n-1}$$
$$+\frac{1}{2}n(n-1)\langle X_s,c(X_s)\rangle\langle X_s,1\rangle^{n-2}]ds.$$

Because b and c have an upper-bound, say $C>0$,

$$P^\mu\langle X_{t\wedge T_k},1\rangle^n\leq\langle\mu,1\rangle^n+\int_0^t C[nP^\mu\langle X_{s\wedge T_k},1\rangle^n+n(n-1)P^\mu\langle X_{s\wedge T_k},1\rangle^{n-1}]ds.\quad(2.1)$$

From this we have

$$P^\mu\langle X_{t\wedge T_k},1\rangle\leq e^{Ct}\langle\mu,1\rangle$$

and thereby, letting $k\to\infty$,

$$P^\mu\langle X_t,1\rangle\leq e^{Ct}\langle\mu,1\rangle.\quad(2.2)$$

Combining (2.1) and (2.2) we can inductively prove $P^\mu\langle X_t,1\rangle^n<\infty,\quad t>0,\quad n\in N.$

Lemma 2.2. *For any fixed $n\in Z$, $M_n(t,\mu,dx_1\cdots dx_n)$ is absolutely continuous. That is, there exists a measurable function $m(t,x_1,x_2,\cdots,x_n)$ such that*

$$M_n(t,\mu,dx_1\cdots dx_n)=m(t,x_1,x_2,\cdots,x_n)dx_1dx_2\cdots dx_n.$$

Proof. From the martingale property of X_t and Lemma 2.1, we have that for $f_i\in D(\triangle)$ such that $\triangle f_i$ is bounded

$$P^\mu\langle X_t,f_1\rangle\cdots\langle X_t,f_n\rangle=\langle\mu,f_1\rangle\cdots\langle\mu,f_n\rangle+\int_0^t\sum_{i=1}^n[\langle X_s,(\triangle+b(X_s))f_i\rangle\prod_{j\neq i}^n\langle X_s,f_j\rangle$$
$$+\sum_{j\neq i}\langle X_s,\frac{1}{2}c(X_s)f_if_j\rangle\prod_{k\neq i,j}\langle X_s,f_k\rangle]ds,$$

i.e.,

$$\int_{(R^d)^n}f_1(x_1)\cdots f_n(x_n)M_n(t,\mu,dx_1\cdots dx_n)$$
$$=\langle\mu,f_1\rangle\cdots\langle\mu,f_n\rangle+P^\mu\int_0^t\sum_{i=1}^n[\langle X_s,(\triangle+b(X_s))f_i\rangle\prod_{j\neq i}^n\langle X_s,f_j\rangle$$
$$+\sum_{j\neq i}^n\langle X_s,\frac{1}{2}c(X_s)f_if_j\rangle\prod_{k\neq i,j}^n\langle X_s,f_k\rangle]ds.$$

In particular, let $f_i(\cdot) = p_h(\cdot, y_i),\ i = 1, 2, \cdots, n$ for some fixed $h > 0$ and y_i. We know that $\triangle f_i$ is bounded, and then formula (2.3) becomes

$$\begin{aligned}
&\int_{(R^d)^n} p_h(x_1, y_1) \cdots p_h(x_n, y_n) M_n(t, \mu, dx_1 \cdots dx_n) \\
&= \langle \mu, p_h(\cdot, y_1)\rangle \cdots \langle \mu, p_h(\cdot, y_n)\rangle \\
&\quad + P^\mu \int_0^t \sum_{i=1}^n [\langle X_s, (\triangle + b(X_s)) p_h(\cdot, y_i)\rangle \prod_{j\neq i}^n \langle X_s, p_h(\cdot, y_j)\rangle \\
&\quad + \sum_{j\neq i}^n \langle X_s, \frac{1}{2} c(X_s) p_h(\cdot, y_i) p_h(\cdot, y_j)\rangle \prod_{k\neq i,j}^n \langle X_s, p_h(\cdot, y_k)\rangle] ds. \qquad (2.4)
\end{aligned}$$

Because of the boundness of b and c and

$$\triangle p_h(\cdot, y)|_x = \triangle p_h(x, \cdot)|_y,$$

we have that if set

$$g_h(t, y_1, \cdots, y_n) := \int_{(R^d)^n} p_h(x_1, y_1) \cdots p_h(x_n, y_n) M_n(t, \mu, dx_1 \cdots dx_n),$$

and

$$C_1 := \sup_{\mu, x} b(\mu, x), \quad C_2 := \sup_{\mu, x} c(\mu, x)/2,$$

then

$$\begin{aligned}
&\frac{d}{dt} g_h(t, y_1, \cdots, y_n) \\
&\le \Big(\sum_{i=1}^n (\triangle_i + C_1)\Big) g_h(t, y_1, \cdots, y_n) \\
&\quad + C_2 \int_{(R^d)^n} \sum_{i=1,\ j\neq i}^n p_h(x_i, y_i) p_h(x_i, y_j) \prod_{k\neq i,j}^n p_h(x_k, y_k) M_{n-1}\Big(t, \prod_{j\neq i} dx_j\Big), \qquad (2.5)
\end{aligned}$$

with the initial value

$$g_h(0, y_1, \cdots, y_n) = \langle \mu, p_h(\cdot, y_1)\rangle \cdots \langle \mu, p_h(\cdot, y_n)\rangle,$$

where $\triangle_i$ stands for the Laplace operator carrying out $g_h(t, y_1, \cdots, y_n)$ in the ith variable. From Lemma 2.1 it is easy to see that $g_h(t, y_1, y_2, \cdots, y_n)$ is smooth enough. Therefore, we can use the Comparison Lemma[4] and have

$$\begin{aligned}
g_h(t, y_1, \cdots, y_n) &\le e^{nC_1 t} \langle \mu, p_{t+h}(\cdot, y_1)\rangle \cdots \langle \mu, p_{t+h}(\cdot, y_n)\rangle \\
&\quad + C_2 \int_0^t e^{nC_1(t-s)} \int_{(R^d)^n} \sum_{i=1,\ j\neq i}^n p_{t-s+h}(x_i, y_i) p_{t-s+h}(x_i, y_j) \\
&\quad \cdot \prod_{k\neq i,j}^n p_{t-s+h}(x_k, y_k) M_{n-1}\Big(s, \prod_{j\neq i} dx_j\Big) ds. \qquad (2.6)
\end{aligned}$$

Noticing that for any open set B,

$$1_B(x) = \lim_{h\to 0} \int_B p_h(x, y) dy,$$

we have that if $B_i,\ i=1,2,\cdots,n$ are open sets, then

$$\begin{aligned}
&M_n(t,\mu;B_1,\cdots,B_n)\\
&=\lim_{h\to 0}\int_{B_1\times B_2\times\cdots\times B_n}\prod_{i=1}^n p_h(x_i,y_i)dy_i M_n(t,\mu;dx_1\cdots dx_n)\\
&=\lim_{h\to 0}\int_{B_1\times B_2\times\cdots\times B_n} g_h(t,y_1,\cdots,y_n)dy_1\cdots dy_n. \qquad (2.7)
\end{aligned}$$

In particular, when $n=1$, (2.6) and (2.7) clearly show that the first moment measure $M_1(t,\mu,dx)$ is absolutely continuous for $t>0$. From this and the inductive reasoning, we can verify that the nth moment is also absolutely continuous for all $n\ge 1$, and the proof of this lemma is complete.

In the proof of Lemma 2.2, it is easy to see that

Corollary 2.1. *If*

$$C_1:=\sup_{\mu,x} b(\mu,x),$$
$$C_2:=\sup_{\mu,x} c(\mu,x)/2,$$

then for $f_i\in B_+(E),\ i=1,2,\cdots,n$ *and* $t\ge 0$,

$$\begin{aligned}
P^\mu\prod_{i=1}^n\langle X_t,f_i\rangle \le\ & e^{C_1nt}\prod_{i=1}^n\langle \mu,P_tf_i\rangle\\
&+C_2\sum_{i=1}^n\sum_{j=1,j\ne i}^n\int_0^t e^{C_1n(t-s)}P_{t-s}f_i(x_i)P_{t-s}f_j(x_i)\\
&\cdot\prod_{k\ne i,j}P_{t-s}f_k(x_k)M_{n-1}\Big(s,\mu;\prod_{i\ne j}^n dx_i\Big)ds, \qquad (2.8)
\end{aligned}$$

where P_t *is the semigroup of Brownian motions in* R^d.

§3. Proof of Theorem

In this section, we give the proof of Theorem 1.1. The main idea here is due to Konno and Shiga[2]. We assume $d=1$.

Let

$$\begin{aligned}
X_t^h(x)&=\langle X_t,p(h,x,\cdot)\rangle\\
&:=\int_E p(h,x,y)X_t(dy).
\end{aligned}$$

We need to verify that for any fixed $T>0$,

$$\sup_{h>0}\int_R P^\mu(X_t^h(x))^2dx<\infty, \qquad (3.1)$$

and

$$\limsup_{h\to 0,\ h'\to 0}\int_R P^\mu|X_t^h(x)-X_t^{h'}(x)|^2dx=0. \qquad (3.2)$$

In fact, by Corollary 2.1 we have

$$\begin{aligned}\int_R P^\mu(X_t^h(x))^2dx &\le \int_R e^{2Ct}\langle\mu, P_tp(h,x,\cdot)\rangle^2dx\\ &\quad + C\int_R\int_0^t e^{2C(t-s)}\langle\mu P_s, (P_{t-s}p(h,x,\cdot))^2\rangle dsdx\\ &\le \text{const.}\Big[\int_R\langle\mu, p(t+h,x,\cdot)\rangle^2dx\\ &\quad + \int_R\int_0^t\langle\mu P_s, (p(t-s+h,x,\cdot))^2\rangle dsdx\Big]\\ &\le \text{const.} < \infty.\end{aligned}$$

Here const. is a constant independent of h.

In a similar manner, we can prove (3.2). That is, $\{X^h(x)\}$ is a Cauchy sequence in the Hilbert space $L^2(P^\mu\times dx)$, so we conclude that there exists a random variable $X(t,x)$ such that

$$\int_R P^\mu X(t,x)^2dx < \infty, \tag{3.3}$$

and

$$\limsup_{h\to 0}\int_R P^\mu|X_t^h(x) - X(t,x)|^2dx = 0. \tag{3.4}$$

Therefore, we have that, for $f\in\mathcal{B}$ and $\int_R f^2(x)dx < \infty$,

$$\begin{aligned}&P^\mu|\langle X_t, f\rangle - \int_R X(t,x)f(x)dx|^2\\ &\le \limsup_{h\to 0}P^\mu\Big[\int_R|X_t^h(x) - X(t,x)|f(x)dx\Big]^2\\ &\le \int_R f^2(x)dx\int_R P^\mu|X_t^h(x) - X(t,x)|^2dx = 0.\end{aligned} \tag{3.5}$$

This finishes the proof.

§4. Remarks

Remark 4.1. In the previous paragraphs we mainly discuss for Brownian motions. Let us trace back to see what we have required in our proof, and get some hint to generalize our results. First of all, in Section 2, we have used the "$\triangle$ (Laplace operator)-symmetry" of transition function of Brownian motions, i.e.,

$$\triangle p_h(\cdot,y)|_x = \triangle p_h(x,\cdot)|_y,$$

and the Comparison Lemma for $\triangle$. Secondly, the Feller property of semigroup P_t is demanded to certify

$$1_A(x) = \lim_{h\to 0}\int_A p_h(x,y)dy.$$

Thirdly, the very important fact in the proof of Theorem 1.1 is that there exists $0<\beta<1$ such that $\forall T>0$,

$$\sup_{0\le t\le T,\ x,y\in R} p_t(x,y)t^\beta < +\infty.$$

To sum up, that is the following assumption.

Assumption 4.1. *For a Feller process in E, there is a σ-finite measure $\mathcal{L}$ on E such that*

(1) *Process ξ has transition density $p_t(x,y)$ with respect to $\mathcal{L}$;*

(2) *There exists $0<\beta<1$ such that $\forall T>0$,*

$$\sup_{0\le t\le T,\ x,y\in E} p_t(x,y)t^{\beta}<+\infty;$$

(3) *The transition density $p_t(x,y)$ is A-symmetric, i.e.,*

$$Ap_t(\cdot,y)|_x = Ap_t(x,\cdot)|_y;$$

(4) *The Comparison Lemma for A holds in certain sense.*

Under Assumption 4.1, we can verify the absolute continuity of X_t (in a similar manner):

Theorem 4.1. *Under Assumption* 4.1*, the corresponding MBPI X_t given by* (1.1) *is almost surely absolutely continuous with respect to the reference measure $\mathcal{L}$ for $t>0$.*

The rigorous proof is left for interested readers. Obviously, Theorem 1.1 is a particular case of Theorem 4.1.

Remark 4.2. Section 2 provides a dull estimate of the moments. How to sharp this estimate is still an open question. Let

$$\tilde{b}=\sup_{\mu\in M} b(\mu),\quad \tilde{c}=\sup_{\mu\in M} c(\mu),$$

and denote the Schrödinger semigroup with potential b by P_t^b , i.e.

$$u(t,x)\overset{\triangle}{=}P_t^b f(x)=P_x e^{\int_0^t b(\xi_s)ds} f(\xi_t)$$

is the unique solution of the following Schrödinger equation:

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x)=Au(t,x)+b(x)u(t,x),\\ u(0,x)=f.\end{cases}$$

Under some conditions, the delicate result should be

$$\begin{aligned}P^{\mu}\prod_{i=1}^{n}\langle X_t,f_i\rangle &\le \prod_{i=1}^{n}\langle \mu, P_t^{\tilde{b}}f_i\rangle\\ &\quad+\sum_{i=1}^{n}\sum_{j=1,j\ne i}^{n}P^{\mu}\int_0^t\langle X_s,\tilde{c}P_{t-s}^{\tilde{b}}f_iP_{t-s}^{\tilde{b}}f_j\rangle\prod_{k\ne i,j}\langle X_s,P_{t-s}^{\tilde{b}}f_k\rangle ds.\end{aligned}\tag{4.1}$$

Particularly, when b and c are independent of μ, we can still use the the conventional way to get the above formula. In fact, El Karoui and Roelly[3] showed that the related superprocess is determined by the Laplace functional

$$P^{\mu}e^{-\langle X_t,f\rangle}=e^{-\langle V_tf,\mu\rangle},\tag{4.2}$$

where V_tf satisfies

$$V_tf+\int_0^t P_s^b c(V_{t-s}f)^2 ds = P_t^b f.\tag{4.3}$$

From this and Taylor's expansion we can easily show (4.1).

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