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A NEW PRINCIPAL PIVOTING SCHEME FOR BOX LINEAR COMPLEMENTARITY PROBLEMS**

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Abstract

Judice and Pires developed in recent years principal pivoting methods for the solving of the so-called box linear complementarity problems (BLCPs) where the constraint matrices are restrictedly supposed to be of P–matrices. This paper aims at presenting a new principal pivoting scheme for BLCPs where the constraint matrices are loosely supposed to be row sufficient. This scheme can be applied to the solving of convex quadratic programs subject to linear constraints and arbitrary upper and lower bound constraints on variables.

Keywords Box linear complementarity problem, Row sufficient, Principal pivoting
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§1. Introduction

Judice and Pires developed in recent years principal pivoting methods[9−11] for solving the so-called box linear complementarity problems (BLCPs) where the constraint matrices are restrictedly supposed to be of P–matrices. (A P–matrix is defined as a square matrix whose principal minors are all positive). The prototype of their methods is Murty's bard–type scheme[14,15], for linear complementarity problems (LCPs), which is known to be characterized in that its pivoting rule is free of minimum–ratio–test and runs mainly on the least–index principle[1−3]. Recently, Murty's scheme was greatly improved by Hertog, Roos and Terlaky in [8], where the constraint matrices of the LCPs concerned are loosely supposed to be sufficient. In [18], we gave out a unified extension of both of the results gained in [11] and [8], and produced a scheme that is free of minimum–ratio–test and is capable of dealing with BLCPs whose constraint matrices are loosely supposed to be sufficient[5,7,8,12,16]. In this paper we are going to present another principal pivoting scheme for BLCPs. Although this scheme is no longer free of minimum–ratio–test, it is capable of tackling BLCPs whose constraint matrices are more loosely supposed to be row sufficient (A square matrix M is defined to be row sufficient, if $(u^\top M)_i u_i \le 0$ for all i implies $(u^\top M)_i u_i = 0$ for every i; M is defined to be column sufficient, if $M^\top$ is row sufficient; and M is defined to be sufficient if it is both row and column sufficient. P–matrices and positive semi–definite matrices are all examples of sufficient matrices[5,6]). Computational tests on this scheme demonstrate that the computational behaviour of this scheme is far superior to that of the scheme proposed in [18].

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Cottle indicated in [5], while revisiting his principal pivoting method[4] in the light of row sufficiency, that Lemke's scheme I for LCPs (see [13]) remains valid in the context of row sufficiency. As is known that Lemke's scheme employs an artificial variable so as to render every pair of complementary variables "in kilter" at the very beginning (and this is so kept throughout). However, this scheme is not, in appearence, a principal pivoting one, because in each iteration there is always a pair of complementary variables which are both to be non–basic. Since the artificial variable appears to be basic most of the time, it varies "passively" rather than "actively" when updating. Similar to Lemke's scheme, an artificial variable is also employed in our scheme for the same purpose. However, ours is based on the principal pivoting approach, i.e., in each iteration every pair of complementary variables is kept to be a pair of basic and nonbasic variables. Therefore, the artificial variable remains invariably nonbasic, so it varies "actively" rather than "passively" when updating.

According to [11], a BLCP can be formulated as such a problem: Find out $x, y \in R^N$ satisfying

$$\begin{cases} y = Mx + q, \\ a \le x \le b, \\ y_i > 0 \implies x_i = a_i \ (\neq -\infty), \\ y_i < 0 \implies x_i = b_i \ (\neq +\infty), \end{cases}$$

where $i \in N = \{1, 2, \dots, n\}, q \in R^N, a \in (R \cup \{-\infty\})^N, b \in (R \cup \{+\infty\})^N, a < b$, and $M = [m_{ij}] \in R^{N \times N}$. Clearly, if $a_i = 0, b_i = +\infty$ for every i, then the BLCP turns out specifically an LCP, i.e.,

$$\begin{cases} y = Mx + q, \\ x \ge 0, \quad y \ge 0, \\ x^\top y = 0. \end{cases}$$

In this paper, a BLCP is somehow formulated generally as such: Find out $x, y \in R^N$ satisfying

$$\begin{cases} w = Mz + q, \\ a \le x \le b, \\ y_i > c_i \implies x_i = a_i \ (\neq -\infty), \\ y_i < c_i \implies x_i = b_i \ (\neq +\infty), \end{cases}$$

where $c \in R^N, a \le b$ (instead of $a < b$), w_i and z_i represent respectively either y_i and x_i, or x_i and y_i (instead of y_i and x_i invariably). Clearly, if M is specifically supposed to be a P–matrix, then the two definitions given above are basically the same. In the following, the above linear system $w = Mz + q$ is to be called specifically a principal dictionary (PD) of the BLCP; and variables, if locating in $w(z)$, are to be called basic (nonbasic). For an $i \in N, x_i$ and y_i are called complementary variables, and x_i and y_i are called in kilter, if $y_i = c_i$ and $a_i \le x_i \le b_i$, or $y_i > c_i$ and $x_i = a_i(\neq -\infty)$, or $y_i < c_i$ and $x_i = b_i(\neq +\infty)$. Obviously, (x, y) forms a solution to the BLCP iff for every $i \in N, x_i$ and y_i are in kilter. In the following discussion, principal dictionaries are invariably denoted by $w = Mz + q$ for the sake of notational simplicity, although the contents of w and z , as well as that of M and q, could be renewed due to updating.

It can be shown (say by the Kuhn–Tucker conditions) that the following box-type convex quadratic program

$$\begin{aligned} \text{minimize} \quad & -c^\top x + d^\top x_{B^*} + \frac{1}{2} x_{B^*}^\top G x_{B^*} + \frac{1}{2} y_B^\top H y_B, \\ \text{subjuct to} \quad & x_B = H y_B - A x_{B^*} + e, \\ & a \le x \le b, \end{aligned}$$

(where B stands for a certain subset of N, $B^* = N - B, d \in R^{B^*}, e \in R^B, A \in R^{B \times B^*}$, and both $G \in R^{B^* \times B^*}$ and $H \in R^{B \times B}$ are symmetric and positive semi–definite) can be reduced to a BLCP, where w, z, q, and M are respectively

$$\begin{pmatrix} x_B \\ y_{B^*} \end{pmatrix}, \quad \begin{pmatrix} y_B \\ x_{B^*} \end{pmatrix}, \quad \begin{pmatrix} e \\ d \end{pmatrix}, \quad \begin{pmatrix} H & -A \\ A^\top & G \end{pmatrix}.$$

Since M is now positive semi–definite hence row sufficient, this BLCP can be processed by our scheme. However, quadratic program in this form cannot be handled by methods given in [9, 10, 11], unless that $c = 0, G$ is symmetric and positive definite (hence a P–matrix in particular), and B is empty (hence H and A disappear, and $M = G$). This means that the "box-type" quadratic program concerned there has to be strictly convex and subject to no linear constraints[11,19].

The following two properties of row sufficient matrices are the only prerequisite required for our discussion. Since they were well proved in [5, 6], their proofs are omitted here in consideration of space-saving.

Property 1.1. *If a PD: $w = Mz + q$ is transformed equivalently to another principal dictionary, then the M of the transformed one remains row sufficient if the original is to be row sufficient.*

Property 1.2. *If $M = [m_{ij}]$ is row sufficient, then $m_{ii} = 0$ for a certain i entails that $m_{ij} < 0 (> 0)$ implies $m_{ji} > 0 (< 0)$.*

§2. The Scheme

A variable x_i is called fixed if $a_i = b_i$, and a variable y_i is called to be fixed if both a_i and b_i are infinite. Obviously, in any of the solutions to the BLCP, the fixed variables are always fixedly valued i.e. for a fixed x_i, it has to be valued in $a_i (= b_i)$; and for a fixed y_i, it has to be valued in c_i. In PDs, fixed variables, if appearing basic and depending on nonbasic variables that are nonfixed, are often "trouble–making" ones in the sense that they might cause degeneracy which none of the standard perturbation techniques can deal with. In order to render a given PD free from such variables, we propose an "improving" procedure as follows:

If the given PD is free from such variables then stop; otherwise, let $w_{\tilde{i}}$ be a fixed variable such that $m_{\tilde{i}\tilde{j}} \neq 0$ and $z_{\tilde{j}}$ is nonfixed. Now if $m_{\tilde{i}\tilde{i}} \neq 0$ holds, then update the PD by pivoting on entry $m_{\tilde{i}\tilde{i}}$ and go back to repeat this procedure again; otherwise (by Properties 1.1 and 1.2, $m_{\tilde{j}\tilde{i}}$ must be nonzero now), update the PD by pivoting respectively on entries $m_{\tilde{i}\tilde{j}}$ and $m_{\tilde{j}\tilde{i}}$ (this is valid because $m_{\tilde{i}\tilde{i}} = 0, m_{\tilde{i}\tilde{j}} \neq 0$, and $m_{\tilde{j}\tilde{i}} \neq 0$ hold now), and go back to repeat this procedure again.

Since each updating of the PD results in reducing the number of the "trouble–making" variables at least by one, the above procedure is finite. With this procedure at hand, we can assume without lass of generality that the following scheme starts off with an initial PD that is free from "trouble–making" variables.

The scheme is as follows:

Step 0 (The Initial Step):

0.1. Set z as such: for $j \in N$, if $z_j = y_j$ then set $z_j := c_j$; if $z_j = x_j$ then set $z_j := a_j$ (if finite) or b_j (if finite), or any preset value (say zero) in case that both a_j and b_j are infinite. Now set accordingly $w := Mz + q$. If there is a fixed basic variable that is not valued as wanted, then the problem is insolvable (because the fixed basic variables are dependent only on fixed nonbasic variables now), stop.

0.2. If every pair of complementary variables is now in kilter, then the current (x, y) forms a solution to the problem, stop; otherwise, add an additional item tp onto the right hand side of the PD, where $p \in R^N$ and t (the so–called artificial variable) are so rendered initially that every pair of complementary variables turns out in kilter with t valued positive.

Step 1 (The Major Step):

Let t decrease inasmuch as every pair of complementary variables is kept in kilter. If t can thus decrease to zero, then obviously the current pair of (x, y) forms a solution to the problem, stop; otherwise, some of the basic variables say $w_{\tilde{i}}$ would "block" the decrease of t, i.e., further decrease of t would cause $x_{\tilde{i}}$ and $y_{\tilde{i}}$ out of kilter.

(now: (i) such a variable is called a blocking variable;

(ii) if the net increment of t is zero, then it is said that degeneracy occurs;

(iii) the current $(x^\top, y^\top, t)^\top$ will be referred to in our proof of Theorem 1 below as a major–step–related vector).

If $m_{\tilde{i}\tilde{i}} \neq 0$, then update the current PD by pivoting on entry $m_{\tilde{i}\tilde{i}}$, and go back to Step 1; otherwise, set $z_{\tilde{i}}$ in a mode of "drive" that it gets ready to increase (decrease) if $z_{\tilde{i}} = x_{\tilde{i}} = a_{\tilde{i}}(b_{\tilde{i}})$ or $w_{\tilde{i}} = x_{\tilde{i}} = a_{\tilde{i}}(b_{\tilde{i}})$ holds currently.

Step 2 (The Transitional Step):

2.1. Let $z_{\tilde{i}}$ "drive" according to the mode set afore inasmuch as every pair of complementary variables is kept in kilter. If $z_{\tilde{i}}$ can thus drive infinitely, then the problem is insolvable (see to Theorem 2 below for the proof), stop; otherwise, either some of the basic variables, say $w_{\tilde{j}}$, or $z_{\tilde{i}}$ itself, would "block" the drive of $z_{\tilde{i}}$, i.e., further drive of $z_{\tilde{i}}$ would cause $x_{\tilde{j}}$ and $y_{\tilde{j}}$, or $x_{\tilde{i}}$ and $y_{\tilde{i}}$, out of kilter.

(now:(i) such a variable is also called a blocking variable;

(ii) if the net increment of $z_{\tilde{i}}$ is zero, then it is said that degeneracy occurs;

(iii) the current $(x^\top, y^\top, t)^\top$ is referred to in our proof of Theorem 1 below as a transitional-step-related vector).

If the blocking variable is $z_{\tilde{i}}$, then go back to Step 1; otherwise, let $w_{\tilde{j}}$ be a blocking variable (hence $m_{\tilde{j}\tilde{i}} \neq 0$ must hold) and go on to the following substep.

2.2. If $m_{\tilde{i}\tilde{j}} = 0$ holds currently (hence $m_{\tilde{j}\tilde{j}} \neq 0$ must be true, because from Properties 1.1 and 1.2, $m_{\tilde{j}\tilde{j}} = 0$ and $m_{\tilde{j}\tilde{i}} \neq 0$ would lead to $m_{\tilde{i}\tilde{j}} \neq 0$), then update the current PD by pivoting on entry $m_{\tilde{j}\tilde{j}}$ and go back to Step 2; otherwise, update the current PD by pivoting respectively on entries $m_{\tilde{i}\tilde{j}}$ and $m_{\tilde{j}\tilde{i}}$ (this is valid because $m_{\tilde{i}\tilde{i}} = 0$, $m_{\tilde{i}\tilde{j}} \neq 0$ and $m_{\tilde{j}\tilde{i}} \neq 0$ hold now), and go back to Step 1.

Remark 2.1. In the case that the constraint matrix $M = [m_{ij}]$ is specifically a P–matrix, now, according to [5, 17], M remains a P–matrix in any of the updated PDs during operation of the scheme. Therefore every m_{ii} remains positive throughout. This means that the operation can never reach Step 2, and must end up with a solution obtained.

Theorem 2.1. *The scheme is finite, if nondegeneracy assumption is to be in force.*

Proof. Suppose that the scheme operates endlessly. Then the operation either passes Step 1 in infinite times or cycles inside Step 2. From this and the nondegeneracy assumption it can be derived that if the former (latter) happens, there must be two major (transitional)–step–related vectors which are basically identical except that the values of t ($z_{\tilde{i}}$) are different. This, however, is contradictory to that $w_{\tilde{i}}(w_{\tilde{j}})$ plays a role of blocking variable.

Remark 2.2. It is not difficult to specify that once the initial PD is "improved" by means of the "improving" procedure proposed afore, all updated PDs would remain "improved". Therefore, with this scheme, the fixed basic variables cannot happen to be blocking ones. Hence, any standard perturbation techniques are applicable to this scheme for securing nondegeneracy.

Remark 2.3. We have discovered additionally. Suppose that the given constraint matrix is sufficient, then, to secure finiteness of the scheme, the nondegeneracy assumption can be replaced by the least–index rule[1,2,3] (i.e., the candidate blocking variable is chosen to be the least–indexed among all the then blocking variables) in so far as that a sufficient number of artificial variables are introduced, that is: one artificial variable (together with a vector possessing a single non–zero component), for each pair of complementary variables that is initially out of kilter. Furthermore, if the given constraint matrix is to be particularly positive semi–definite, then only one artificial variable (as is described in Step 0.2), is required. Since the proof of this argument is technically lengthy, we cannot but drop it here out of simplicity consideration.

Theorem 2.2. *The scheme is valid.*

Proof. This is mainly to prove that if the operation ends up at Step 2.1, then the problem is insolvable. Suppose this is not true. Then let $x^{'}$ and $y^{'}$ be a solution to this problem. Now, let the then t decrease to zero and denote the corresponding x and y by $x^{''}$ and $y^{''}$. First, since the then $w_{\tilde{i}}$ plays a role of blocking variable, when t reaches zero it must happen that either

(i) $w_{\tilde{i}} = x^{''}_{\tilde{i}} < a_{\tilde{i}}$ ($b_{\tilde{i}}$),

or

(ii) $w_{\tilde{i}} = y^{''}_{\tilde{i}} < c_i$ ($> c_i$) and $z_{\tilde{i}} = x_{\tilde{i}} = a_{\tilde{i}}$ ($b_{\tilde{i}}$).

Secondly, since the then $z_{\tilde{i}}$ can drive infinitely, this implies that if $m_{j\tilde{i}} > 0$ (< 0) then either $z_j = x^{''}_j = a_j$ (b_j) and $w_j = y^{''}_j > c_j$ ($< c_j$), or

$$z_j = y_j = c_j \text{ and } b_j = +\infty \ (a_j = -\infty).$$

By referring to Properties 1.1 and 1.2, it can be concluded that if $m_{\tilde{i}j} < 0$ (> 0) then either $z_j = x^{''}_j = a_j$ (b_j) and $w_j = y^{''}_j > c_j$ ($< c_j$), or

$$z_j = y_j = c_j \quad \text{and} \quad b_j = +\infty \ (a_j = -\infty).$$

By taking notice to the fact that $(x^{'}, y^{'})$, and $(x^{''}, y^{''})$ as well, must satisfy the $\widetilde{i}$-th equation of the PD then concerned (with $t = 0$), it follows that if the above mentioned case (i) happens, then

$$x'_{\tilde{i}} \leq x''_{\tilde{i}} < a_{\tilde{i}} \ (x'_{\tilde{i}} \geq x''_{\tilde{i}} > b_{\tilde{i}}),$$

and if the above mentioned case (ii) happens, then

$$y'_{\tilde{i}} \leq y''_{\tilde{i}} < c_{\tilde{i}} \ \text{ and } \ b_{\tilde{i}} = +\infty \ \ (y'_{\tilde{i}} \geq y''_{\tilde{i}} > c_{\tilde{i}} \ \text{ and } \ a_{\tilde{i}} = -\infty).$$

This, however, is contradictory to the supposition that (x', y') forms a solution to the problem.

References

[1] Bland, R. G., New finite pivoting rules for simplex method, *Math.Oper.Res.,* **2** (1977), 103–107.

[2] Chang, Y. Y., Least index resolution of degeneracy in linear complementarity problems, *Technical Report,* **79–14** (1979), Dept. of Operations Research, Stanford Univ..

[3] Chang, Y. Y. & Cottle, R. W., Least–index resolution of degeneracy in quadratic programming, *Math. Programming,* **18** (1980), 127–137.

[4] Cottle, R. W., The principal pivoting method of quadratic programming, Mathematics of Decision Sciences, Part 1 (Dantzig, C. B. & Veinott, A. F., eds.), Amer. Math. Soc., Providence, 1968, pp. 142–162.

[5] Cottle, R. W., The pricipal pivoting method revisited, *Math. Programming,* **48** (1990), 369–386.

[6] Cottle, R. W., Pang, J. S. & Venkateswaran, V., Sufficient matrices and the linear complementarity problem, *Linear Algebra Appl.,* **114/115** (1989), 231–249.

[7] Fukuda, K. & Terlaky, T., Linear complementarity and oriented methods, *J. Oper. Res. Society of Japan,* **35:** 1 (1992), 45–61.

[8] Hertog, D. D., Roos, C. & Terlaky, T., The linear complementarity problem, sufficient matrices, and the criss–cross method, *Linear Algebra Appl.,* **187** (1993), 1–14.

[9] Judice, J. J. & Pires, F. M., A bard–type method for a generalized linear complementarity problem with a nonsingular M–matrix, *Naval Research Logistics Quarterly,* **37** (1990), 279–297.

[10] Judice, J. J. & Pires, F. M., Direct methods for convex quadratic programs subject to box constraints, *Investigacao Operational,* **9** (1989), 23–56.

[11] Judice, J. J. & Pires, F. M., Basic–set algorithm for a generalized linear complementarity problem, *JOTA,* **74:**3 (1992), 391–411.

[12] Klafszky, E. & Terlaky, T., Some generalization of the criss–cross method for linear complementarity problem of oriented matroids, *Combinatorica,* **9:**2 (1989), 189–198.

[13] Lemke, C. E., Bimatrix equilibrium points and mathematical programming, *Management Sci.,* **11** (1965), 681–689.

[14] Murty, K. G., A note on a bard–type scheme for solving the complementarity problem, *Opsearch,* **11:**2–3 (1974), 123–130.

[15] Murty, K. G., Linear complementarity, linear and nonlinear programming, *Heldermann–Verlag,*, Berlin, Germany, 1988.

[16] Terlaky, T. & Wang, Z. M., A general scheme for solving linear complementarity problems in the setting of oriented matroids, Proc. of SSICC'92 (Yap, H. P., Ku, T. H., Lloyd, E. K. & Wang, Z. M., eds.), World Scientific, Singapore, 1993, 244–255.

[17] Tucker, A. W., Principal pivotal transforms of square matrices, *SIAM Review,* **5** (1963).

[18] Wang, Z. M., A extended criss–cross method for solving box linear complementarity problems, to appear in *JOTA*.

[19] Yang, E. K. & Totte, J. W., A class of methods for solving large, convex quadratic programs subject to box constraints, *Math. Programming,* **51:**2 (1991), 223–228.