

## BINARY NONLINEARIZATION FOR THE DIRAC SYSTEMS\*\*

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### Abstract

A Bargmann symmetry constraint is proposed for the Lax pairs and the adjoint Lax pairs of the Dirac systems. It is shown that the spatial part of the nonlinearized Lax pairs and adjoint Lax pairs is a finite dimensional Liouville integrable Hamiltonian system and that under the control of the spatial part, the time parts of the nonlinearized Lax pairs and adjoint Lax pairs are interpreted as a hierarchy of commutative, finite dimensional Liouville integrable Hamiltonian systems whose Hamiltonian functions consist of a series of integrals of motion for the spatial part. Moreover an involutive representation of solutions of the Dirac systems exhibits their integrability by quadratures. This kind of symmetry constraint procedure involving the spectral problem and the adjoint spectral problem is referred to as a binary nonlinearization technique like a binary Darboux transformation.

**Keywords** Zero curvature representation, Nonlinearization method, Liouville integrable system, Soliton hierarchy

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### §1. Introduction

Symmetry constraints become prominent in recent few years due to the important roles they play in soliton theory. For  $1+1$  dimensional soliton hierarchies, a very successful symmetry constraint method is the nonlinearization technique proposed by Cao<sup>[1,2]</sup> and Cao and Geng<sup>[3,4]</sup>. A large class of finite dimensional Liouville integrable Hamiltonian systems is thus generated which are connected with soliton hierarchies<sup>[1-6]</sup> and the nonlinearization technique is systematically extended by Zeng and Li<sup>[7,8]</sup> for quite a few soliton hierarchies. At the same time, a symmetry constraint procedure for bi-Hamiltonian soliton hierarchies is proposed by Antonowicz and Wojciechowski<sup>[9]</sup> and bi-Hamiltonian structures for the constrained systems can be worked out<sup>[9,10]</sup>. By observing that stationary soliton flows may be interpreted as finite dimensional Hamiltonian systems<sup>[11]</sup> on introducing the so-called Jacobi-Ostrogradsky coordinates<sup>[12]</sup>, a natural generalization of nonlinearization technique to higher order symmetry constraints is made by Zeng<sup>[13]</sup> for the KdV and Boussinesq hierarchies.

For  $1+2$  dimensional soliton hierarchies, a sort of symmetry constraints similar to ones in the nonlinearization technique has also been presented by several authors<sup>[14]</sup>, although relatively little is known about the general construction at present. Some interrelations of

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1 + 2 dimensional soliton hierarchies to 1 + 1 dimensional soliton ones and various structures of constrained systems have vigorously been exposed<sup>[15]</sup>. The theory progress along the above two lines has been making and has aroused increasing interest in recent times.

This paper is devoted to the symmetry constraints in the first line. Based upon the idea of nonlinearization technique, we would like to propose a binary nonlinearization technique for soliton hierarchies possessing Lax pairs. We shall express, in terms of the eigenfunctions and the adjoint eigenfunctions, the variational derivative of the spectral parameter with respect to the potential, and thus the binary nonlinearization technique involves two sets of dependent variables, which is different from the nonlinearization technique. By considering the constraint problem of the Lax pairs and the adjoint Lax pairs, we shall exhibit the idea of binary nonlinearization technique and provide new examples for finite dimensional Liouville integrable Hamiltonian systems. In Section 2, we shall give the concrete construction of the Dirac systems and their Hamiltonian structures. Furthermore we want to establish some properties which will be used later. In Section 3, we shall exhibit systematically the binary nonlinearization procedure for the Lax pairs and the adjoint Lax pairs of the Dirac systems.

## §2. The Dirac Systems and Their Hamiltonian Structures

We consider the Dirac spectral problem<sup>[16]</sup>

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad (2.1a)$$

where the spectral operator is as follows:

$$U = U(u, \lambda) = r\sigma_1 + \lambda i\sigma_2 + q\sigma_3 = \begin{pmatrix} q & \lambda + r \\ -\lambda + r & -q \end{pmatrix}, \quad (2.1b)$$

and the  $\sigma_j$ ,  $1 \leq j \leq 3$ , are  $2 \times 2$  Pauli matrices. To derive the Dirac systems associated with (2.1), we first solve the adjoint representation equation (see [17])  $V_x = [U, V]$  of  $\phi_x = U\phi$ . Set

$$V = a\sigma_1 + bi\sigma_2 + c\sigma_3 = \begin{pmatrix} c & a+b \\ a-b & -c \end{pmatrix}. \quad (2.2)$$

Noting that

$$[U, V] = (-2\lambda c + 2qb)\sigma_1 + (-2rc + 2qa)i\sigma_2 + (2\lambda a - 2rb)\sigma_3,$$

we see that the adjoint representation equation  $V_x = [U, V]$  becomes

$$\begin{cases} a_x = -2\lambda c + 2qb, \\ b_x = -2rc + 2qa, \\ c_x = 2\lambda a - 2rb, \end{cases}$$

which is equivalent to

$$\begin{cases} a_0 = c_0 = 0, \quad b_{0x} = 0, \\ a_{ix} = -2c_{i+1} + 2qb_i, \\ b_{ix} = -2rc_i + 2qa_i, \quad i \geq 0, \\ c_{ix} = 2a_{i+1} - 2rb_i, \end{cases} \quad (2.3)$$

on setting  $a = \sum_{i \geq 0} a_i \lambda^{-i}$ ,  $b = \sum_{i \geq 0} b_i \lambda^{-i}$ ,  $c = \sum_{i \geq 0} c_i \lambda^{-i}$ . We choose  $b_0 = -1$  and assume that  $a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0$ ,  $i \geq 1$  (or equivalently select constants of integration to be

zero). At this point, the recursion relation (2.3) uniquely determines a series of differential functions with respect to  $u$ . For instance, we have

$$\begin{aligned} a_1 &= -r, \quad c_1 = -q, \quad b_1 = 0; \\ a_2 &= -\frac{1}{2}q_x, \quad c_2 = \frac{1}{2}r_x, \quad b_2 = -\frac{1}{2}q^2 - \frac{1}{2}r^2; \\ a_3 &= \frac{1}{4}r_{xx} - \frac{1}{2}q^2r - \frac{1}{2}r^3, \quad c_3 = \frac{1}{4}q_{xx} - \frac{1}{2}q^3 - \frac{1}{2}qr^2, \quad b_3 = \frac{1}{2}(qr_x - q_xr). \end{aligned}$$

From  $(V^2)_x = [U, V^2]$ , we see that  $(\frac{1}{2}\text{tr}V^2)_x = (c^2 + a^2 - b^2)_x = 0$ . Thus by  $(\frac{1}{2}\text{tr}V^2)|_{u=0} = 1$  we have  $c^2 + a^2 - b^2 = 1$ . Further we obtain

$$b_n = \frac{1}{2} \sum_{i=1}^{n-1} (b_i b_{n-i} - a_i a_{n-i} - c_i c_{n-i}), \quad n \geq 2.$$

By the mathematical induction, it follows from (2.3) and the above equality that  $a_i, b_i, c_i$  are all differential polynomial functions of  $u$ . The compatibility conditions of Lax pairs

$$\phi_x = U\phi, \quad \phi_{t_n} = V^{(n)}\phi, \quad V^{(n)} = (\lambda^n V)_+, \quad n \geq 0, \quad (2.4)$$

where the symbol  $+$  stands for the selection of the polynomial part of  $\lambda$ , engender a hierarchy of the Dirac systems

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = K_n = \begin{pmatrix} -2a_{n+1} \\ 2c_{n+1} \end{pmatrix}, \quad n \geq 0. \quad (2.5)$$

The first nonlinear Dirac system in the hierarchy (2.5) reads as

$$\begin{cases} q_{t_2} = -\frac{1}{2}r_{xx} + q^2r + r^3, \\ r_{t_2} = \frac{1}{2}q_{xx} - q^3 - qr^2. \end{cases}$$

This system is different from the coupled nonlinear Schrödinger systems in AKNS hierarchy because it contains the cubic terms  $q^3, r^3$ .

In what follows, we want to give the Hamiltonian structures of the Dirac systems (2.5) by means of the trace identity proposed in [18]. To this end, we need the following quantities which are easy to prove:

$$\langle V, \frac{\partial U}{\partial \lambda} \rangle = -2b, \quad \langle V, \frac{\partial U}{\partial q} \rangle = 2c, \quad \langle V, \frac{\partial U}{\partial r} \rangle = 2a, \quad (2.6)$$

where  $\langle A, B \rangle = \text{tr}(AB)$ . Now applying the trace identity<sup>[18]</sup>

$$\frac{\delta}{\delta u} \langle V, \frac{\partial U}{\partial \lambda} \rangle = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left( \langle V, \frac{\partial U}{\partial q} \rangle, \langle V, \frac{\partial U}{\partial r} \rangle \right)^T, \quad \gamma = \text{const.},$$

we obtain at once

$$\frac{\delta}{\delta u} (-2b) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma (2c, 2a)^T.$$

Equating the coefficients of  $\lambda^{-n-1}$  on two sides of the above equality, we have

$$\frac{\delta}{\delta u} (-b_{n+1}) = (\gamma - n)(c_n, a_n)^T, \quad n \geq 0.$$

By taking simply  $n = 1$ , we find that the constant  $\gamma = 0$ . In this way we obtain an important equality

$$\frac{\delta b_{n+1}}{\delta u} = n(c_n, a_n)^T, \quad n \geq 0. \quad (2.7)$$

In addition, for  $n \geq 1$  we have

$$\begin{aligned} \begin{pmatrix} -2a_{n+1} \\ 2c_{n+1} \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2c_{n+1} \\ 2a_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2q\partial^{-1}r & -\frac{1}{2}\partial + 2q\partial^{-1}q \\ \frac{1}{2}\partial - 2r\partial^{-1}r & 2r\partial^{-1}q \end{pmatrix} \begin{pmatrix} 2c_n \\ 2a_n \end{pmatrix}. \end{aligned}$$

Therefore the hierarchy (2.5) may be cast into the following Hamiltonian form

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = K_n = \begin{pmatrix} -2a_{n+1} \\ 2c_{n+1} \end{pmatrix} = JG_n = JL^n \begin{pmatrix} 2c_1 \\ 2a_1 \end{pmatrix} = J \frac{\delta H_n}{\delta u}, \quad n \geq 0, \quad (2.8)$$

where the Hamiltonian operator  $J$ , the recursive operator  $L$  and the Hamiltonian functions  $H_n$  are determined by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} -2q\partial^{-1}r & -\frac{1}{2}\partial + 2q\partial^{-1}q \\ \frac{1}{2}\partial - 2r\partial^{-1}r & 2r\partial^{-1}q \end{pmatrix}, \quad H_n = \frac{2b_{n+2}}{n+1}, \quad n \geq 0. \quad (2.9)$$

By a direct calculation similar to [19], we can get

$$V_{t_n}^{(m)} - V_{t_m}^{(n)} + [V^{(m)}, V^{(n)}] = (V^{(m)})'[K_n] - (V^{(n)})'[K_m] + [V^{(m)}, V^{(n)}] = 0,$$

which implies the commutability of the flows of (2.8). Therefore each system in the hierarchy (2.8) has an infinite number of symmetries  $\{K_m\}_{m=0}^{\infty}$ . Besides, we can directly verify

$$V_{t_n} = [V^{(n)}, V], \quad n \geq 0, \quad (2.10)$$

when  $u_{t_n} = K_n$ , i.e.  $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$ ,  $n \geq 0$ . In fact, we easily find that  $V_{t_n} - [V^{(n)}, V]$  satisfies the adjoint representation equation of  $\phi_x = U\phi$  and that  $V_{t_n} - [V^{(n)}, V]$  vanishes at  $u = 0$ . Thus (2.10) holds for  $n \geq 0$  because the adjoint representation equation  $V_x = [U, V]$  has uniqueness, namely if  $V_x = [U, V]$  and  $V$  vanishes at  $u = 0$ , then  $V$  itself vanishes.

### §3. Binary Nonlinearization

Let us begin with the binary nonlinearization of the Lax pairs and adjoint Lax pairs of the Dirac systems. Associated with Lax pairs (2.4), the Dirac systems have the adjoint Lax pairs

$$\begin{cases} \psi_x = -U^T \psi = -U^T(u, \lambda) \psi, \end{cases} \quad (3.1a)$$

$$\begin{cases} \psi_{t_n} = -(V^{(n)})^T \psi = -(V^{(n)})^T(u, \lambda) \psi, \end{cases} \quad (3.1b)$$

where  $T$  means the transpose of matrix and  $\psi = (\psi_1, \psi_2)^T$ . It follows from  $\phi_x = U\phi$ ,  $\psi_x = -U^T \psi$ , that

$$\frac{\delta \lambda}{\delta u} = \frac{1}{E} (\phi_1 \psi_1 - \phi_2 \psi_2, \phi_1 \psi_2 + \phi_2 \psi_1)^T, \quad E = \int_{-\infty}^{\infty} (\phi_1 \psi_2 - \phi_2 \psi_1) dx. \quad (3.2)$$

When the zero boundary conditions  $\lim_{|x| \rightarrow +\infty} \phi = \lim_{|x| \rightarrow +\infty} \psi = 0$  hold, we have

$$L \frac{\delta \lambda}{\delta u} = \lambda \frac{\delta \lambda}{\delta u}, \quad \text{or} \quad L^* J \frac{\delta \lambda}{\delta u} = \lambda J \frac{\delta \lambda}{\delta u}, \quad (3.3)$$

where  $L$  is defined as in (2.9), and  $L^*$  is the adjoint operator of  $L$ , which is a hereditary symmetry. For a general spectral problem, this kind of property (3.3) has been discussed in [20].

Now introducing  $N$  distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_N$ , we obtain the following two spatial and time systems:

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_x = U(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = -U^T(u, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N; \end{cases} \quad (3.4a)$$

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_{t_n} = V^{(n)}(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_n} = -(V^{(n)})^T(u, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N. \end{cases} \quad (3.5a)$$

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_{t_n} = V^{(n)}(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_n} = -(V^{(n)})^T(u, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N. \end{cases} \quad (3.5b)$$

Because  $U_{t_n} - V^{(n)} + [U, V^{(n)}] = 0$  if and only if  $(-U^T)_{t_n} - (-(V^{(n)})^T)_x + [-U^T, -(V^{(n)})^T] = 0$ , the compatibility condition of (3.4) and (3.5) is still the  $n$ -th Dirac systems  $u_{t_n} = K_n$ . Let us take the Bargmann constraint (requiring the  $G$ -vector field to be a linear function but not a differential function with respect to  $u$ ) for the Dirac systems

$$JG_0 = J \sum_{j=1}^N 2E_j \frac{\delta \lambda_j}{\delta u}, \quad E_j = \int_{-\infty}^{\infty} (\phi_{1j} \psi_{2j} - \phi_{2j} \psi_{1j}) dx. \quad (3.6)$$

This kind of constraints is, in fact, symmetry constraints because the  $J \frac{\delta \lambda_j}{\delta u}$  are common symmetries of the Dirac systems. The constraint (3.6) allows us to impose that

$$\begin{pmatrix} c_1 \\ a_1 \end{pmatrix} = (\langle \Phi_1, \Psi_1 \rangle - \langle \Phi_2, \Psi_2 \rangle, \langle \Phi_1, \Psi_2 \rangle + \langle \Phi_2, \Psi_1 \rangle)^T,$$

which implies that

$$\begin{cases} q = -\langle \Phi_1, \Psi_1 \rangle + \langle \Phi_2, \Psi_2 \rangle, \\ r = -\langle \Phi_1, \Psi_2 \rangle - \langle \Phi_2, \Psi_1 \rangle. \end{cases} \quad (3.7a)$$

$$(3.7b)$$

Here  $\Phi_i = (\phi_{i1}, \dots, \phi_{iN})^T$ ,  $\Psi_i = (\psi_{i1}, \dots, \psi_{iN})^T$ ,  $i = 1, 2$ ,  $\langle y, z \rangle = \sum_{j=1}^N y_j z_j$ ,  $y, z \in R^N$ .

Note that (3.7) has two particular properties: including both the eigenfunctions  $\Phi_1, \Phi_2$  and the adjoint eigenfunctions  $\Psi_1, \Psi_2$ , and nonlinearity with respect to  $\Phi_i, \Psi_i$ . Therefore we refer to (3.7) as a binary nonlinear constraint. In general, nonlinear constraints only involve eigenfunctions of spectral problems associated with integrable systems (see [1-8]). We denote by  $\tilde{A}$  the expression of  $A$  under the constraint (3.7). The property (3.3) ensures that

$$\begin{pmatrix} \tilde{c}_n \\ \tilde{a}_n \end{pmatrix} = \tilde{L}^{n-1} \begin{pmatrix} \tilde{c}_1 \\ \tilde{a}_1 \end{pmatrix} = \begin{pmatrix} \langle A^{n-1} \Phi_1, \Psi_1 \rangle - \langle A^{n-1} \Phi_2, \Psi_2 \rangle \\ \langle A^{n-1} \Phi_1, \Psi_2 \rangle + \langle A^{n-1} \Phi_2, \Psi_1 \rangle \end{pmatrix}, \quad n \geq 1, \quad (3.8a)$$

and that from (2.3),

$$\tilde{b}_n = \partial^{-1}(2\tilde{q}\tilde{a}_n - 2\tilde{r}\tilde{c}_n) = \langle A^{n-1} \Phi_1, \Psi_2 \rangle - \langle A^{n-1} \Phi_2, \Psi_1 \rangle, \quad n \geq 1. \quad (3.8b)$$

Here  $\tilde{b}_1 \neq 0$ . But  $\tilde{V}_x = [\tilde{U}, \tilde{V}]$  still holds. By substituting (3.7) into the Lax pairs and the adjoint Lax pairs: (3.4) and (3.5), we acquire the nonlinearized Lax pairs and adjoint Lax pairs

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_x = U(\tilde{u}, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = -U^T(\tilde{u}, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N; \end{cases} \quad (3.9a)$$

$$(3.9b)$$

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_{t_n} = V^{(n)}(\tilde{u}, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_n} = -(V^{(n)})^T(\tilde{u}, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j = 1, 2, \dots, N. \end{cases} \quad (3.10a)$$

$$(3.10b)$$

The spatial part of the nonlinearized Lax pairs and adjoint Lax pairs, i.e. the system (3.9) is a finite dimensional system but the time parts of the nonlinearized Lax pairs and adjoint Lax pairs, i.e. the systems (3.10) for  $n \geq 0$  are all systems of evolution equations in  $1 + 1$  dimensions. In the following, we shall verify that the system (3.9) is a Liouville integrable (see [21]) Hamiltonian system and that under the control of (3.9), the systems (3.10) are also Liouville integrable Hamiltonian systems.

The system (3.9) may be cast into the following Hamiltonian form

$$\Phi_{ix} = \frac{\partial H}{\partial \Psi_i}, \quad \Psi_{ix} = -\frac{\partial H}{\partial \Phi_i}, \quad i = 1, 2, \quad (3.11a)$$

where the Hamiltonian function

$$\begin{aligned} H = & -\frac{1}{2} (\langle \Phi_1, \Psi_1 \rangle - \langle \Phi_2, \Psi_2 \rangle)^2 \\ & - \frac{1}{2} (\langle \Phi_1, \Psi_2 \rangle + \langle \Phi_2, \Psi_1 \rangle)^2 - \langle A\Phi_1, \Psi_2 \rangle + \langle A\Phi_2, \Psi_1 \rangle. \end{aligned} \quad (3.11b)$$

Let us now construct integrals of motion for (3.11). An obvious equality

$$(\tilde{V}^2)_x = [\tilde{U}, \tilde{V}^2]$$

leads to

$$F_x = \left( \frac{1}{2} \text{tr} \tilde{V}^2 \right)_x = \frac{d}{dx} (\tilde{c}^2 + \tilde{a}^2 - \tilde{b}^2) = 0.$$

Thus  $F$  is a generating function of integrals of motion for (3.11). Since  $F = \sum_{n \geq 0} F_n \lambda^{-n}$ , we obtain the following expressions

$$F_n = \sum_{i=0}^n (\tilde{c}_i \tilde{c}_{n-i} + \tilde{a}_i \tilde{a}_{n-i} - \tilde{b}_i \tilde{b}_{n-i}).$$

Further by (3.8), we have

$$F_0 = -1, \quad F_1 = 2(\langle \Phi_1, \Psi_2 \rangle - \langle \Phi_2, \Psi_1 \rangle), \quad (3.12a)$$

$$\begin{aligned} F_n = & \sum_{i=1}^{n-1} (\tilde{c}_i \tilde{c}_{n-i} + \tilde{a}_i \tilde{a}_{n-i} - \tilde{b}_i \tilde{b}_{n-i}) - 2\tilde{b}_0 \tilde{b}_n \\ = & \sum_{i=1}^{n-1} \left[ (\langle A^{i-1} \Phi_1, \Psi_1 \rangle - \langle A^{i-1} \Phi_2, \Psi_2 \rangle) (\langle A^{n-i-1} \Phi_1, \Psi_1 \rangle - \langle A^{n-i-1} \Phi_2, \Psi_2 \rangle) \right. \\ & + (\langle A^{i-1} \Phi_1, \Psi_2 \rangle + \langle A^{i-1} \Phi_2, \Psi_1 \rangle) (\langle A^{n-i-1} \Phi_1, \Psi_2 \rangle + \langle A^{n-i-1} \Phi_2, \Psi_1 \rangle) \\ & - (\langle A^{i-1} \Phi_1, \Psi_2 \rangle - \langle A^{i-1} \Phi_2, \Psi_1 \rangle) (\langle A^{n-i-1} \Phi_1, \Psi_2 \rangle - \langle A^{n-i-1} \Phi_2, \Psi_1 \rangle) \Big] \\ & + 2(\langle A^{n-1} \Phi_1, \Psi_2 \rangle - \langle A^{n-1} \Phi_2, \Psi_1 \rangle), \quad n \geq 2. \end{aligned} \quad (3.12b)$$

Here  $F_n$ ,  $n \geq 0$ , are all polynomials of  $4N$  dependent variables  $\phi_{ij}$ ,  $\psi_{ij}$ . Note that we have (2.10). With the same deduction, we find that  $F = \frac{1}{2} \text{tr} \tilde{V}^2$  is also a generating function of integrals of motion for (3.10). Moreover, we have

$$\left( \frac{\partial F}{\partial \Phi_1}, \frac{\partial F}{\partial \Phi_2} \right)^T = \left( \text{tr} \left( \tilde{V} \frac{\partial}{\partial \Phi_1} \tilde{V} \right), \text{tr} \left( \tilde{V} \frac{\partial}{\partial \Phi_2} \tilde{V} \right) \right)^T, \quad (3.13)$$

$$\begin{aligned}
\operatorname{tr}\left(\tilde{V}\frac{\partial}{\partial\Phi_1}\tilde{V}\right) &= \operatorname{tr}\sum_{i=0}^{\infty}\begin{pmatrix}\tilde{c}_i & \tilde{a}_i+\tilde{b}_i \\ \tilde{a}_i-\tilde{b}_i & -\tilde{c}_i\end{pmatrix}\lambda^{-i}\sum_{j=0}^{\infty}\frac{\partial}{\partial\Phi_1}\begin{pmatrix}\tilde{c}_j & \tilde{a}_j+\tilde{b}_j \\ \tilde{a}_j-\tilde{b}_j & -\tilde{c}_j\end{pmatrix}\lambda^{-j} \\
&= \operatorname{tr}\sum_{\substack{i\geq 0 \\ j\geq 1}}\begin{pmatrix}\tilde{c}_i & \tilde{a}_i+\tilde{b}_i \\ \tilde{a}_i-\tilde{b}_i & -\tilde{c}_i\end{pmatrix}\begin{pmatrix}A^{j-1}\Psi_1 & 2A^{j-1}\Psi_2 \\ 0 & -A^{j-1}\Psi_1\end{pmatrix}\lambda^{-(i+j)} \\
&= 2\sum_{\substack{i\geq 0 \\ j\geq 1}}[\tilde{c}_iA^{j-1}\Psi_1+(\tilde{a}_i-\tilde{b}_i)A^{j-1}\Psi_2]\lambda^{-(i+j)}. \tag{3.14a}
\end{aligned}$$

Similarly we can obtain

$$\operatorname{tr}\left(\tilde{V}\frac{\partial}{\partial\Phi_2}\tilde{V}\right) = 2\sum_{\substack{i\geq 0 \\ j\geq 1}}[(\tilde{a}_i+\tilde{b}_i)A^{j-1}\Psi_1-\tilde{c}_iA^{j-1}\Psi_2]\lambda^{-(i+j)}. \tag{3.14b}$$

The equalities (3.13) and (3.14) lead to

$$\frac{1}{2}\begin{pmatrix}\frac{\partial F_{n+1}}{\partial\Phi_1} \\ \frac{\partial F_{n+1}}{\partial\Phi_2}\end{pmatrix} = -(-\tilde{V}^{(n)})^T\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix}, \quad n \geq 0.$$

The similar deduction can give rise to

$$\frac{1}{2}\begin{pmatrix}\frac{\partial F_{n+1}}{\partial\Psi_1} \\ \frac{\partial F_{n+1}}{\partial\Psi_2}\end{pmatrix} = \tilde{V}^{(n)}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}, \quad n \geq 0.$$

At this stage, the systems (3.10) may be readily rewritten as

$$\begin{aligned}
\Psi_{it_n} &= -\frac{\partial(\frac{1}{2}F_{n+1})}{\partial\Phi_i}, \\
\Phi_{it_n} &= \frac{\partial(\frac{1}{2}F_{n+1})}{\partial\Psi_i}, \quad i = 1, 2. \tag{3.15}
\end{aligned}$$

Associated with the symplectic structure on  $R^{4N}$

$$\omega^2 = d\Psi_1 \wedge d\Phi_1 + d\Psi_2 \wedge d\Phi_2,$$

we can generate the Poisson bracket

$$\{P, Q\} = \omega^2(IdQ, IdP), \tag{3.16}$$

where  $IdR$  denotes a Hamiltonian vector field of a smooth function  $R$  on  $R^{4N}$ , defined by  $IdR \lrcorner \omega^2 = -dR$ , with  $\lrcorner$  being the left interior product. Since  $F$  is a generating function of integrals of motion for (3.10), we obtain

$$\{F_{m+1}, F_{n+1}\} = 2\frac{\partial}{\partial t_n}F_{m+1} = 0, \quad m, n \geq 0. \tag{3.17}$$

This elucidates that  $\{F_n\}_{m=0}^{\infty}$  constitutes a Poisson algebra with regard to (3.16). Evidently we have

$$\begin{aligned}
\left.\frac{\partial F_n}{\partial\Phi_1}\right|_{\Phi_1=\Phi_2=0} &= 2A^{n-1}\Psi_2, \\
\left.\frac{\partial F_n}{\partial\Phi_2}\right|_{\Phi_1=\Phi_2=0} &= -2A^{n-1}\Psi_1, \quad n \geq 1. \tag{3.18}
\end{aligned}$$

Therefore there must exist one region  $\Omega \subseteq R^{4N}$  on which the  $2N$  1-forms  $dF_1, \dots, dF_{2N}$  are every linearly independent since the Vandermonde determinant  $V(\lambda_1, \lambda_2, \dots, \lambda_N)$  is nonzero. In this way we have shown that the spatial part of the nonlinearized Lax pairs and adjoint Lax pairs and the time parts of the nonlinearized Lax pairs and adjoint Lax pairs under the control of the spatial part are all finite dimensional integrable Hamiltonian systems in the Liouville sense (see [21]).

When  $\lim_{|x| \rightarrow +\infty} \Psi_i = \lim_{|x| \rightarrow +\infty} \Phi_i = 0$ ,  $i = 1, 2$  (at this stage,  $\tilde{b}_1 = \frac{1}{2}F_1 = 0$  since  $F_{1x} = 0$ ) and  $\Phi_i(x, t_n)$ ,  $\Psi_i(x, t_n)$  solve simultaneously (3.11) and (3.15), we see that

$$u = (-\langle \Phi_1, \Psi_1 \rangle + \langle \Phi_2, \Psi_2 \rangle, -\langle \Phi_1, \Psi_2 \rangle - \langle \Phi_2, \Psi_1 \rangle)^T$$

is a solution to the  $n$ -th Dirac system  $u_{t_n} = K_n$ . This allows us to conclude that the  $n$ -th Dirac system  $u_{t_n} = K_n$  has the following involutive representation of solutions

$$\begin{cases} q(x, t_n) = -\langle g_H^x g_{F_{n+1}}^{t_n} \Phi_1(0, 0), g_H^x g_{F_{n+1}}^{t_n} \Psi_1(0, 0) \rangle \\ \quad + \langle g_H^x g_{F_{n+1}}^{t_n} \Phi_2(0, 0), g_H^x g_{F_{n+1}}^{t_n} \Psi_2(0, 0) \rangle, \\ r(x, t_n) = -\langle g_H^x g_{F_{n+1}}^{t_n} \Phi_1(0, 0), g_H^x g_{F_{n+1}}^{t_n} \Psi_2(0, 0) \rangle \\ \quad - \langle g_H^x g_{F_{n+1}}^{t_n} \Phi_2(0, 0), g_H^x g_{F_{n+1}}^{t_n} \Psi_1(0, 0) \rangle, \end{cases} \quad (3.19a)$$

$$\begin{cases} q(x, t_n) = -\langle g_H^x g_{F_{n+1}}^{t_n} \Phi_1(0, 0), g_H^x g_{F_{n+1}}^{t_n} \Psi_1(0, 0) \rangle \\ \quad + \langle g_H^x g_{F_{n+1}}^{t_n} \Phi_2(0, 0), g_H^x g_{F_{n+1}}^{t_n} \Psi_2(0, 0) \rangle, \\ r(x, t_n) = -\langle g_H^x g_{F_{n+1}}^{t_n} \Phi_1(0, 0), g_H^x g_{F_{n+1}}^{t_n} \Psi_2(0, 0) \rangle \\ \quad - \langle g_H^x g_{F_{n+1}}^{t_n} \Phi_2(0, 0), g_H^x g_{F_{n+1}}^{t_n} \Psi_1(0, 0) \rangle, \end{cases} \quad (3.19b)$$

with  $g_H^x, g_{F_{n+1}}^{t_n}$  being the Hamiltonian phase flows (see [21]) associated with the Hamiltonian functions  $H, F_{n+1}$ , but  $\Phi_i(0, 0), \Psi_i(0, 0)$ ,  $i = 1, 2$ , being any fixed constant vector of  $R^N$ . This kind of involutive representation of solutions to the Dirac systems also show, to some extent, the characteristic of integrability of the Dirac systems in view of the Liouville integrability of the flows  $g_H^x, g_{F_{n+1}}^{t_n}$ .

#### §4. Conclusions and Remarks

We introduced a kind of symmetry constraint on the Dirac integrable systems. Moreover we exhibited an explicit Poisson algebra on the symplectic manifold  $(R^{4N}, \omega^2)$  and, further, an involutive representation of solutions to the Dirac systems. The proposed symmetry constraint includes the eigenfunctions and the adjoint eigenfunctions and is nonlinear with respect to them. The corresponding reductions of the Lax pairs and adjoint Lax pairs and the manipulation with the integrability of the constrained systems constitute a binary nonlinearization problem.

We remark that when the zero boundary conditions on the eigenfunctions and the adjoint eigenfunctions are not imposed, the spatial part of the nonlinearized Lax pairs and adjoint Lax pairs is invariant and thus still a Hamiltonian system with the original Hamiltonian function. However, the time parts of the nonlinearized Lax pairs and adjoint Lax pairs will vary with the boundary conditions. When the zero boundary conditions are not satisfied, they become very complicated Hamiltonian systems. Nevertheless, their Hamiltonian functions are some polynomials of  $F_n$ ,  $n \geq 1$ , which may be determined by an obtained equality  $c^2 + a^2 - b^2 = 1$ . Furthermore we can consider higher order constraints

$$JG_n = J \sum_{j=1}^N 2E_j \frac{\delta \lambda_j}{\delta u} \quad (n \geq 1).$$



This kind of symmetry constraints is somewhat different from the Bargmann constraint because the  $G_n$  involve the differential of  $u$  with respect to  $x$ . To discuss them, we need to introduce new dependent variables, i.e., the so-called Jacobi-Ostrogradsky coordinates. These problems are left to a further investigation.

There exist also certain relations between the nonlinearization technique and the binary nonlinearization technique. For example, if we take the possible reduction

$$\Psi_1 = -\Phi_2, \quad \Psi_2 = \Phi_1,$$

the involutive system (3.12) simplifies to

$$\begin{aligned} F_1 &= 2(\langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle), \\ F_n &= \sum_{i=1}^{n-1} \left[ (\langle A^{i-1} \Phi_1, \Phi_1 \rangle - \langle A^{i-1} \Phi_2, \Phi_2 \rangle) (\langle A^{n-i-1} \Phi_1, \Phi_1 \rangle - \langle A^{n-i-1} \Phi_2, \Phi_2 \rangle) \right. \\ &\quad - (\langle A^{i-1} \Phi_1, \Phi_1 \rangle + \langle A^{i-1} \Phi_2, \Phi_2 \rangle) (\langle A^{n-i-1} \Phi_1, \Phi_1 \rangle + \langle A^{n-i-1} \Phi_2, \Phi_2 \rangle) \\ &\quad \left. + 4 \langle A^{i-1} \Phi_1, \Phi_2 \rangle \langle A^{n-i-1} \Phi_1, \Phi_2 \rangle \right] + 2(\langle A^{n-1} \Phi_1, \Phi_1 \rangle + \langle A^{n-1} \Phi_2, \Phi_2 \rangle), \quad n \geq 2. \end{aligned}$$

This is an involutive system on the symplectic manifold  $(R^{2N}, \omega^2 = d\Phi_1 \wedge d\Phi_2)$ , and it may be generated by the nonlinearization technique.

We should note that the idea of binary nonlinearization is quite broad; it can be applied to other integrable systems (see, for example, [22]). Therefore a large class of finite dimensional Liouville integrable Hamiltonian systems may be raised by means of our binary nonlinearization technique and the involutive representation of solutions can exhibit the integrability by quadratures for the considered systems.

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