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# EXPONENTIAL TRICHOTOMY, ORTHOGONALITY CONDITION AND THEIR APPLICATION***

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**Abstract**

Exponential trichotomy theory is developed and the Fredholm Alternative Lemma is proved for the system with exponential trichotomies. An application of these theories is also given to obtain the persistence condition for heteroclinic orbits connecting nonhyperbolic equilibria, which extends the corresponding result of [11].

**Keywords** Exponential trichotomy, Bounded solutions, Orthogonality, Heteroclinic orbits
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In this paper, we develop the theory of exponential trichotomies and show the orthogonality condition associated with a system having exponential trichotomies.

Exponential trichotomy theory is an essential foundation to study the persistence and bifurcation problems of heteroclinic orbits connecting hyperbolic or nonhyperbolic equilibria (cf. [1-11]). But, the theory developed up to now is far from perfect to meet the needs for solving the problems just mentioned. We will first confine ourselves to the establishment of a theory of bounded solutions for the adjoint equation which is a key point to solve the homoclinic and heteroclinic bifurcation problems (cf. [10, 11]).

Let us consider a linear system in $\mathbf{R}^n$

$$\dot{x} \;=\; A(t)x + h(t), \tag{$*$}$$

where $A(t)$ is a continuous and uniformly bounded matrix-valued function.

When $A(t)$ and $h(t)$ are $T$-periodic, the orthogonality condition is also known as the Fredholm Alternative Lemma which plays an important part in the theory of linear periodic systems. Palmer[5] extended this lemma to the case where system $(*)$ has exponential dichotomies on both half lines. Hale and Lin[3, Lemma 4.5] made a further extension to the functional differential equations with so-called shifted exponential dichotomies on $\mathbf{R}^+$ and $\mathbf{R}^-$. This orthogonality condition has been extensively used to show the existence of the bounded solutions, particularly, the existence of heteroclinic orbits connecting two hyperbolic saddles (cf. [1,5,6,8,9]). In this paper, we prove the lemma in a still further

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generalized version which can be applied to show the existence of the bounded solutions in case that the associated system $(*)$ has exponential trichotomies both in $\mathbf{R}^+$ and $\mathbf{R}^-$. Notice that a related result (somewhat similar to, but not quite the same as our Theorem 2.1) has been given in [2] without proof. As compared with it, ours is much refined and hence more adaptable to the exponential trichotomy theory. Our proof follows essentially the idea of [5], which is completely different from that of [3], and, therefore, is rather elementary.

Using this generalized orthogonality condition and the theory of bounded solutions for the adjoint equation developed in this paper, we can easily extend the work of [6-9] (all only concerned with hyperbolic equilibria) to the case that is associated with nonhyperbolic equilibria. Because of the limited space, we only give one of the possible applications to get the persistence condition for heteroclinic orbits joining nonhyperbolic equilibria, and leave the further study to the forthcoming papers.

## §1. Exponential Trichotomy and Bounded Solutions of the Adjoint System

In this section, we first introduce the definition of the exponential trichotomy with the version of paper [4].

Let $X(t,s)$ be the solution map (i.e., the fundamental solution matrix with $X(s,s)=I$) for the linear homogeneous equation associated with system $(*)$.

**Definition 1.1.** *We say that system $(*)$ , or X(t,s), has an exponential trichotomy in a time interval $J$ if there exist projections $P_s(t)$, $P_c(t)$ and $P_u(t)=I-P_s(t)-P_c(t)$ satisfying*

$$X(t,s)P_v(s) \;=\; P_v(t)X(t,s), \qquad v=s,c,u,$$

*for $t\geq s$ in $J$, and there are constants $K\geq 1$ and $\alpha>\sigma>0$ such that*

$$\begin{aligned}
|X(t,s)P_s(s)| &\leq K\,e^{-\alpha(t-s)}, &\quad t\geq s \quad &\text{in } J,\\
|X(t,s)P_c(s)| &\leq K\,e^{\sigma|t-s|}, &\quad t,s \quad &\text{in } J,\\
|X(s,t)P_u(t)| &\leq K\,e^{-\alpha(t-s)}, &\quad t\geq s \quad &\text{in } J.
\end{aligned}$$

The constants $\alpha$ and $\sigma$ are called the exponents of the trichotomy, and the projection spaces $\Re P_s(t)$, $\Re P_c(t)$, $\Re P_u(t)$ are called the stable space, centre space and unstable space, respectively.

We say that system $(*)$ has an exponential dichotomy in $J$ if it has an exponential trichotomy with $P_c(t)=0$ and $P_s(t)+P_u(t)=I$.

Now we consider the adjoint system of the linear homogeneous equation associated with system $(*)$

$$\dot{x} \;=\; -A^*(t)x, \tag{1.1}$$

where the sign $*$ denotes the transposition.

It is well known that $Y(t,s)=X^{*-1}(t,s)=X^*(s,t)$ is the solution map of system (1.1). It turns out that, if system $(*)$ has an exponential trichotomy in $J$ with constants $K$, $\alpha$, $\sigma$, and projections $P_s(t)$, $P_c(t)$ and $P_u(t)$, then the adjoint system (1.1) also has an exponential trichotomy in $J$ with the same constants, and the corresponding projections $P_u^*(t)$, $P_c^*(t)$

and $P_s^*(t)$. More precisely, we have

$$Y(t,s)P_v^*(s) = P_v^*(t)Y(t,s), \qquad t \geq s \quad \text{in } J, \quad v = s,c,u,$$

$$\begin{aligned} |Y(t,s)P_u^*(s)| &\leq K\,e^{-\alpha(t-s)}, && t \geq s \quad \text{in } J, \\ |Y(t,s)P_c^*(s)| &\leq K\,e^{\sigma|t-s|}, && t,s \quad \text{in } J, \\ |Y(s,t)P_s^*(t)| &\leq K\,e^{-\alpha(t-s)}, && t \geq s \quad \text{in } J. \end{aligned}$$

We now consider the number of the linearly independent bounded solutions of the adjoint system (1.1) and the space spanned by these solutions. For this, we need the following lemma which can be verified quite easily.

**Lemma 1.1.** *Suppose that $P$ is a projection operator in a Hilbert space $H$. Then, $\Re P = (\Re(I-P^*))^{\perp}$. Here, the sign $\perp$ denotes the orthogonal complement.*

Set

$$E(b,J) = \Big\{x \in C^o : \sup_{t\in J}\{|x(t)|e^{b|t|}\} < \infty\Big\},$$

$$E(b,r,J) = \{x \in C^r : x, \cdots, x^{(r)} \in E(b,J)\}.$$

Then $E(b,J)$ and $E(b,r,J)$ are Banach spaces with norms

$$\|x\|_o = \sup_{t\in J}\{|x(t)|e^{b|t|}\} \qquad \text{and} \qquad \|x\|_r = \sum_{k=0}^{r} \|x^{(k)}\|_o,$$

respectively. And denote

$$\dim \Re P_s^i(0) = s^i, \qquad \dim \Re P_c^i(0) = c^i, \qquad \dim \Re P_u^i(0) = u^i,$$

for $i = +,-$,

$$d = n + c - c^+ - s^+ - c^- - u^-,$$

$$d_1 = d + c^-, \qquad d_2 = d + s^+ + c^-,$$

$$d_3 = d + c^+ + c^-, \qquad d_4 = d + s^+ + u^-,$$

where the number $c$ will be assigned different value in different cases.

**Theorem 1.1.** *Suppose that $(*)$ has exponential trichotomies both in $\mathbf{R}^+$ and $\mathbf{R}^-$ with constants $K_i, \alpha_i, \sigma_i$ and projections $P_s^i(t)$, $P_c^i(t)$, $P_u^i(t)$, respectively, for $i = +,-$. Let $\alpha = \min\{\alpha_+,\ \alpha_-\}$, $\sigma = \max\{\sigma_+,\ \sigma_-\}$. Then, the following five conclusions are valid.*

(i) *If $\dim(\Re(P_s^+(0) + P_c^+(0)) \cap \Re(P_c^-(0) + P_u^-(0))) = c$, then*

$$\dim(\Re P_u^{+*}(0) \cap \Re P_s^{-*}(0)) = d, \tag{1.2}$$

*that is, (1.1) has exactly $d$ linearly independent bounded solutions $\psi_1(t), \cdots, \psi_d(t)$ in $E(\alpha,1,\mathbf{R})$.*

(ii) *If $\dim(\Re(P_s^+(0) + P_c^+(0)) \cap \Re P_u^-(0)) = c$, then*

$$\dim(\Re P_u^{+*}(0) \cap \Re(P_s^{-*}(0) + P_c^{-*}(0))) = d_1, \tag{1.3}$$

*that is, (1.1) has exactly $d_1$ linearly independent bounded solutions $\psi_1(t), \cdots, \psi_{d_1}(t)$ in $E(\alpha,1,\mathbf{R}^+) \cap E(-\sigma,1,\mathbf{R}^-)$.*

(iii) *If $\dim(\Re P_c^+(0) \cap \Re P_u^-(0)) = c$, then*

$$\dim(\Re(P_u^{+*}(0) + P_s^{+*}(0)) \cap \Re(P_s^{-*}(0) + P_c^{-*}(0))) = d_2, \tag{1.4}$$

*that is,* (1.1) *has exactly* $d_2$ *linearly independent bounded solutions* $\psi_1(t), \cdots, \psi_{d_2}(t)$ *in* $E(-\alpha, 1, \mathbf{R}^+) \cap E(-\sigma, 1, \mathbf{R}^-)$.

(iv) *If* $\dim(\Re P_s^+(0) \cap \Re P_u^-(0)) = c$, *then*

$$\dim(\Re(P_u^{+*}(0) + P_c^{+*}(0)) \cap \Re(P_s^{-*}(0) + P_c^{-*}(0))) \;=\; d_3, \tag{1.5}$$

*that is,* (1.1) *has exactly* $d_3$ *linearly independent bounded solutions* $\psi_1(t), \cdots, \psi_{d_3}(t)$ *in* $E(-\sigma, 1, \mathbf{R}^+) \cap E(-\sigma, 1, \mathbf{R}^-)$.

(v) *If* $\dim(\Re P_c^+(0) \cap \Re P_c^-(0)) = c$, *then*

$$\dim(\Re(P_u^{+*}(0) + P_s^{+*}(0)) \cap \Re(P_s^{-*}(0) + P_u^{-*}(0))) \;=\; d_4, \tag{1.6}$$

*that is,* (1.1) *has exactly* $d_4$ *linearly independent bounded solutions* $\psi_1(t), \cdots, \psi_{d_4}(t)$ *in* $E(-\alpha, 1, \mathbf{R})$.

**Proof.** We only show the conclusion (i). The others can be proved in a similar way. By the fact

$$\dim(\Re(P_s^+(0) + P_c^+(0)) \oplus \Re(P_c^-(0) + P_u^-(0))) = s^+ + c^+ + c^- + u^- - c,$$

we have

$$\text{codim}\ (\Re(P_s^+(0) + P_c^+(0)) \oplus \Re(P_c^-(0) + P_u^-(0))) \;=\; d.$$

Then, it follows from Lemma 1.1 that

$$\begin{aligned}
&\dim(\Re P_u^{+*}(0) \cap \Re P_s^{-*}(0)\\
&= \dim((\Re(I - P_u^+(0)))^\perp \cap (\Re(I - P_s^-(0)))^\perp)\\
&= \dim((\Re(P_s^+(0) + P_c^+(0)))^\perp \cap (\Re(P_c^-(0) + P_u^-(0)))^\perp)\\
&= \text{codim}\ (\Re(P_s^+(0) + P_c^+(0)) \oplus \Re(P_c^-(0) + P_u^-(0))) = d.
\end{aligned}$$

Therefore, system (1.1) has exactly $d$ linearly independent bounded solutions $\psi_1(t), \cdots, \psi_d(t)$ in $E(\alpha, 1, \mathbf{R})$.

If we notice that $\Re P_v^{i*}(t)$ is a linear subspace for $i = +, -$ and $v = s, c, u$, and that

$$\Re(P + Q) \;=\; \Re P \oplus \Re Q$$

for any two projections satisfying $PQ = 0$, then the following four propositions can be easily deduced from Lemma 1.1 and Theorem 1.1.

**Theorem 1.2.** *Suppose that the conditions contained in Theorem* 1.1 *are valid, and*

$$\dim(\Re(P_s^+(0) + P_c^+(0)) \cap \Re(P_c^-(0) + P_u^-(0))) = c,$$

$$\dim(\Re(P_s^+(0) + P_c^+(0)) \cap \Re P_u^-(0)) = c.$$

*Then, system* (1.1) *has exactly* $d$ *linearly independent bounded solutions* $\psi_1(t), \cdots, \psi_d(t)$ *in* $E(\alpha, 1, \mathbf{R})$, *and exactly* $c^-$ *linearly independent bounded solutions* $\psi_{d+1}(t), \cdots, \psi_{d_1}(t)$ *in*

$$E(\alpha, 1, \mathbf{R}^+) \cap (E(-\sigma, 1, \mathbf{R}^-) - E(\alpha, 1, \mathbf{R}^-)).$$

*Moreover, we can choose* $\psi_1(t), \cdots, \psi_{d_1}(t)$ *such that*

$$\begin{aligned}
&\text{span}\ \{\psi_1(t), \cdots, \psi_{d_1}(t)\} &&\subset \Re P_u^{+*}(t) = (\Re(P_s^+(t) + P_c^+(t)))^\perp &&\textit{for}\quad t \ge 0,\\
&\text{span}\ \{\psi_1(t), \cdots, \psi_d(t)\} &&\subset \Re P_s^{-*}(t) = (\Re(P_c^-(t) + P_u^-(t)))^\perp &&\textit{for}\quad t \le 0,\\
&\text{span}\ \{\psi_{d+1}(t), \cdots, \psi_{d_1}(t)\} &&\subset \Re P_c^{-*}(t) = (\Re(P_s^-(t) + P_u^-(t)))^\perp &&\textit{for}\quad t \le 0.
\end{aligned}$$

**Theorem 1.3.** *Suppose that the conditions contained in Theorem* 1.1 *are valid, and*

$$\dim(\Re(P_s^+(0)+P_c^+(0))\cap\Re P_u^-(0))=c,$$

$$\dim(\Re P_c^+(0)\cap\Re P_u^-(0))=c.$$

*Then, system* (1.1) *has exactly* $d_1$ *linearly independent bounded solutions* $\psi_1(t),\cdots,\psi_{d_1}(t)$ *in* $E(\alpha,1,\mathbf{R}^+)\cap E(-\sigma,1,\mathbf{R}^-)$*, and exactly* $s^+$ *linearly independent bounded solutions* $\psi_{d_1+1}(t)$, $\cdots,\psi_{d_2}(t)$ *in*

$$(E(-\alpha,1,\mathbf{R}^+)-E(\alpha,1,\mathbf{R}^+))\cap(E(-\sigma,1,\mathbf{R}^-).$$

*Moreover, we can choose* $\psi_1(t),\cdots,\psi_{d_2}(t)$ *such that*

$$\begin{aligned}
&\mathrm{span}\,\{\psi_1(t),\cdots,\psi_{d_1}(t)\} &&\subset \Re P_u^{+*}(t)=(\Re(P_s^+(t)+P_c^+(t)))^\perp &&\textit{for}\quad t\ge 0,\\
&\mathrm{span}\,\{\psi_{d_1+1}(t),\cdots,\psi_{d_2}(t)\} &&\subset \Re P_s^{+*}(t)=(\Re(P_c^+(t)+P_u^+(t)))^\perp &&\textit{for}\quad t\ge 0,\\
&\mathrm{span}\,\{\psi_1(t),\cdots,\psi_{d_2}(t)\} &&\subset \Re(P_s^{-*}(t)+P_c^{-*}(t))=(\Re P_u^-(t))^\perp &&\textit{for}\quad t\le 0.
\end{aligned}$$

**Theorem 1.4.** *Suppose that the conditions contained in Theorem* 1.1 *are valid, and*

$$\dim(\Re(P_s^+(0)+P_c^+(0))\cap\Re P_u^-(0))=c,$$

$$\dim(\Re P_s^+(0)\cap\Re P_u^-(0))=c.$$

*Then, system* (1.1) *has exactly* $d_1$ *linearly independent bounded solutions* $\psi_1(t),\cdots,\psi_{d_1}(t)$ *in* $E(\alpha,1,\mathbf{R}^+)\cap E(-\sigma,1,\mathbf{R}^-)$*, and exactly* $c^+$ *linearly independent bounded solutions* $\psi_{d_1+1}(t)$, $\cdots,\psi_{d_3}(t)$ *in*

$$(E(-\sigma,1,\mathbf{R}^+)-E(\alpha,1,\mathbf{R}^+))\cap(E(-\sigma,1,\mathbf{R}^-).$$

*Moreover, we can choose* $\psi_1(t),\cdots,\psi_{d_3}(t)$ *such that*

$$\begin{aligned}
&\mathrm{span}\,\{\psi_1(t),\cdots,\psi_{d_1}(t)\} &&\subset \Re P_u^{+*}(t)=(\Re(P_s^+(t)+P_c^+(t)))^\perp &&\textit{for}\quad t\ge 0,\\
&\mathrm{span}\,\{\psi_{d_1+1}(t),\cdots,\psi_{d_3}(t)\} &&\subset \Re P_c^{+*}(t)=(\Re(P_s^+(t)+P_u^+(t)))^\perp &&\textit{for}\quad t\ge 0,\\
&\mathrm{span}\,\{\psi_1(t),\cdots,\psi_{d_3}(t)\} &&\subset \Re(P_s^{-*}(t)+P_c^{-*}(t))=(\Re P_u^-(t))^\perp &&\textit{for}\quad t\le 0.
\end{aligned}$$

**Theorem 1.5.** *Suppose that the conditions contained in Theorem* 1.1 *are valid, and*

$$\dim(\Re(P_s^+(0)+P_c^+(0))\cap\Re(P_c^-(0)+P_u^-(0)))=c,$$

$$\dim(\Re P_c^+(0)\cap\Re P_c^-(0))=c.$$

*Then, system* (1.1) *has exactly* $d$ *linearly independent bounded solutions* $\psi_1(t),\cdots,\psi_d(t)$ *in* $E(\alpha,1,\mathbf{R})$*, and exactly* $s^++u^-$ *linearly independent bounded solutions* $\psi_{d+1}(t),\cdots,\psi_{d_4}(t)$ *in*

$$E(-\alpha,1,\mathbf{R})-E(\alpha,1,\mathbf{R}).$$

*Moreover, we can choose* $\psi_1(t),\cdots,\psi_{d_4}(t)$ *such that*

$$\begin{aligned}
&\mathrm{span}\,\{\psi_1(t),\cdots,\psi_d(t)\} &&\subset \Re P_u^{+*}(t)=(\Re(P_s^+(t)+P_c^+(t)))^\perp &&\textit{for}\quad t\ge 0,\\
&\mathrm{span}\,\{\psi_1(t),\cdots,\psi_d(t)\} &&\subset \Re P_s^{-*}(t)=(\Re(P_c^-(t)+P_u^-(t)))^\perp &&\textit{for}\quad t\le 0,\\
&\mathrm{span}\,\{\psi_{d+1}(t),\cdots,\psi_{d_4}(t)\} &&\subset \Re P_s^{+*}(t)=(\Re(P_c^+(t)+P_u^+(t)))^\perp &&\textit{for}\quad t\ge 0,\\
&\mathrm{span}\,\{\psi_{d+1}(t),\cdots,\psi_{d_4}(t)\} &&\subset \Re P_u^{-*}(t)=(\Re(P_s^-(t)+P_c^-(t)))^\perp &&\textit{for}\quad t\le 0.
\end{aligned}$$

## §2. Orthogonality Conditions

We now turn our attention to the orthogonality condition associated with system $(*)$.

Let $L$ be the linear operator defined by

$$(Lx)(t) \;=\; \dot{x}(t) - A(t)x(t)$$

for $x \in C^1(\mathbf{R}, \mathbf{R}^n)$, and $L_1, L_2, L_3$ be the restrictions of L in $E(-\sigma, 1, \mathbf{R}),\ E(-\sigma, 1, \mathbf{R}^+) \cap E(\alpha, 1, \mathbf{R}^-),\ E(\alpha, 1, \mathbf{R})$, respectively. Denote

$$\begin{aligned} E_1^o &= E(-\sigma, \mathbf{R}), & E_1 &= E(\alpha, 1, \mathbf{R}),\\ E_2^o &= E(-\sigma, \mathbf{R}^+) \cap E(\alpha, \mathbf{R}), & E_2 &= E(\alpha, 1, \mathbf{R}^+) \cap E(-\sigma, 1, \mathbf{R}),\\ E_3^o &= E(\alpha, \mathbf{R}), & E_3 &= E(-\sigma, 1, \mathbf{R}), \end{aligned}$$

$$\begin{aligned} &I_1 = \dim \Re(P_s^+(t) + P_c^+(t)) + \dim \Re(P_c^-(t) + P_u^-(t)) - n,\\ &I_2 = \dim \Re(P_s^+(t) + P_c^+(t)) + \dim \Re P_u^-(t) - n,\\ &I_3 = \dim \Re P_s^+(t) + \dim \Re P_u^-(t) - n. \end{aligned}$$

The linear operator $L$ is referred to as a Fredholm operator if $\Re(L)$ is closed and has finite codimension. The index of $L$ as a Fredholm operator is defined as $\dim \mathcal{N}(L) -$ codim $\Re(L)$.

**Theorem 2.1.** *Suppose that the hypotheses of Theorem* 1.1 *hold. Then,* $h \in \Re(L_i)$ *if and only if* $h \in E_i^o$ *and*

$$\int_{-\infty}^{\infty} \psi^*(t)\, h(t)\, dt \;=\; 0 \tag{2.1}$$

*for all bounded solutions* $\psi(t)$ *of* (1.1) *in* $E_i$*. Moreover,* $L_i$ *is a Fredholm operator with index* $I_i,\ i = 1, 2, 3$.

**Proof.** We only consider the case $i = 1$. The proof of the other cases is similar. The proof given below is essentially an analogue and generalization of that given by Palmer in [5] for the case that has an exponential dichotomy.

Assume that $h \in \Re(L_1)$. Then there exists an $x \in E(-\sigma, 1, \mathbf{R})$ satisfying

$$h(t) \;=\; \dot{x}(t) - A(t)x(t).$$

So, $h \in E(-\sigma, \mathbf{R})$. Now if $\psi(t)$ is a bounded solution of (1.1) in $E(\alpha, 1, \mathbf{R})$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \psi^*(t)\, h(t)\, dt \;&=\; \int_{-\infty}^{\infty} (\psi^*(t)\dot{x}(t) - \psi^*(t)A(t)x(t))\, dt\\ &= \int_{-\infty}^{\infty} (\psi^*(t)\dot{x}(t) + \dot{\psi}^*(t)x(t))\, dt\\ &= \psi^*(t)x(t)|_{-\infty}^{\infty} \;=\; 0. \end{aligned}$$

The last equality holds just because $\psi^*(t)x(t) \to 0$ exponentially as $|t| \to \infty$, owing to the fact that $\psi(t)e^{\alpha|t|}$ and $x(t)e^{-\sigma|t|}$ are bounded. Thus, we have shown that if $h \in \Re(L_1)$, then the orthogonality condition (2.1) holds for all bounded solutions $\psi(t)$ of the adjoint system (1.1) in $E_1 = E(\alpha, 1, \mathbf{R})$.

Conversely, suppose that $h \in E(-\sigma, \mathbf{R})$ and that (2.1) is valid for all bounded solutions $\psi(t)$ of (1.1) in $E(\alpha, 1, \mathbf{R})$. It should be clear that, for each $\psi(t) \in E(\alpha, 1, \mathbf{R})$, there exists a vector $\eta \in \mathbf{R}^n$ such that

$$\psi(t) \;=\; \begin{cases} Y(t,0)P_u^{+*}(0)\eta & \text{for} \quad t \geq 0, \\ Y(t,0)P_s^{-*}(0)\eta & \text{for} \quad t \leq 0. \end{cases} \tag{2.2}$$

This means $P_u^{+*}(0)\eta = P_s^{-*}(0)\eta$. Equivalently, we have

$$(I - P_s^{+*}(0) - P_c^{+*}(0))\eta \;=\; (I - P_u^{-*}(0) - P_c^{-*}(0))\eta,$$

that is,

$$(P_s^{+*}(0) + P_c^{+*}(0))\eta \;=\; (P_u^{-*}(0) + P_c^{-*}(0))\eta.$$

Then, we obtain

$$\eta^*(P_s^{+}(0) + P_c^{+}(0) - P_u^{-}(0) - P_c^{-}(0)) \;=\; 0. \tag{2.3}$$

Substituting (2.2) into (2.1) and using $Y(t,s) = X^*(s,t)$, we get

$$\eta^* v \;=\; 0, \tag{2.4}$$

where

$$v \;=\; \int_0^\infty P_u^{+}(0)X(0,t)h(t)\,dt \;+\; \int_{-\infty}^0 P_s^{-}(0)X(0,t)h(t)\,dt. \tag{2.5}$$

By (2.3) and (2.4), it follows that there exists a vector $\xi \in \mathbf{R}^n$ satisfying

$$(P_s^{+}(0) + P_c^{+}(0) - P_u^{-}(0) - P_c^{-}(0))\xi \;=\; v.$$

Making use of (2.5) and $P_j^i(0)X(0,t) = X(0,t)P_j^i(t)$ for $i = +,-;\ j = s,u$, we have

$$\begin{aligned} &(P_s^{+}(0) + P_c^{+}(0))\xi - \int_0^\infty X(0,s)P_u^{+}(s)h(s)\,ds \\ &= (P_u^{-}(0) + P_c^{-}(0))\xi - \int_{-\infty}^0 X(0,s)P_s^{-}(s)h(s)\,ds. \end{aligned} \tag{2.6}$$

Then, it can be verified that the function $x(t)$, defined for $t \geq 0$ as

$$\begin{aligned} &X(t,0)(P_s^{+}(0) + P_c^{+}(0))\xi + \int_0^t X(t,s)(P_s^{+}(s) \\ &+ P_c^{+}(s))h(s)ds - \int_t^\infty X(t,s)P_u^{+}(s)h(s)\,ds \end{aligned}$$

and for $t \leq 0$ as

$$\begin{aligned} &X(t,0)(P_u^{-}(0) + P_c^{-}(0))\xi + \int_0^t X(t,s)(P_u^{-}(s) \\ &+ P_c^{-}(s))h(s)ds + \int_{-\infty}^t X(t,s)P_s^{-}(s)h(s)\,ds \end{aligned}$$

is in $E(-\sigma, 1, \mathbf{R})$ and is a solution of the inhomogeneous linear system $(*)$. It means $h \in \Re(L_1)$, as expected.

Now we show that the linear operator $L_1$ is Fredholm. By (2.1), each bounded solution $\psi(t)$ of (1.1) in $E(\alpha, 1, \mathbf{R})$ defines a bounded linear functional on $E(-\sigma, \mathbf{R})$ through

$$h \to \int_{-\infty}^\infty \psi^*(t)\,h(t)\,dt.$$

This correspondence gives an isomorphism between

$$\Re P_u^{+*}(t) \cap \Re P_s^{-*}(t) \;=\; (\Re(P_s^{+}(t) + P_c^{+}(t)))^\perp \cap (\Re(P_c^{-}(t) + P_u^{-}(t)))^\perp$$

and a finite dimensional subspace of the dual space $(E(-\sigma,\mathbf{R}))^*$. This means that $\Re(L_1)$ is a subspace of $E(-\sigma,\mathbf{R})$ with a finite codimension which is equal to

$$\operatorname{codim}\Re(L_1) \;=\; \dim((\Re(P_s^+(t)+P_c^+(t)))^\perp \cap (\Re(P_c^-(t)+P_u^-(t)))^\perp).$$

Due to the basic functional theory, this also means that $\Re(L_1)$ is closed. So $L_1$ is Fredholm.

By definition, the index of $L_1$ is

$$\begin{aligned}
&\dim\mathcal{N}(L_1) - \operatorname{codim}\Re(L_1)\\
&= \dim(\Re(P_s^+(t)+P_c^+(t)) \cap \Re(P_c^-(t)+P_u^-(t)))\\
&\quad - \dim((\Re(P_s^+(t)+P_c^+(t)))^\perp \cap (\Re(P_c^-(t)+P_u^-(t)))^\perp),\\
&= \dim(\Re(P_s^+(t)+P_c^+(t)) \cap \Re(P_c^-(t)+P_u^-(t)))\\
&\quad - \operatorname{codim}\,(\Re(P_s^+(t)+P_c^+(t)) \oplus \Re(P_c^-(t)+P_u^-(t)))\\
&= \dim(\Re(P_s^+(t)+P_c^+(t)) \cap \Re(P_c^-(t)+P_u^-(t)))\\
&\quad - n + \dim(\Re(P_s^+(t)+P_c^+(t)) \oplus \Re(P_c^-(t)+P_u^-(t)))\\
&= \dim\Re(P_s^+(t)+P_c^+(t)) + \dim\Re(P_c^-(t)+P_u^-(t)) - n,
\end{aligned}$$

as asserted.

**Remark 2.1.** If system $(*)$ has exponential dichotomies on both half lines, then, in the above propositions, we have

$$P_c^+ = P_c^- = 0,\quad E(-\sigma,1,J) = E(0,1,J),\quad E(-\sigma,J) = E(0,J)$$

for $J = \mathbf{R}^+,\ \mathbf{R}^-$.

**Corollary 2.1.**[5] Suppose that $(*)$ has exponential dichotomies on both half lines. Then the linear operator

$$L:\ E(0,1,\mathbf{R}) \to E(0,\mathbf{R})$$

is Fredholm and has index $\dim\Re P_s^+(t) + \dim\Re P_u^-(t) - n$. Moreover, $h \in \Re(L)$ if and only if $h \in E(0,\mathbf{R})$ and the orthogonality condition (2.1) holds for all bounded solutions $\psi(t)$ of the adjoint system (1.1).

## §3. Application

As a simple application, we give the conditions for the persistence of heteroclinic orbits which extends the result of [11]. Consider the following systems

$$\dot{x} = G(x,t,\alpha,\beta,\mu), \tag{3.1}$$

$$\dot{x} = F(x), \tag{3.2}$$

where $x \in \mathbf{R}^n,\ \alpha,\beta \in \mathbf{R},\ \mu \in \mathbf{R}^m,\ G \in C^r,\ r \geq 2,\ G(x,t,0,0,0) = F(x)$, and $G$ is $T$-periodic in $t$.

Assume that system (3.2) has a heteroclinic orbit $\Gamma = \Gamma(t)$ to two equilibria, $p = \Gamma(-\infty)$ and $q = \Gamma(+\infty)$. Denote

$$\begin{aligned}
&\dim W_q^s = s^+, \quad \dim W_q^u = u^+, \quad \dim W_q^c = c^+,\\
&\dim W_p^s = s^-, \quad \dim W_p^u = u^-, \quad \dim W_p^c = c^-,
\end{aligned}$$

where $W_x^u$, $W_x^s$ and $W_x^c$ are respectively the unstable, stable and center manifolds of $x$. We still need the following hypotheses.

(H1) $c^+ = 1,\ c^- \le 1,\ u^- + c^- \le u^+$.

(H2) $\dim(T_{\Gamma(t)}(W_p^c \cap W_q^c)) = \dim(T_{\Gamma(t)}W_p^{cu} \cap T_{\Gamma(t)}W_q^{cs}) = 1$ for $c^- = 1$,

$$\dim(T_{\Gamma(t)}(W_p^u \cap W_q^c)) = \dim(T_{\Gamma(t)}W_p^u \cap T_{\Gamma(t)}W_q^{cs}) = 1 \quad \text{for } c^- = 0,$$

where $W_p^{cu}$ and $W_q^{cs}$ are the center-unstable and the center-stable manifolds of $p$ and $q$, resperctively.

(H3) $p$ and $q$ are transcritical equilibia of system (3.1) with control parameters $\alpha$ and $\beta$, respectively.

When $u^- + c^- > u^+$, we see that $W_p^{cu}$ and $W_q^{cs}$ intersect transversally. So, it is trivial. For the definition of control parameters, one can refer to [10,11]. Here, we may as well assume that, when $\alpha > 0$ (resp. $\beta > 0$), the equilibrium $p$ (resp. $q$) splits into two $T$-periodic orbits.

Set $A(t) = DF(\Gamma(t))$. From [10, 11], it follows that the linear variational system

$$\dot{x} \ = \ A(t)x \tag{3.3}$$

and its adjoint system (1.1) have exponential trichotomies in both $\mathbf{R}^+$ and $\mathbf{R}^-$ with the same constants $K \ge 1,\ \alpha > \sigma > 0$ and the projections

$$P_s^i(t),\ P_c^i(t),\ P_u^i(t) \text{ and } P_u^{i*}(t),\ P_c^{i*}(t),\ P_s^{i*}(t) \quad \text{for } i = +, -.$$

Moreover,

$$\Re P_c^+(t) = T_{\Gamma(t)}W_q^c, \qquad \Re(P_s^+(t) + P_c^+(t)) = T_{\Gamma(t)}W_q^{cs},$$

$$\Re P_c^-(t) = T_{\Gamma(t)}W_p^c, \qquad \Re(P_u^-(t) + P_c^-(t)) = T_{\Gamma(t)}W_p^{cu}.$$

Let $d = n - s^+ - c^- - u^-$. By Theorem 1.1 (i), system (1.1) has exactly $d$ linearly independent bounded solutions $\psi_1(t), \cdots, \psi_d(t)$ in $E(\alpha, 1, \mathbf{R})$.

Suppose that system (3.1) has a heteroclinic orbit $\Gamma_1 :\ \{x(t + t_o) = \Gamma(t) + y(t, \alpha, \beta, \mu) : t \in \mathbf{R}\}$ connecting two $T$-periodic orbits near $p$ and $q$, and satisfying $y(t, 0, 0, 0) = 0$. Then, $y \in E(-\sigma, 1, \mathbf{R}),\ \ y = O(\alpha) + O(\beta) + O(|\mu|)$ and

$$\dot{y} \ = \ A(t)y + h(t, \alpha, \beta, \mu), \tag{3.4}$$

where

$$h(t, \alpha, \beta, \mu) = \alpha G_\alpha(z) + \beta G_\beta(z) + G_\mu(z)\mu + \text{h.o.t.},$$
$$z = (\Gamma(t), t + t_o, 0, 0, 0).$$

Applying Theorem 2.1 to operator $L_1$, we see that $\Gamma_1$ exists if and only if

$$I_i \ \equiv \ \int_{-\infty}^{\infty} \psi_i^*(t)h(t, \alpha, \beta, \mu)\, dt \ = \ 0 \tag{3.5}$$

for $i = 1, \cdots, d$. Let

$$M_i^j(t_o) \ = \ \int_{-\infty}^{\infty} \psi_i^*(t)G_j(z)\, dt, \tag{3.6}$$

for $j = \alpha, \beta, \mu$. Then,

$$I_i \ = \ \alpha M_i^\alpha(t_o) + \beta M_i^\beta(t_o) + M_i^\mu(t_o)\mu + \text{h.o.t.}$$

Thus, a direct application of the implicit function theorem yields the following result.

**Theorem 3.1.** *Suppose that hypotheses* (H1)-(H3) *hold, and there exists* $t_o$ *such that*

$$\text{rank}((M_1^\alpha(t_o), M_1^\beta(t_o), M_1^\mu(t_o)), \cdots, (M_d^\alpha(t_o), M_d^\beta(t_o), M_d^\mu(t_o))) \ = \ d.$$

*Then, in the* $(\alpha, \beta, \mu)$ *space, there exists an* $(m-d+2)$*-dimensional surface* $H$*, with* $\alpha > 0$*,* $\beta > 0$ *and* $|\alpha| + |\beta| + |\mu| << 1$*, such that* (3.1) *has a heteroclinic orbit near* $\Gamma$ *when* $(\alpha, \beta, \mu) \in H$*.*

**Remark 3.1.** In order to make (3.6) computable, a suitable coordinates change is necessary (cf. [10] and forthcoming papers). In order to obtain the transversality condition for the persistent heteroclinic orbits, we need to analyse the structures of the perturbed center-stable and center-unstable manifolds.

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