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WEAK CONVERGENCE FOR NON-UNIFORM φ-MIXING RANDOM FIELDS**

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Abstract

Let $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbf{Z}^d\}$ be a nonuniform φ-mixing strictly stationary real random field with $E\xi_{\mathbf{0}} = 0, E|\xi_{\mathbf{0}}|^{2+\delta} < \infty$ for some $0 < \delta < 1$. A sufficient condition is given for the sequence of partial sum set-indexed process $\{Z_n(A),\ A \in \mathcal{A}\}$ to converge to Brownian motion. By a direct calculation, the author shows that the result holds for a more general class of set index $\mathcal{A}$, where $\mathcal{A}$ is assumed only to have the metric entropy exponent $r, 0 < r < 1$, and the rate of nonuniform φ-mixing is weakened. The result obtained essentially improve those given by Chen[1] and Goldie, Greenwood[6], etc.

Keywords Weak convergence of partial-sum processes, Set-indexed process, Nonuniform φ-mixing, Random fields

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§1. Introduction

Let $\mathbf{Z}$ be the set of all integers, $\mathbf{N}$ be the set of all positive integers and $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbf{Z}^d\}$ be a strictly stationary real random field on a d-dimensional integer lattice . Let $\mathcal{A}$ be a totally bounded subset of the class $\mathcal{B}^d$ of Borel subsets of $I^d = [0,1]^d$ with the metric $d_L(A,B) = |A \triangle B|$, its closure $\overline{\mathcal{A}}$ is complete and totally bounded, hence compact. For $m \in \mathbf{N}$,

$$J_m = \{(l_1/m, \cdots, l_d/m), l_j \in \mathbf{N}, 1 \le l_j \le m, j = 1, \cdots, d\},$$

$$C_{m,\mathbf{j}} = \prod_{i=1}^{d} (j_i - 1/m, j_i], \qquad \mathbf{j} = (j_1, \cdots, j_d) \in J_m.$$

From the random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbf{Z}^d\}$, we define the partial-sum process of n-th level as

$$Z_n(A) = n^{-d/2} \sum_{\mathbf{j} \in J_n} \frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} (\xi_{n\mathbf{j}} - E\xi_{n\mathbf{j}}) \quad A \in \mathcal{A},\ n \in \mathbf{N}, \tag{1.1}$$

where $n\mathbf{j} = (nj_1, \cdots, nj_d), |\cdot|$ is the Lebesgue measure. For sets $E, F \subseteq R^d$ the separation distance is

$$\rho(E,F) = \inf_{\mathbf{x} \in E, \mathbf{y} \in F} \|\mathbf{x} - \mathbf{y}\| = \inf_{\mathbf{x} \in E, \mathbf{y} \in F} \max_{1 \le i \le d} |x_i - y_i|.$$

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Let $C(\overline{\mathcal{A}})$ be the space of continuous real-valued functions on $\overline{\mathcal{A}}$ with the supnorm $\|\cdot\|$, i.e., $\|f\|_{\overline{\mathcal{A}}} = \sup\limits_{A\in\overline{\mathcal{A}}} |f(A)|$. Let $CA(\overline{\mathcal{A}})$ be the set of everywhere additive elements of $C(\overline{\mathcal{A}})$, i.e., elements f such that $f(A\cup B) = f(A) + f(B) - f(A\cap B)$ whenever $A, B, A\cup B, A\cap B \in \overline{\mathcal{A}}$. Suppose that $\mathcal{A}$ satisfies a metric entropy condition with exponent r, i.e., for every $\varepsilon > 0$ there is a finite family $\mathcal{N}(\mathcal{A},\varepsilon) \subset \mathcal{B}^d$, which we take to have minimal cardinality $\exp(H(\varepsilon))$, such that for every $A \in \mathcal{A}$ there exist $A_-, A^+ \in \mathcal{N}(\mathcal{A},\varepsilon)$ with $A_- \subseteq A \subseteq A^+$ and $|A^+ \setminus A_-| \le \varepsilon$, and the function $H(\cdot)$ is called the metric entropy (with inclusion), its exponent is $H(\varepsilon) = O(\varepsilon^{-r})$.

Definition 1.1 *The random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ is said to be φ-mixing, if*

$$\varphi(x) = \sup_{\substack{\mathbf{I},J\subset R^d\\ \rho(\mathbf{I},J>x}} \sup_{\substack{E\in\sigma(\xi_{\mathbf{t}},\mathbf{t}\in I),P(E)>0\\ F\in\sigma(\xi_{\mathbf{s}},\mathbf{s}\in I),P(F)>0}} \max(|P(E|F) - P(E)|, |P(F|E) - P(F)|)$$

and $\varphi(x) \longrightarrow 0$ as $x \longrightarrow \infty$. The random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbf{Z}^d\}$ is said to be nonuniform φ-mixing, if for $\Lambda_i \subset \mathbf{Z}^d, |\Lambda_i| < \infty, i = 1,2$, there exists a nonnegative real function $\varphi_{|\Lambda_1|}(\cdot)$ depending only on $|\Lambda_1|$ such that

$$\sup_{E\in\sigma(\Lambda_1),F\in\sigma(\Lambda_2),P(F)>0} |P(E|F) - P(E)| \le \varphi_{|\Lambda_1|}(\rho(\Lambda_1,\Lambda_2))$$

and $\varphi_{|\Lambda_1|}(x) \to 0$ as $x \to \infty$, where $|\Lambda|$ is the cardinality of Λ.

Dobrushin[3] showed that the φ-mixing condition is not satisfied even for some simple examples of Gibbs random fields. Dobrushin and Nahapetian[4] introduced the nonuniform φ-mixing condition. Chen[1] gave a sufficient condition for a sequence of partial-sum set-indexed processes with nonuniform φ-mixing condition to converge to Brownian motion, when the indexed set $\mathcal{A} = \mathcal{C} = \{(\mathbf{a}, b], \mathbf{a}, b \in [0,1]^d\}$, as follows.

Theorem C. *Let $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ be a strictly stationary nonuniform φ-mixing random field and satisfy*

(i) *there exists a non-negative real function $\varphi(\cdot)$ on R^1, such that for any $\Lambda \subset \mathbf{Z}^d$, $|\Lambda| < \infty$, $\varphi_{|\Lambda|}(\cdot) \le |\Lambda|\varphi(\cdot)$, and for some $\delta > 0$*

$$\lim_{r\to\infty} \sup_{r\in R^+} (\varphi(r))^{1/2} r^{3d+4d/\delta} < \infty, \tag{1.2}$$

(ii) $E\xi_{\mathbf{0}} = 0$, $E|\xi_{\mathbf{0}}|^{2+\delta} < \infty$, *where* $\mathbf{0} = (0, \cdots, 0)$,

(iii)

$$0 < \sigma^2 := \sum_{\mathbf{t}\in\mathbf{Z}^d} \mathrm{Cov}(\xi_{\mathbf{0}}, \xi_{\mathbf{t}}) < \infty. \tag{1.3}$$

Then Z_n converges weakly in $CA(\mathcal{C})$ to a Brownian motion with parameter σ, as $n \to \infty$.

In this paper, we improve Theorem C by a direct calculation, and prove that the conclusion holds for the more general indexed set $\mathcal{A}$, where $\mathcal{A}$ is assumed only to have the metric entropy exponent $r, 0 < r < 1$, and the rate of nonuniform φ-mixing is weakened.

Theorem 1.1. *Let $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ be a strictly stationary nonuniform φ-mixing random field, and satisfy*

(i) $\varphi_{|\Lambda|}(\cdot) \le |\Lambda|\varphi(\cdot)$ *and* $\varphi(x) = O(x^{-2d-1-2d/\delta})$ *for some* $\delta > 0$,

(ii) $E\xi_{\mathbf{0}} = 0, E|\xi_{\mathbf{0}}|^{2+\delta} < \infty$,

(iii) $\mathcal{A}$ *has exponent of metric entropy (with inclusion)* $r < 1$.

Then Z_n converges weakly in $CA(\mathcal{A})$ to a Brownian motion with parameter σ, as $n \to \infty$, where σ^2 is defined as (1.3).

Remark 1.1. From Theorem 1.1 it follows that a uniform central limit theorem for certain Gibba fields which has been given in [1] is also true for indexed set $\mathcal{A}$, if $\mathcal{A}$ has exponent of metric entropy $r < 1$.

§2. Finite-Dimensional Convergence

For the finite dimensional distributions of $Z_n(\cdot)$ to converge to the corresponding finite dimensional distributions of Brownian motion, we have

Theorem 2.1. *Let $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ be a strictly staionary nonuniform φ-mixing random field with $E\xi_{\mathbf{0}} = 0, E|\xi_{\mathbf{0}}|^{2+\delta} < \infty$ for some $\delta > 0$. The set-indexed partial-sum processes $\{Z_n(A), A \in \mathcal{B}^d\}$ are defined in* (1.1)*. Suppose that*

$$\varphi_{|\Lambda|}(\cdot) \leq |\Lambda|\varphi(\cdot), \varphi(x) = O(x^{-2(d+\varepsilon)}), \qquad \text{for some} \quad \varepsilon > 0.$$

Then the finite dimensional distributions of $\{Z_n\}$ on $\mathcal{B}^d$ converge to the corresponding finite dimensional distributions of Brownian motion with parameter σ, where $0 < \sigma^2 = \sum\limits_{\mathbf{t} \in \mathbf{Z}^d} E\xi_{\mathbf{0}}\xi_{\mathbf{t}} < \infty$.

The proof of Theorem 2.1 needs the following lemmas.

Lemma 2.1. *Let $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$, $\{Z_n(A), A \in \mathcal{B}^d\}$ be as in Theorem* 2.1*. Suppose that*

$$\varphi_{|\Lambda|}(\cdot) \leq |\Lambda|\varphi(\cdot) \quad \text{and} \quad \varphi(x) = O(x^{-(d+1+\theta)}) \quad \text{for some} \quad \theta > 0.$$

Then for any $A \in \mathcal{B}^d$

$$\|Z_n(A)\|_{2+\delta} \leq c\sigma_0|nA|^{1/2}, \tag{2.1}$$

where $\sigma_0^2 = E\xi_{\mathbf{0}}^2$.

Proof. First, imitating the proof of Theorem 3.1 of [5], we prove that for $\delta = 0$

$$\sigma(2m) \leq 2^{1/2}(1 + 2m^{1/2}\varphi^{1/2}(d^{-1/2}m^p - 2))^{1/2}\sigma(m) + 3\overline{\sigma}(cm^s)$$

where $\sigma(m) = \sup\limits_{A \in \mathcal{B}^d, |A|=m} \|Z_n(A)\|_2$, $\overline{\sigma}(m) = \sup\limits_{m' \leq m} \sigma(m')$, take $0 < p < 1/d$ in the bisection lemma to be such that the exponent $s = (q + pd)/(q+1)$ does not have any positive power equal to $1/2$. For any given $h > 1$, we write $h = 2^k m, k \in \mathbf{N}, 1/2 < m \leq 1$, note that $\sigma(m) \leq 1$, and for nonuniform φ-mixing we have

$$\sigma(2^{j+1}m) \leq \alpha_j\sigma(2^j m) + \beta_j,$$

where

$$\alpha_j = 2^{1/2}(1 + 2 \cdot 2^{j/2}m^{j/2}\varphi(d^{-1/2}2^{jp}m^p - 2))^{1/2},$$
$$\beta_j = 3\,\overline{\sigma}(c2^{js}).$$

By iteration,

$$\sigma(h) \leq \prod_{j=0}^{k-1}\alpha_j + \sum_{j=0}^{k-1}\beta_j\prod_{i=j+1}^{k-1}\alpha_i. \tag{2.2}$$

Furthermore, we take $p, \frac{1}{d} > p > \frac{1}{d+1}$, such that $s = \frac{q+pd}{q+1}$ does not also have any positive power equal to 1/2. There exists a $j_0 \geq 1$ such that $d^{1/2}2^{jp}2^{-p} > 2^{j/(d+1)}$ for $j \geq j_0$, so

$$\begin{aligned}
\varphi &:= \sum_{j=j_0}^{\infty} 2^{j/2}\varphi^{1/2}(d^{-1/2}2^{jp}m^p - 2) \\
&\leq \sum_{j=1}^{\infty} 2^{j/2}\varphi^{1/2}(2^{j/(d+1)}) \\
&\leq c\sum_{j=1}^{\infty} 2^{\frac{j}{2}}2^{-\frac{1}{2}\cdot\frac{j}{d+1}(d+1+\theta)} \\
&= c\sum_{j=1}^{\infty}\Big(2^{\theta/(2(d+1))}\Big)^{-j} < \infty.
\end{aligned}$$

Thus we obtain

$$\sigma(h) \leq ce^{c\varphi}\Big(2^{\frac{k}{2}} + 3\sum_{j=0}^{k-1} 2^{\frac{k-j-1}{2}}\overline{\sigma}(c2^{js})\Big).$$

The remainder of the proof is the same as in the proof of Theorem 3.1 of [5]. This proves that (2.1) holds for $\delta = 0$.

For the case of $0 < \delta \leq 1$. By using

$$|1+x|^{2+\delta} \leq 1 + 9|x| + 9|x|^{1+\delta} + |x|^{2+\delta} \tag{2.3}$$

and the notation of Theorem 3.1 of [5]

$$Z_n(A) = Z_n(A''_+) + Z_n(A''_-) - Z_n(A'_+) - Z_n(A'_-) + Z_n(A \cap S), \tag{2.4}$$

where $|A| = 2m, S$ is a slice of A, $|A''_+| = |A''_-| = m$, A''_+ and A''_- are situated on the different sides of S, the separation distance $\rho(A''_+, A''_-) \geq d^{-1/2}m^p, |A'_+|, |A'_-|$, and $A \cup S$ do not exceed cm^s. Denote

$$\tau(h) = \sup_{|A|=h, A\in\mathcal{B}^d} \|Z_n(A)\|_{2+\delta}, \quad \overline{\tau}(h) = \sup_{h'\leq h}\tau(h').$$

From (2.4)

$$\tau(2m) \leq \|Z_n(A''_+) + Z_n(A''_-)\|_{2+\delta} + 3\overline{\tau}(cm^s). \tag{2.5}$$

By (2.3) we have

$$\begin{aligned}
&E|Z_n(A''_+) + Z_n(A''_-)|^{2+\delta} \\
&\leq 2\tau^{2+\delta}(m) + 9(E|Z_n(A''_+)||Z_n(A''_-)|^{1+\delta} + E|Z_n(A''_+)|^{1+\delta}|Z_n(A''_-)|).
\end{aligned} \tag{2.6}$$

By the property of nonuniform φ-mixing, we have

$$\begin{aligned}
&E|Z_n(A''_+)||Z_n(A''_-)|^{1+\delta} \\
&\leq E|Z_n(A''_+)|E|Z_n(A''_-)|^{1+\delta} + 2\varphi_{|A''_+|}^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\tau^{2+\delta}(m) \\
&\leq \Big(1 + 2m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\Big)\tau^{2+\delta}(m).
\end{aligned} \tag{2.7}$$

Similarly we have

$$\begin{aligned}
&E|Z_n(A''_+)|^{1+\delta}|Z_n(A''_-)| \\
&\leq \Big(1 + 2m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\Big)\tau^{2+\delta}(m).
\end{aligned} \tag{2.8}$$

Inserting (2.7), (2.8) into (2.6) yields

$$\|Z_n(A''_+) + Z_n(A''_-)\|_{2+\delta} \le c\Big(1 + m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\Big)^{\frac{1}{2+\delta}}\tau(m),$$

whence

$$\tau(2m) \le c\Big(1 + m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\Big)^{\frac{1}{2+\delta}}\tau(m) + 3c\overline{\tau}(cm^s).$$

By iteration, for $h = 2^k m, k \in \mathbf{N}, 1/2 < m \le 1$ we have

$$\tau(h) \le \prod_{j=0}^{k-1}\alpha_j + \sum_{j=0}^{k-1}\beta_j\prod_{i=j+1}^{k-1}\alpha_i,$$

where

$$\alpha_j \le c\Big\{1 + 2^{\frac{j}{2+\delta}}m^{\frac{j}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}2^{jp}m^p - 2)\Big\}^{\frac{1}{2+\delta}}.$$

So

$$\begin{aligned}\varphi' &= \sum_{j=j_0}^{\infty} 2^{\frac{j}{2+\delta}}\varphi^{\frac{1}{2+\delta}}\Big(d^{-\frac{1}{2}}2^{jp}m^p - 2\Big)\\ &\le c\sum_{j=1}^{\infty}\Big(2^{\varepsilon/(2(2+\delta)(d+1)}\Big)^{-j} < \infty.\end{aligned}$$

The remainder of the proof is the same as in the case of $\delta = 0$. The proof of Lemma 2.1 is completed.

Lemma 2.2. *Let the $\xi_{n,\mathbf{j}}$ and the partial-sum processes Z_n satisfy*

(i) $E\xi_{n,\mathbf{j}} = 0 \quad \forall n, \mathbf{j}$,

(ii) *the set $\{\xi_{n,\mathbf{j}}, \mathbf{j} \in J_n, n \ge 1\}$ is uniformly integrable,*

(iii) $\sup_n \sum_{j=1}^{\infty} \varphi^{1/4}(2^j) < \infty$,

(iv) $0 < \sigma^2 = \sum_{i \in Z^d} E\xi_{\mathbf{0}}\xi_{\mathbf{i}} < \infty$,

(V) $EZ_n^2(C) \longrightarrow |C| (n \to \infty)$ *for any $C \in \mathcal{B}^d$.*

Then the finite dimensional distributions of $\{Z_n\}$, on $\mathcal{B}^d$, converge to the corresponding finite dimensional distribution of Brownian motion with parameter σ.

This is Theorem 4.1 of [5].

Proof of Theorem 2.1. Without loss of generality, we assume $\sigma^2 = 1$. Put $\xi_{n,\mathbf{j}} = n^{-d/2}\xi_{n\mathbf{j}}$. It is clear that the conditions (i), (ii) of Lemma 2.2 are satisfied. By the strict stationarity of $\xi_{\mathbf{t}}$, the condition (iii) of Lemma 2.2 ia also satisfied. To check (iv) and (v) of Lemma 2.2, we have

$$EZ_n^2(A) = n^{-d}\sum_{\mathbf{j}\in J_n}\frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}\sum_{\mathbf{k}\in J_n}\frac{|A \cap C_{n,\mathbf{k}}|}{|C_{n,\mathbf{k}}|}E\xi_{n\mathbf{j}}\xi_{n\mathbf{k}}.$$

Denote $I(\mathbf{j}) = n(\mathbf{j} - 1/n, \mathbf{j}]$, $\mathbf{j} \in J_n, |I(\mathbf{j})| = 1$. Note that

$$\begin{aligned}\gamma(n\mathbf{j} - n\mathbf{k}) = E\xi_{n\mathbf{j}}\xi_{n\mathbf{k}} &\le 2\varphi^{1/2}_{|I(\mathbf{j})|}(|n\mathbf{j} - n\mathbf{k}|)\sigma_0^2\\ &\le 2\varphi^{1/2}(|n\mathbf{j} - n\mathbf{k}|)\sigma_0^2.\end{aligned}$$

Since

$$\sum_{\mathbf{k}\in\mathbf{Z}^d}\varphi^{1/2}(|\mathbf{k}|)=\sum_{r=1}^{\infty}\sum_{2^{r-1}\le|\mathbf{k}|<2^r}\varphi^{1/2}(|\mathbf{k}|)$$
$$\le\sum_{r=1}^{\infty}c2^{(r-1)d}\varphi^{1/2}(2^{r-1})$$
$$\le c\sum_{r=1}^{\infty}\Big(2^{\varepsilon}\Big)^{-r}<\infty,$$

the condition (iv) of Lemma 2.2 is satisfied. At last, by the same discussion as in the proof of Corollary 1.4 of [5], the condition (v) of Lemma 2.2 is also satisfied.

§3. Proof of Theorem 1.1

We use the notation of [1]. From the proof of Lemma 3.1 of [1], now we need only prove

$$\lim_{\nu\downarrow 0}\limsup_{n\to\infty}P\Big\{\|Z_n(A\cap I_{n,i},0,a)\|_{\mathcal{A}_0}>\lambda\Big\}=0, \tag{3.1}$$

where $\mathcal{A}_0=\{A\backslash B: A,B\in\mathcal{A},|A\backslash B|\le\nu\},\ p_n=\Big[n^{\frac{2+\delta}{2(1+\delta)}}\Big]$,

$$Z_n(A\cap I_{n,i},0,a)=\sum_{\mathbf{l}\in J_{p_n}}\sum_{\mathbf{j}\in S(n,\mathbf{l},i)}\frac{|A\cap I_{n,\mathbf{l},i}\cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}(\eta_{n\mathbf{j}}(0,a)-E\eta_{n,\mathbf{j}}(0,a)$$
$$=\sum_{\mathbf{l}\in J_{p_n}}V_{n\mathbf{l}}(A\cap I_{n,i},0,a),$$
$$\eta_{n,\mathbf{j}}(0,a)=n^{-(d/2)}\xi_{n\mathbf{j}}I(n^{d\delta/(2(1+\delta))}n^{-d/2}|\xi_{n\mathbf{j}}|<a),\quad \mathbf{j}\in J_n,$$

where $S(n,\mathbf{l},i)=\{\mathbf{j}\in J_n: I_{n,\mathbf{l},i}\cap C_{n,\mathbf{j}}\ne\emptyset\}$. Denote $V_{n\mathbf{l}}=V_{n\mathbf{l}}(A\cap I_{n,i},0,a)$. By the property of nonuniform φ-mixing, we have

$$Ee^{\alpha\sum\limits_{\mathbf{l}\in J_{p_n}}V_{n\mathbf{l}}}\le Ee^{\alpha V_{n\mathbf{l}}}Ee^{\alpha\sum\limits_{\mathbf{k}\ne\mathbf{l},\mathbf{k}\in J_{p_n}}V_{n\mathbf{k}}}$$
$$+2\varphi_{|S(n,\mathbf{l},i)|}\Big(\frac{n}{2p_n}\Big)Ee^{\alpha\sum\limits_{\mathbf{k}\ne\mathbf{l}}V_{n\mathbf{k}}}\|V_{n\mathbf{l}}\|_{\infty}. \tag{3.2}$$

Note that $|V_{n\mathbf{l}}|\le 2a$ and $e^{\alpha V_{n\mathbf{l}}}\le 1+\alpha V_{n\mathbf{l}}+\alpha^2V_{n\mathbf{l}}^2$ when $\alpha a\le 1/4$. From Lemma 2.4 of [1] it follows that

$$EV_{n\mathbf{l}}^2\le C|A\cap I_{n\mathbf{l},i}|\qquad\text{for}\quad i=1,2,\cdots,2^d.$$

Therefore for $\alpha a\le 1/4$

$$Ee^{\alpha V_{n\mathbf{l}}}\le e^{E\alpha^2V_{n\mathbf{l}}^2}\le e^{C\alpha^2|A\cap I_{n\mathbf{l},i}|}. \tag{3.3}$$

Inserting (3.3) into (3.2) yields

$$Ee^{\alpha\sum\limits_{\mathbf{l}\in J_{p_n}}V_{n\mathbf{l}}}\le wEe^{\alpha\sum\limits_{\mathbf{k}\ne\mathbf{l},\ \mathbf{k}\in J_{p_n}}V_{n\mathbf{k}}},$$

where

$$w\le e^{C\alpha^2|A\cap I_{n\mathbf{l},i}|}\Big(1+2e^{1/2}|S(n,\mathbf{l},i)|\varphi\Big(\frac{n}{2p_n}\Big)\Big)$$
$$\le e^{C\alpha^2|A\cap I_{n\mathbf{l},i}|}\Big(1+2e^{1/2}\frac{n^d}{(2p_n)^d}\varphi\Big(\frac{n}{2p_n}\Big)\Big).$$

By iteration,

$$\begin{aligned} Ee^{\alpha \sum\limits_{1\in J_{p_n}} V_{n1}} &\le e^{C\alpha^2|A\cap I_{n,i}|}\Big(1+2e^{1/2}\frac{n^d}{(2p_n)^d}\varphi\Big(\frac{n}{2p_n}\Big)\Big)^{p_n^d} \\ &\le \exp\Big(c\alpha^2|A|+cn^d\Big(\frac{n}{2p_n}\Big)^{-2d-2d/\delta-1}\Big) \\ &= \exp\Big(c\alpha^2|A|+cn^{-\frac{\delta}{2(1+\delta)}}\Big) \\ &\le c\ \exp(\alpha^2|A|) \end{aligned} \tag{3.4}$$

for large n.

Now we return to the estimate of the left hand side of (3.1). Since $0\le r<1$, we can take $s>0$ such that $r<1/(1+s)$. Set

$$\begin{aligned} &\delta_j=\nu/2^j, && j=0,1,\cdots, \\ &\lambda_j=\lambda_0 e^{-j(1+r-r(2+s))/(2+s)}, && j=1,2,\cdots, \\ &\lambda_0=\lambda(1-e^{-(1+r-r(2+s))/(2+s)}), && \\ &a_j=e^{-j(1+r)/(2+s)}a, && j=1,2,\cdots, \\ &a=c\nu^{1/(1+\delta)}, && c=(6\tau/\lambda_0)^{1/(1+s)}, \end{aligned}$$

where $\tau=E|\xi_{\mathbf{0}}|^{2+\delta}$. For any $A\in\mathcal{A}_0$ there exist $A_j, A_j^+\in\mathcal{A}_0(\delta_j)$ such that $A_j\subseteq A\subseteq A_j^+$ and $|A_j^+\backslash A_j|\le\delta_j$. Then

$$\begin{aligned} &Z_n(A\cap I_{n,i},0,a) \\ &= Z_n(A_0\cap I_{n,i},0,a)+\sum_{j=0}^{\infty}\{Z_n(A_{j+1}\cap I_{n,i},0,a_j)-Z_n(A_j\cap I_{n,i},0,a_j)\} \\ &\quad+\sum_{j=0}^{\infty}\{Z_n(A\cap I_{n,i},a_j,a_{j-1})-Z_n(A_j\cap I_{n,i},a_j,a_{j-1})\}. \end{aligned}$$

So if $\|Z_n(\cdot\cap I_{n,i},0,a)\|_{\mathcal{A}_0}$ is to exceed λ, at least one of the following must hold:

(a) for some $A\in\mathcal{A}_0(\delta_0)$, $\quad|Z_n(A_0\cap I_{n,i},0,a)|>\lambda_0$;

(b) for some j, for some $A_j\in\mathcal{A}_0(\delta_j),\ A_{j+1}\in\mathcal{A}_0(\delta_{j+1})$,

$$|A_j\triangle A_{j+1}|\le 2\delta_j,$$

$$|Z_n(A_{j+1}\cap I_{n,i},0,a_j)-Z_n(A_j\cap I_{n,i},0,a_j)|>2\lambda_j;$$

(c) for some j, for some $A_j, A_j^+\in\mathcal{A}_0(\delta_j),\ A_j\subseteq A\subseteq A_j^+$,

$$|A_j^+\backslash A_j|\le\delta_j,$$

$$|Z_n(A\cap I_{n,i},a_j,a_{j-1})-Z_n(A_j\cap I_{n,i},a_j,a_{j-1})|>\lambda_j.$$

The number of pairs A_j, A_j^+ in $\mathcal{A}_0(\delta_j)$ is $\le\exp(4H(\delta_j/2))$, while the number of pairs $A_j\in\mathcal{A}_0(\delta_j),\ A_{j+1}\in\mathcal{A}_0(\delta_{j+1})$ is $\le\exp(4H(\delta_{j+1}/2))$. Since $H(x)=O(x^{-r})$ is nonincreasing, we have

$$P\{\|Z_n(\cdot\cap I_{n,i},0,a)\|_{\mathcal{A}_0}>\lambda\}\le p_0+\sum_{j=0}^{\infty}r_j+\sum_{j=1}^{\infty}s_j,$$

where

$$p_0 \leq 2\exp\{2H(\delta_0/2)\} \max_{|A_0|\leq 2\delta_0} P\{|Z_n(A_0 \cap I_{n,i}, 0, a)| > \lambda_0\},$$

$$r_j \leq 4\exp\{4H(\delta_{j+1}/2)\} \max_{|A_{j+1}\triangle A_j|\leq 2\delta_j} (P\{|Z_n(A_{j+1}\backslash A_j) \cap I_{n,i}, 0, a)| > \lambda_j\}$$

$$+ P\{|Z_n(A_j\backslash A_{j+1}) \cap I_{n,i}, 0, a)| > \lambda_j\}),$$

$$s_j \leq \exp\{4H(\delta_j/2)\} \max_{A_j\subseteq A_j^+, |A_j^+\backslash A_j|\leq 2\delta_j} P\Big\{ \sup_{A_j\subseteq A\subseteq A_j^+} |Z_n(A\cap I_{n,i}, a_j, a_{j-1})$$

$$- Z_n(A_j \cap I_{n,i}, a_j, a_{j-1})| > \lambda_j\Big\}.$$

By (3.4), taking $\alpha = 1/4a_0$, we have

$$p_0 \leq 2\exp\{2H(\delta_0/2)\}\exp\Big(-\frac{\lambda_0}{4a_0} + c\frac{\delta_0}{a_0^2}\Big)$$
$$\leq 2\exp\Big\{c2^{r+1}\delta_0^{-r} - \frac{\lambda_0}{4a_0} + c\frac{\delta_0}{a_0}\Big\}.$$

Similarly

$$r_j \leq 4\exp\Big\{c2^{r+1}\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c\frac{\delta_j}{a_j}\Big\},$$
$$s_j \leq \exp\Big\{c2^{r+1}\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c\frac{\delta_j}{a_j}\Big\}.$$

Thus

$$p_0 + \sum_{j=0}^{\infty} r_j + \sum_{j=1}^{\infty} s_j$$
$$\leq 6\sum_{j=0}^{\infty}\exp\Big\{c\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c\frac{\delta_j}{a_j}\Big\}$$
$$\leq 6\sum_{j=0}^{\infty}\exp\Big\{\Big(c\nu^{-r} - c'\nu^{-\frac{1}{1+s}} + c\nu^{-\frac{1-s}{1+s}}2^{-\frac{i\delta r}{2+s}}\Big)2^{jr}.$$

Because of $r < 1/(1+s)$, the coefficient of 2^{ir} may be made negative as large as required, by choosing ν small enough, and (3.1) follows. The remainder of the proof is the same as the proof of Theorem 1.1 of [1]. Theorem 1.1 is proved.

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