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A REMARK ON HOFER-ZEHNDER SYMPLECTIC CAPACITY IN SYMPLECTIC MANIFOLDS $M\times R^{2n}$**

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Abstract

The author studies the Hofer-Zehnder capacity and the Weinstein conjecture in $M \times R^{2n}$ and extends the results in [6] and [13] to the case $\omega_{|\pi_2(M)} \neq 0$.

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§1. Introduction

In 1985 M. Gromov proved in his seminar paper "Pseudoholomorphic Curves in Sympletic Manifold"[8], among other things a striking rigidity result, the so-called Gromov's Sympletic Squeezing theorem. Consider the vectorspace C^n equipped with its usual Hermitian inner product. Denote by $\langle\cdot,\cdot\rangle$ =Re$(\cdot,\cdot)$ the associated real inner product and by $\sigma = -\text{Im}(\cdot,\cdot)$ the usual induced symplectic form. If $B^{2n}(r)$ denotes the Euclidean r-ball and $Z^{2n}(\varepsilon) \times C^{n-1}$ the symplectic cylinder of radius ε, M.Gromov proved that $B^{2n}(R)$ admits a symplectic embedding into $Z^{2n}(\varepsilon)$ iff $R \leq \varepsilon$.

In the same paper, M.Gromov gave a definition of symplectic radius for a symplectic manifold, the so-called Gromov Capacity, i.e.

$$c_G(M,\omega) = \sup\{\pi r^2 | \text{ there exist a symplectic embedding } B^{2n}(r) \to M\}.$$

Coming from the variational theory of Hamiltonian dynamics, I. Ekeland and H. Hofer observed in [2,3] that the study of periodic solutions of Hamiltonian Systems can be effectively used to prove the squeezing theorem, and more important, gave new Symplectic invariant, the so-called Ekeland-Hofer symplectic capacity c_{EH}. Recently, H. Hofer and E. Zehnder in [13] propose the axioms of capacity for a general symplectic manifold (M,ω) (with or without boundary), and construct a new capacity for a general symplectic manifold, the so-called Hofer-Zehnder capacity c_{HZ}.

The construction of Hofer-Zehnder capacity as given in [13] goes as follows. Denote by $H(M,\omega)$ the subset of $C^\infty(M,R)$ consisting of all smooth maps satisfying

- There exists a compact subset $K \subset M\backslash\partial M$ depending on H such that $H_{|(M\backslash K)} = m(H)$ is constant.
- There is a nonempty open set U depending on H such that $H_{|U} \equiv 0$.

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• $0 \leq H(x) \leq m(H)$ for all $x \in M$.

We shall call $H \in H(M,\omega)$ admissible if it has the property that all T-periodic solutions of Hamiltonian systems $X = X_H(x)$ on M with $0 < T \leq 1$ are constant. We shall write $H_{ad}(M,\omega)$ for the set of admissible $H \in H(M,\omega)$. We define $c_{HZ}:\ \aleph \to R^*_+$ by

$$c_{HZ}(M,\omega) = \sup\{m(H) | H \in H_{ad}(M,\omega)\}.$$

It has been shown in [13] that c_{HZ} defines a symplectic capacity having the additional property that $c_{HZ}(B^{2n}(r);\sigma) = c_{HZ}(Z^{2n}(r);\sigma) = \pi r^2$. In this paper, we shall study the Hofer-Zehnder capacity and Weinstein conjecture in $(M \times R^{2n}, \omega \oplus \sigma)$ and extend the above result to the case including the $\omega_{|\pi_2(M)} \neq 0$.

Definition 1.1. *Let (M,ω) be a compact symplectic manifold. We define*

$$l(M,\omega) =: \ \inf\{\langle\omega,\alpha\rangle | \langle\omega,\alpha\rangle > 0,\ \alpha \in [S^2, M]\}.$$

Here $[S^2, M]$ stands for the set of free homotopy classes from S^2 to M.

Theorem 1.1. *Let (M,ω) be a compact symplectic manifold. Suppose that $l(M,\omega) > 0$ and $0 < \pi r^2 < l(M,\omega)$. Suppose that $H : M \times Z(r) \to R^1$, a smooth function, satisfies:*

$H_{|U} \equiv 0$ and $H_{|(M\times Z(r)\setminus K)} = m(H)$ for an open subset U and a compact subset K of U and suppose that $\pi r^2 < m(H) < l(M,\omega)$. Then, the Hamiltonian systems $\dot{x} = x_H(x)$ on $M \times Z(r)$ has at least one nontrivial 1-periodic solution.

From the above theorem, we obtain the main result in this paper which extends the result by H. Hofer and E. Zehnder in [13], i.e.,

Theorem 1.2. *Let (M,ω) be a compact symplectic manifold. Suppose that $l(M,\omega) > 0$ and $0 < \pi R^2 < l(M,\omega)$. Then, we have $c_{HZ}(M \times B(r);\omega \oplus \sigma) = \pi r^2$.*

Proof. Suppose that $H:\ M \times Z(r) \to R^1$ satisfies:

• $H_{|U} \equiv 0$ for an open subset of $M \times Z(r)$.

• $H_{M\times Z(r)\setminus K} \equiv m(H)$ for a compact subset of $M \times Z(r)$.

• $0 \leq H(x) \leq m(H)$ and $m(H) > \pi r^2$.

Now, if $m(H) < l\ (M,\omega)$, by Theorem 1.1, the Hamiltonian systems $\dot{x} = x_H(x)$ on $M \times Z(r)$ has a nontrivial 1-periodic solution. So, we can assume that $m(H) \geq l(M,\omega)$. Then there exists $0 < \lambda < 1$ such that $\pi r^2 < m(\lambda H) < l(M,\omega)$. By Theorem 1.1, we know that the Hamiltonian systems $\dot{x} = x_{\lambda H}(x)$ on $M \times Z(r)$ has a nontrivial 1-periodic solution which corresponds to a nontrivial λ-periodic solution of $\dot{x} = x_H(x)$ on $M \times Z(r)$ with $0 < \lambda < 1$. This completes the proof of Theorem 1.2.

Let $Z_0, \cdots, Z_n$ be coordinates on C^{n+1} and denote by $\pi : C^{n+1}\backslash\{0\} \to CP^n$ the standard projection map. Let $U \subset CP^n$ be an open set and $Z:\ U \to C^{n+1}\backslash\{0\}$ a lifting of U, i.e., a holomorphic map with $\pi \circ Z = id$. Consider the differential form

$$\omega_0 = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log||z||^2.$$

One can verify that ω_0 is independent of the lifting chosen and ω_0 is a globally defined Kahler form. Thus ω_0 defines a hermitian metric on CP^n, called the Fubini-Study metric ds^2 on CP^n. It is well-known that the De Rham class $[\omega_0] \in H^2(CP^n, Z)$ is the class of a hyperplane H and the first Chern class of tangent bundle of CP^n is $(n+1)[\omega_0]$ (see [7]). So, we will take $\omega = (n+1)\omega_0$ as the standard symplectic structure on CP^n. In this case, one can easily check $l(CP^n,\omega) = n+1$, i.e., the smallest one of all positive first Chern numbers.

Now, we consider the sympletic product of complex projective spaces. Let

$$(M,\omega) = (CP^{m_1-1}\times\cdots\times CP^{m_k-1}, \omega_1\oplus\cdots\oplus\omega_k).$$

By the property of the Chern class and our choice on ω_i, we know that the De Rham cohomology class $[\omega_1\oplus\cdots\oplus\omega_k]$ is the first Chern class of $T(CP^{m_1-1}\times\cdots\times CP^{m_k-1})$. By the integerability of the first Chern class on $H^2(M,Z)\simeq\pi_2(M)$, we know $l(M,\omega)>0$. In addition, the all Chern numbers form a subgroup Γ of the integer group Z. By an easy calculation, we know that $l(M,\omega)$ is the smallest positive number in Γ and $l(M,\omega) = (m_1,\cdots,m_k)$, where $(m_1,\cdots,m_k)$ denotes the greatest common divisor of $m_1,\cdots,m_k$.

Corollary 1.1. *For $n>1$, consider on CP^{n-1} the standard symplectic form ω such that $\omega[u]=n$ for a generator u of $H^2(CP^{n-1})$. Suppose $0<\pi r^2<(m_1,\cdots,m_k)$. Then*

$$c_{HZ}(CP^{m_1-1}\times\cdots\times CP^{m_k-1}\times B(r), \omega_1\oplus\cdots\oplus\omega_k\oplus\sigma) = \pi r^2,$$

$$c_{HZ}(CP^{m_1-1}\times\cdots\times CP^{m_k-1}\times Z(r), \omega_1\oplus\cdots\oplus\omega_k\oplus\sigma) = \pi r^2,$$

where $(m_1,\cdots,m_k)$ denotes the greatest common divisor of $m_1,\cdots,m_k$.

Now, let us recall some notations. Consider a compact $(2n-1)$ dimensional manifold S equipped with a 1-form λ such that $\lambda\wedge(d\lambda)^{n-1}$ is a volume form. One calls λ a contact form.

The manifold S carries a natural line bundle defined by

$$\mathcal{L}_s = \{(x,\xi)\in TS | d\lambda(x)(\xi,\eta)=0 \text{ for all } \eta\in T_xS\}.$$

Hence $\mathcal{L}_s\subset TS$ and since $\lambda\wedge(d\lambda)^{n-1}$ is a volume, $\lambda(x,\xi)\neq 0$ for $(x,\xi)\subset\mathcal{L}_S$, if $\xi\neq 0$. Denote by $D(S)$ the closed integral curves for the (integrable) distribution $\mathcal{L}_S\to S$. A. Weinstein conjectured that $D(S)\neq 0$ at least if $H^1(S;R)=0$.

Let us say that a compact smooth hypersurface S in a symplectic manifold (M,ω) is of contact type provided there exists a 1-form λ on S such that $d\lambda = i^*\omega$, where $i: S\to M$ is the inclusion and $\lambda(x,\xi)\neq 0$ for non-zero elements in $\mathcal{L}_S$.

As a corollary of Theorem 1.2 and the Theorem 4 in [13], we obtain

Theorem 1.3. *Let (M,ω) be a compact symplectic manifold and supposed that $l(M,\omega)>0, \sum\subset M\times Z(\sqrt{l(M,\omega)/\pi})$ is a compact hypersurface of contact type. Then $\sum$ has at least one closed characteristic.*

Remark 1.1. Theorem 1.3 implies that the Weinstein conjecture holds in

$$M\times Z(\sqrt{l(M,\omega)/\pi}).$$

Remakr 1.2. In the case $\omega|_{\pi_2(M)}=0, l(M,\omega)=\infty$. Theorem 1.3 implies the results proved by A. Floer, H. Hofer, C. Viterbo in [6].

Remark 1.3. In [16], we have proved the Theorem 1.1 under the assumption $0<\pi r^2<\frac{1}{2}l(M,\omega)$ by the similar method. We do not know if the restriction $0<\pi r^2<l(M,\omega)$ in Theorem 1.1 is necessary or not.

Corollary 1.2. *For $n>1$, consider on CP^{n-1} the standard symplectic structure ω such that $\pi r_0^2=\int_{CP^1}\omega$. Let $\sum\subset M\times Z(r_0)$ be a compact hypersurface of contact type. Then $\sum$ has at least one closed characteristic.*

We shall prove Theorem 1.1 by slight modification of the method proposed by A. Floer, H. Hofer, C. Viterbo in [6]. In the first two sections, we reformulate the proof of Theorem

1.1 to the compactness of the set of solutions of the associated nonlinear Cauchy-Riemann equations. We shall estimate the energy or area of a possible splitting holomorphic sphere by the geometric radius r in $M\times Z(r)$ (see Lemma 2.1). Then, by the standard argument on the blowing-up sequence and the assumption $\pi r^2 < l(M,\omega)$, we can remove the possibility of blow up or bubble. Thus, we yield the local compactness theorem. By the nonlinear Fredholm alternative (see §4 or [6], [8], [16]), we know that the global compactness theorem does not hold. The failure of global compactness implies the existence of nontrivial periodic solutions (see §3.2). We complete the proof of Theorem 1.1 by the standard argument as in [6].

Note. We recently noticed that the method of this paper can be improved to prove the weinstein conjectures in $T^*N\times R^{2n}(n\geq 1)$, where $(T^*N, d\theta)$ is the cotangent bundle of N with Liouville form $\theta=\sum\limits_{i=1}^{n} y_i dx_i$. The main fact we used in this case is that the J-holomorphic can not touch the contact boundary from inside. For the details and another method, see [17].

§2. An Extension of the Hamiltonian Function

We first assume that H satisfies the assumption of Theorem, i.e., we assume that H vanishes in a neighbourhood U of u_0 and denote by K a compact set such that $K\subset M\times Z(r)$ and $H(x)=m(H)$ for $x\in(M\times Z(r)\backslash K)$. We shall extend the function $H_{|K}$ in a suitable way to a function $\bar{H}$ defined on $M\times R^{2n}$.

Pick $0<\varepsilon_1<\frac{1}{2}$ such that

$$\pi(1+\varepsilon_1)r^2 < m(H) < l(M,\omega) \tag{2.1}$$

and pick a smooth $f:\ [0,\infty)\to R$ such that

$$\begin{aligned}
&\bullet\ f(t)=m(H), \quad t\leq r,\\
&\bullet\ f''(t)>0, \quad t>r,\\
&\bullet\ f(t)=\pi(1+\varepsilon_1)t^2, \quad t \quad \text{large},\\
&\bullet\ 0\leq f(t)\leq 2\pi(1+\varepsilon_1)t, \quad t\geq 0,\\
&\bullet\ f'(t_0)=2\pi t_0 \quad \text{for} \quad t_0>r
\end{aligned} \tag{2.2}$$

implies $t_0=\sqrt{2}r$.

Now, let $R_2>r$ be such that

$$\mathrm{Supp}(H-m(H))\subset M\times B(R_2) \tag{2.3}$$

and choose a smooth function $g:\ [0,\infty)\to R$ satisfying

$$\begin{aligned}
&g(s)=0, && 0\leq s\leq R_2,\\
&g(s)=\tfrac{1}{2}\pi s^2, && s \text{ large},\\
&0<g'(s)<2\pi s, && s>R_2.
\end{aligned} \tag{2.4}$$

The extension H of H_K is now defined as follows. Set $z=(x,y), x\in M, y\in R^{2n}, y=(y_1,y_2), y_1\in R^2, y_2\in R^{2n-2}$,

$$\overline{H}=\begin{cases} H(z), & \text{if } z\in(M\times Z(r))\cap(M\times B(R_2)),\\ f(|y_1|)+g(|y_2|), & \text{if } z\notin(M\times Z(r))\cap(M\times B(R_2)). \end{cases} \tag{2.5}$$

Clearly $\overline{H}\in C^\infty(M\times R^{2n})$ and $\overline{H}(x)=H(x)$ if $x\in K$.

In the following, we still denote $\overline{H}$ by H for convenience.

One can use H to associate the so-called Hamiltonian Systems (H) on $M \times R^{2n}$:

$$\begin{cases} \dot{z} = X_H(z), \\ z(0) = z(1). \end{cases} \tag{2.6}$$

As in [6,13], the 1-periodic solutions of $\dot{z} = x_H(z)$ falls in

- non-constant 1-periodic solution in K,
- constant,
- $z(t) = (x, y(t))$ where $y(t) = y_0 e^{2\pi i t}, y_0 = (y_1, 0, \cdots, 0)$ and $f(|y_0|) = 2\pi |y_0|$. Thus, $|y_0| = \sqrt{2}r$.

We shall call "triviall solution" any of the last two kinds of solutions. Then Theorem 1.1 is equivalent to

Proposition 2.1 *The Hamiltonion system* (H)*:*

$$\begin{cases} \dot{z} = X_H(z), \\ z(0) = z(1), \end{cases} \tag{2.7}$$

with H as in Definition 1.2*, has at least one non-trivial solution.*

Set $z = (z_1, z_2) \in \Omega(M \times R^{2n})$ (loop space), $z_1 \in \Omega(M), z_2 \in \Omega(R^{2n}), \overline{z} = (\overline{z}_1, \overline{z}_2) : D \to M \times R^{2n}, \overline{z}_{|\partial D} = z$. For any given point $x \in M$, we define an action functional on $\{x\} \times R^{2n} \subset M \times R^{2n}$ as follows:

$$A_H(x)(z_2) = \int_D \overline{z}_2^* \sigma - \int_{\partial D} H(x, y). \tag{2.8}$$

Since the standard symplectic form σ on R^{2n} is exact, we can rewrite $A_H(x)(\cdot)$ as

$$A_H(x)(y) = \frac{1}{2} \langle J\dot{y},\ y \rangle - \int_0^1 H(x, y). \tag{2.9}$$

Lemma 2.1. *For any trivial solution (x, y) of Hamiltonian system* (H)*, we have the estimates*

$$-m(H) \leq A_H(x)(y) \leq 0. \tag{2.10}$$

Proof. See the proof of Lemma 4.6 in [15].

§3. A Holomorphic Curve Approach to the Problem

Let $(V, \overline{\omega})$ be a symplectic manifold, such that $(V, \overline{\omega}) = (M \times R^{2n}, \omega \oplus \sigma)$. Let J be an almost complex structure with ω, J_0 a standard complex structure on R^{2n}. Set $\overline{J} = J \oplus J_0$. Thus, $\overline{J}$ is an almost complex structure on V compatible with ω.

Let Γ be the set obtained by gluing $(-\infty, 0) \times S^1$ and D with its orientation reversed along $\{0\} \times S^1 \simeq \partial D, \Gamma_s = \Gamma - (-\infty s) \times S^1, \Gamma_- = \Gamma - \Gamma_0 = (-\infty, 0) \times S^1$.

Then, Γ has a canonical complex structure, and with this structure, Γ is conformally diffeomorphic to $S^2 - 0 \simeq C$ identifying D with the disk of radius $\frac{1}{2\pi}$ in C (or the southern hemisphere in S^2) and $(-\infty, 0] \times S^1$ to $C - D(0, \frac{1}{2\pi})$ (or the north hemisphere minus the north pole) by polar coordinates $(r, t) \mapsto e^{-r} e^{2\pi i t}$.

Let $U = H^{-1}(0), u_0 \in \sum, D(u_0, \varepsilon) \subset U$. $\gamma \subset S(u_0, \frac{\varepsilon}{4})$ a geodesic circle and $u_1 = \gamma(0), a(u_1, \gamma(t)) = \int_0^t ||\gamma\prime(\tau)|| d\tau$ a arclength of γ starting at $\gamma(0) = u_1$ and ending at $\gamma(t)$.

Similar to [6], we can formulate the proof of Proposition 1.1 to the study of compactness of the set of solutions of nonlinear Cauchy-Riemann equations associated by (H), i.e.,

$$\overline{\partial}u(z) = g\lambda_c\left(z, u(z)\right) \qquad \text{on} \quad \Gamma, \tag{3.1}$$

where

$$g\lambda_c(z,u) = \begin{cases} -\lambda c, & \text{on} \quad D, \\ -\bigtriangledown H\left(u(z)\right), & \text{on} \quad \Gamma_-, \end{cases} \quad 0 \le \lambda \le 1,$$

u is homotopic to zero and satisfies

$$\lim_{s\to-\infty} u(s,t) = u(-\infty) \in \gamma \qquad \text{in} \quad H^1(S^1) \tag{3.2}$$

and

$$\int_{-\infty}^{-1}\int_0^1 \left|\frac{\partial u}{\partial s}\right|^2 dtds = \delta\left\{\int_{-1}^{0}\int_0^1 \left|\frac{\partial u}{\partial s}\right|^2 dtds + a\left(u_1, u(-\infty)\right)\right\}, \tag{3.3}$$

and $\overline{\partial}$ stands for Cauchy-Riemann operater. For the details, refer to [6].

One can easily verify that these constraints are independent. Thus it is a $2n$-dimensional condition. Since $\lim\limits_{s\to-\infty} u(s,t) = u(-\infty) \in \gamma$, we can extend the map $u : \Gamma \mapsto V$ to the map $\overline{u} : \Gamma \cup \{-\infty\} = S^2 \mapsto V$ by setting $\overline{u}(-\infty) = \lim\limits_{s\to-\infty} u(s,t)$. Thus, we shall identifying the map $u : \Gamma \mapsto V$ with the map $\overline{u} : S^2 \mapsto V$ in the following.

Lemma 3.1. *If C is a vector field on $(V, \overline{\omega}) = (M \times R^{2n}, \omega \oplus \sigma)$ induced by a constant vector field $C = (c_1, 0, \cdots, 0)$ on C^n, and if $u = (x,y)$ is a solution of* (2.1), (2.2), (2.3), *then we have the following prior-estimates:*

$$E(u) := \int_{-\infty}^{0}\int_0^1 |\frac{\partial u}{\partial s}|^2 dtds + \frac{1}{2}\int_D |\bigtriangledown x|^2 \le \pi r^2.$$

Proof. See the proof of Lemma 4.2 in [16].

Lemma 3.2. *If c is a vector field on $(V, \omega) = (M \times R^{2n}, \omega \oplus \sigma)$ induced by a constant vector field on c, then, for $|\lambda| = 1$ and $|c|$ large enough, the equations* (3.1), (3.2), (3.3) *have no solution.*

Proof. See the proof of Lemma 4.1 in [16].

§4. Compactness of the Set of Solutions

4.1. Local Compactness Theorem

Lemma 4.1. *Let $(\lambda_n, u_n) = (\lambda_n, x_n, y_n)$ be a sequence of solution in* (3.1), (3.2), (3.3). *Then, for any $s \in (-\infty, 0)$, after taking a subsequence, $||y_n||_{W^{1,p}}(\Gamma_s) \le c(s)$ for all n's, c(s) depends only on s.*

Proof. For a solution u_n of (3.1), by the definition,

$$\int_{\infty}^{s}\int_0^1 u_n^*\omega - \inf H\left(u_n(s,\cdot)\right) \ge 0. \tag{4.1}$$

Since u_n is homotopic to zero

$$\int_\Gamma u_n^*\omega = \int_{-\infty}^{s}\int_0^1 u_n^*\omega + \int_{\Gamma_s} u_n^*\omega = 0. \tag{4.2}$$

Taking (4.2) into (4.1), by the definition of metric, we have

$$\frac{1}{4}\int_{\Gamma_s}\left(|\overline{\partial}u_n|^2 - |\partial u_n|^2\right) - \int H\left(U_n(s,\cdot)\right) \ge 0 \tag{4.3}$$

and

$$\int_{\Gamma_s}\left(|\overline{\partial}u_n|^2+4\int H(u_n(s,\cdot)\right)\le\int_{\Gamma_s}|\overline{\partial}u|^2=\int_{\Gamma_s}|\overline{\partial}x_n|^2+\int_{\Gamma_s}|\overline{\partial}y_n|^2$$
$$\le c_1(s)+c_2(s)\int_{\Gamma_s}|y_n|^2. \quad (4.4)$$

The last inequality follows from the equation satisfied by u_n, and the fact that

$$|g_{\lambda_n c}\,(z,u_n(z))\,|\le\pi(1+\varepsilon_1)|y_n|. \quad (4.5)$$

Thus

$$\int_{\Gamma_s}|y_n|^2\le c_1(s)+c_2(s)\int_{\Gamma_s}|y_n|^2, \quad (4.6)$$

$$\int_{\Gamma_s}|\overline{\partial}y_n|^2\le c_1(s)+c_2(s)\int_{\Gamma_s}|y_n|^2. \quad (4.7)$$

Hence

$$||y_n||_{w^{1,2}(\Gamma_s)}\le c_3(s)||y_n||L^2(\Gamma_s+c_4(s). \quad (4.8)$$

Let now $\overline{y}_n=\frac{1}{||y_n||_{L^2(\Gamma_s)}}y_n=\frac{1}{\rho_n}y_n$. Then y_n is bounded in $W^{1,2}(\Gamma_s)$ and hence has a subsequence converging to $\overline{y}_\infty$ in $L^p(\Gamma_s)$ (compactness of $W^{1,2}\subset L^p$) and we can assume weak convergence in $W^{1,2}(\Gamma_s)$. Since $\overline{y}_n$ satisfies

$$\partial y_n=-\frac{\lambda_n}{\rho_n}\quad\text{on } D$$
$$=-2\pi(1+\varepsilon_1)\overline{y}_n+\varepsilon_n\quad\text{on }\Gamma_s\cap\Gamma_-, \quad (4.9)$$

y_n is a weak solution

$$\overline{\partial}y_\infty=\begin{cases}0 & \text{on } D,\\ -2\pi(1+\varepsilon_1)\overline{y}_\infty & \text{on }\Gamma_s\cap\Gamma_-.\end{cases} \quad (4.10)$$

Also consider

$$0\le\frac{1}{\rho_n^2}A_H(u_n)(s)$$
$$=\frac{1}{4\rho_n^2}\left(\int_{\Gamma_s}(|\overline{\partial}x_n|^2-|\partial x_n|^2+|\overline{\partial}y_n|^2-|\partial y_n|^2)\right)-\frac{1}{\rho_n^2}\int H\left(u_n(s,\cdot)\right)$$
$$\le\frac{1}{\rho_n^2}\pi r^2. \quad (4.11)$$

For $\overline{s}<s$, elliptic estimates imply strong convergence $\overline{y}_n\mapsto\overline{y}_\infty$ in Γ_s, so that (4.11) implies

$$\frac14\int_{\Gamma_s}(|\overline{\partial}y_\infty|^2-|\partial\overline{y}_n|^2)-\int\pi(1+\varepsilon_1)|\overline{y}_n(\overline{s},\cdot)|^2$$
$$\to\frac14\int_{\Gamma_{|s}}\left(|\overline{\partial}\overline{y}_\infty|^2-|\partial y_\infty|^2\right)-\int\pi(1+\varepsilon_1)|\overline{y}_\infty(\overline{s},\cdot)|^2=A_{H_\infty}(\overline{y}_\infty)(\overline{s}) \quad (4.12)$$

and $\frac{1}{\rho_n^2}\int_{\Gamma_s}|\overline{\partial}x_n|^2$ goes to zero because $\overline{\partial}x_n-\bigtriangledown_xH(x_n,y_n)$ is bounded.

So $A_{H_\infty}(\overline{y}_\infty)(\overline{s})=\frac{1}{\rho_n^2}\int_{\Gamma_s}|\overline{\partial}x_n|^2$. But the right hand side increases with $\overline{s}$, whereas the left hand side is decreasing which follows from:

$$A_{H_\infty}(y_\infty)(\overline{s})=\frac14\int_{\Gamma_s}\left(|\overline{\partial}\overline{y}_\infty|^2-|\overline{\partial}\overline{y}_\infty|^2\right)-\int\pi(1+\varepsilon_1)|\overline{y}_\infty(\overline{s},\cdot)|^2$$
$$=\int_{\Gamma_s}\overline{y}_\infty^*\omega-\int\pi(1+\varepsilon_1)|\overline{y}_\infty(\overline{s},\cdot)|^2. \quad (4.13)$$

We calculate the derivative $A'_{H_\infty}(\overline{y}_\infty)(s)$ of $A_{H_\infty}(\overline{y}_\infty)(s)$ as usual. We obtain

$$A_{H_\infty}(\overline{y}_\infty)(\overline{s}) = -\int_s^0 \int_0^1 |\partial \overline{y}_\infty|^2 dtds. \qquad (4.14)$$

Therefore $A_{H_\infty}(y_\infty)(s)$ is a constant. Hence $\overline{y}_\infty(s,\cdot)$ is a solution of $\dot{z} = 2\pi(1+\varepsilon_1)z$ and $\overline{y}_\infty(\overline{s},\cdot) = 0$ for $\overline{s} < s$.

This, by Fubini's Theorem, contradicts $||\overline{y}_\infty||_{L^2(\Gamma_s)} = 1$. As a result, $||y_n||w^{1,2}(\Gamma_s)$ is bounded by a constant $c(\overline{s})$.

Now, we can prove the locall compactness theorem, i.e.

Theorem 4.1. *Let (λ_n, u_n) be a sequence of solutions of* (3.1)-(3.3). *Then, after taking subsequence, (λ_n, u_n) converges to (λ, u) in $W^{1,p}_{loc}$ and u satisfies $\overline{\partial}u(z) = g\lambda_c(z, u(z))$.*

Proof. See the proof of Theorem 4.1 in [16].

4.2. Global Compactness Theorem

Now if we assume that the Hamiltonian system $(\overline{H})$ has no nontrivial solution, then we can obtain the global compactness theorem from the local one, i.e.,

Therem 4.2. *Suppose that the Hamiltonian system $(\overline{H})$ has only trivial solutions. Let (λ_n, u_n) be a sequence of solutions on* (3.1)-(3.3). *Then, after taking a subsequence (λ_n, u_n) converges to some (λ, u) in $W^{1,p}(S^2, V)$ such that u satisfies* (3.1)-(3.3).

Proof. Since $H \equiv 0$ on set U and $u_0 \in U$, we can take disk $D(u_0, \varepsilon) \subset U$. We can of course assume that $z_n = (r_n, \theta_n) \in (-\infty, -1) \times S^1$ and also that $u_n(z_n)$ goes to a limit $\overline{u}_0 \in \gamma \subset S(u_0, \frac{\varepsilon}{4})$. Since the set of limit points of sequences $u_n(z_n)$ is obviously connected and contains u_0, we can even assume that r_n is the smallest (remember that $r_n \leq -1$) real number such that $u_n((-\infty, r_n) \times S^1) \subset D(u_0, \frac{\varepsilon}{2})$.

If r_n are bounded from below by a constant r_0, the theorem can be easily checked. So, we assume $r_n \to -\infty (n \to \infty)$. We now set $v_n(r,\theta) = u_n(r + r_n, \theta + \theta_n)$ for $(r,\theta) \in \overline{\partial} v_n = -\bigtriangledown H(v_n)$ on $(-\infty, -r_n) \times S^1$ and $E(v_n) = \int_{-\infty}^{-r_n}\int_0^1 |\frac{\partial v_n}{\partial_s}|^2 dtds \leq E(u_n) \leq \pi r^2 < l(M,\omega)$. Thus, the same proof as used in Theorem 4.1 shows that v_n converges on all compact subsets to some v defined on $(-\infty, +\infty) \times S^1$ and of course

$$\overline{\partial} v = -\bigtriangledown H(v), \quad E(v) = \int_{-\infty}^{+\infty}\int_0^1 \Big|\frac{\partial v}{\partial s}\Big| dtds \leq \pi r^2 < l(M,\omega).$$

We shall prove that v is a constant by four steps.

Set $v = (v_1, v_2)$, where v_1, v_2 are M, R^{2n} components of v respectively.

(1) For a sequence s_n going to infinity, then, after taking a subsequence, $v(s_n,\cdot)$ converges to a trivial solution $(x, y(t))$ of (H) in $H^1(S^1)$ and

$$\lim_{n\to\infty} \Big\| \frac{\partial v_1}{\partial t}(s_n, t) \Big\|_{L^2(S^1)} = 0.$$

To prove the assertion (1), we assume the contrary, i.e., we assume that there exists a sequence $\tau_\alpha \to \infty$ so that $v(\tau_\alpha)$ does not accumulate at $H^1(S^1)$. Then, the sequence $v_\alpha(\tau, t) = v(\tau_\alpha + \tau, t)$ on $\big[-\frac{\tau_\alpha}{2}, \frac{\tau_\alpha}{2}\big]$ satisfies the hyperthesis of Theorem 4.1 with

$$E(v_\alpha) = \int_{-\frac{\tau_\alpha}{2}}^{\frac{\tau_\alpha}{2}}\int_0^1 \Big|\frac{\partial v_\tau}{\partial \tau}\Big|^2 = \int_{\frac{\tau_\alpha}{2}}^{\frac{3\tau_\alpha}{2}}\int_0^1 \Big|\frac{\partial v}{\partial s}\Big|^2 \to 0 \ \ (\alpha \to \infty).$$

Hence u_α converges locally to some constant trajectory, by the assumption of Theorem 4.2, which is a trivial solution of (H). This contradicts the above assumption.

(2) There exist two sequences of numbers such that

$$0 \leq \lim_{n\to\infty} \int_{s_n\prime}^{s_n\prime\prime} \int_0^1 v_1^*\omega < l(M,\omega). \tag{4.15}$$

To prove (4.15), we define $A_H(v):\ R^1 \times R^1 \mapsto R^1$ as follows:

$$A_H(v)(s_1,s_2) = \int_{s_1}^{s_2} \int_0^1 v^*\omega - \int_0^1 \left(v\left((s_2,\cdot)\right) - H\left(v(s_1,\cdot)\right)\right) dt. \tag{4.16}$$

Clearly $A_H(v):\ R^1 \times R^1 \mapsto R^1$ is a smooth map. We differentiate (4.16) on variable s_2 and then integrate it as usual. Thus we obtain

$$A_H(v)(s_1,s_2) = \int_{s_1}^{s_2} \int_0^1 |\frac{\partial v}{\partial s}|^2 dtds \leq \delta\left[\pi r^2 + \text{arclengh}(\gamma)\right]. \tag{4.17}$$

We choose two sequences $s_n\prime, s_n\prime\prime$ of numbers such that $s\prime_n \to -\infty, s_n\prime\prime \to \infty$ satisfying assertion (1). Since $v_n\left((-\infty,0)\times S^1\right) = u_n\left((-\infty,r_n)\times S^1\right) \subset D(u_0,\frac{\varepsilon}{2})$ and $H \equiv 0$ on $D(u_0,\varepsilon)$, $v(s_n\prime,\cdot)$ converges to a constant by assertion (1) and $\overline{\partial} v = 0$ on $(-\infty,0)\times S^1$.

$$\begin{aligned} &\lim_{n\to\infty} \int_{s_n\prime}^{s_n\prime\prime} v_2^*\omega - \int_0^1 \left(H\left(v(s_n\prime\prime,\cdot)\right) - H\left(v(s_n\prime,\cdot)\right)\right) \\ &= \lim_{n\to\infty}\left[-\frac{1}{2} < -J\dot{v}(s_n\prime\prime,\cdot)v(s_n\prime\prime,\cdot) > - \int_0^1 H\left(v(s_n\prime\prime,\cdot)\right)\right]. \end{aligned} \tag{4.18}$$

By Lemma 1.1 and assertion (1)

$$-m(H) \leq \lim_{n\to\infty}\left[\int_{s_n\prime}^{s_n\prime\prime}\int_0^1 v_2^*\omega - \int_0^1 \left(H\left(v(s_n\prime\prime,\cdot)\right) - H\left(v(s_n\prime,\cdot)\right)\right)\right] \leq 0. \tag{4.19}$$

Note that $v^*\omega = v_1^*\omega + v_2^*\omega$. Combining (4.16—4.19), we get

$$0 \leq \lim_{n\to\infty} \int_{s_n\prime}^{s_n\prime\prime} \int_0^1 v_1^*\omega \leq \delta\left[\pi r^2 + \text{arclenth}(\gamma)\right] + m(H) < l(M,\omega) \tag{4.20}$$

for δ small enough.

(3) $\lim\limits_{n\to\infty} \int_{s_n\prime}^{s_n\prime\prime} \int_0^1 v_1^*\omega = 0$.

If assertion (3) does not hold, by assertion (2), we assume

$$0 \leq \lim_{n\to\infty} \int_{s_n\prime}^{s_n\prime\prime} \int_0^1 v_1^*\omega < l(M,\omega). \tag{4.21}$$

By assertion (1), we can fill the small loops $v_1(s_n\prime,\cdot), v_1(s_n\prime\prime,\cdot)$ by small disks $D_n\prime, D_n\prime\prime$ respectively in such a way that v on $[s_n\prime, s_n\prime] \times S^1$ gives together with the filled disks a smooth map $\overline{v}_{1,n}:\ D_n\prime \cup \left([s_n\prime, s_n\prime\prime]\times S^1\right) \cup D_n\prime\prime \mapsto M$ and by (4.21)

$$0 < \int_{s^2} \overline{v}_{1,n}^*\omega < l(M,\omega)$$

contradicting the definition of $l(M,\omega)$.

(4) v is a constant.

The assertions (1)-(3) imply $\int_{-\infty}^{\infty}\int_0^1 |\frac{\partial v}{\partial s}|^2 dtds = 0$, i.e., v is a constant trajectory. Then, $H \equiv 0$ on $D(u_0,\varepsilon)$ and $v(-\infty,0)\times S^1) \subset D(u_0,\frac{\varepsilon}{2})$ implies that v is a constant. Since $v_n(0,0) = u_n(z_n)$ goes to $\overline{u}_0$, this constant has to be $\overline{u}_0$. This gives a contradiction by an argument used by A. Floer, H. Hofer, C. Viterbo in [6]. This completes the proof of Theorem 4.2.

§5. A Fredholm Formulation of Our Problem

Let $\mathcal{B} = \{u \in W^{1,p}(S^2, V) | u(-\infty) \in \gamma$ and u homotopic to zero$\}$ and $p > 2$, $\varepsilon \to \mathcal{B}$ be the Banach vector bundle with fibre ε_u, where ε_u stands for the set of v which are L^p–sections of the bundle over S^2 with fibre $\overline{C}(T_z S^2, T_{u(z)})$. $R_\lambda(u) = \overline{\partial} u(z) - g\lambda_c(z, u(z))$ defines a c^∞–section R_λ of $\varepsilon \to \mathcal{B}$ with Fredholm index zero. Then, the degree theory on Fredholm section can be used to complete the proof of Theorem 1.1 as in [4]. For the details, one can refer to [6]. This completes the proof of Theorem 1.1.

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