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BOUNDED SOLUTIONS AND PERIODIC SOLUTIONS OF VISCOUS POLYTROPIC GAS EQUATIONS

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Abstract

A piston problem of viscous polytropic gas equations is discussed. It is shown that the global solution is bounded uniformly in time if the piston motion is bounded and that if the piston motion is periodic in time, then there exists a periodic solution to the piston problem with the same period.

Keywords Viscous polytropic gas, Uniformly bounded solution, Periodic solution
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§1. Introduction

The one-dimensional motion for the viscous gas is well formulated by the following stytem in Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \left(\dfrac{\mu u_x}{v}\right)_x + f, \end{cases} \tag{1.1}$$

where v is the specific volume, u is the velocity, p is the pressure, μ is the viscosity coefficient and f is the external force. In what follows we will assume that the gas is polytropic, i.e., $p(v) = av^{-\gamma}$, where $a > 0,\ \gamma > 1$ are positive constants, and that the viscosity coefficient is constant μ =const.> 0.

The sysyem (1.1) is considered on a fixed domain Q in the Lagrangian mass coordinate

$$Q = \{t \geq 0,\ 0 \leq x \leq 1\}$$

with the boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = u_1(t), \tag{1.2}$$

where $u_1(t)$ is a given function (piston velocity).

We treat two initial boundary value problems.

(i) The external force problem, i.e., the system (1.1), (1.2) with $u_1(t) = 0$.

(ii) The piston problem, i.e., the system (1.1), (1.2), with $f = 0$.

We only give the treatment for piston problem in this paper, the external force problem can be handled similarly. When gas is assumed polytropic, it is known by Kanel[2] and Kanzhikhov[3] that the initial boundary value problem with fixed boundary has a unique

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global in time solution which decays to the constant equilibrium state as time tends to infinity. While in the case when a piston acts on the gas, the obtained global in time solution has only a bound which depends on time (see [3] and [1]), therefore the asymptotic behavior is unknown. In the case of $p(v) = av^{-1},\ a > 0$, Matsumura and Nishida[4] showed that if the piston motion is bounded with respect to time, then the bounded solution exists globally in time, and if the piston motion is periodic in time, there exists at least a periodic solution with the same period. In contrast with the case of $p(v) = av^{-1}$ in [4], certain essential difficulties occur when we deal with the case of $p(v) = av^{-\gamma},\ a > 0,\ \gamma > 1$. For the case of $p(v) = av^{-1}$, the positive lower and upper bounds, independent of t, can be obtained easily, while in the case of $p(v) = av^{-\gamma},\ a > 0,\ \gamma > 1$, it seems that the boundness of the piston motion along is not sufficient to ensure that the specific volume v can be confined in a bounded interval $[\underline{v}, \bar{v}]$ with positive $\underline{v}$ and $\bar{v}$, independent of t. However, if the piston motion is suitably small, we can show that the bounded solution exists globally in time. For the periodic solution, we prove that if the piston motion is periodic in time, then there exists at least a periodic solution with the same period provided the piston motion $|u_1(t)| + \left|\frac{du_1(t)}{dt}\right|$ is suitably small.

§2. Uniformly Bounded Solution of the Piston Problem

In this section we discuss the piston problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + \left(\dfrac{a}{v^\gamma}\right)_x = \left(\dfrac{\mu u_x}{v}\right)_x, \quad t \geq 0, \quad x \in [0,1], \end{cases} \tag{2.1}$$

$$u(0,t) = 0, \quad u(1,t) = u_1(t), t \geq 0, \tag{2.2}$$

with the initial data

$$(v,u)(x,0) = (v_0,u_0)(x), \quad 0 \leq x \leq 1, \tag{2.3}$$

where we assume that

$$B_1^{-1} \leq v_0(x) \leq B_1 \ \text{ for a positive constant } B_1, \tag{2.4}$$

$$v_0 \in C^{1+\alpha}(0,1), \quad u_0 \in C^{2+\alpha}(0,1) \text{ for some } 0 < \alpha < 1 \tag{2.5}$$

and the compatibility condition

$$u_0(0) = 0, \quad u_0(1) = u_1(0). \tag{2.6}$$

Furthermore, we assume

$$\int_0^1 u_0(x)dx = 1 \tag{2.7}$$

for convenience. Our main result in this section is

Theorem 2.1. *If the piston does not collide the fixed boundeary $x = 0$ and does not go to positive infinity neither, i.e.,*

$$X_0^{-1} < X(t) \equiv 1 + \int_0^1 u_1(s)ds < X_0$$

for a positive constant X_0 and $\sup\limits_{t\geq 0}\left\{|u_1(t)| + \left|\frac{du_1(t)}{dt}\right|\right\}$ is suitably small, then the piston problem (2.1)-(2.3) *with* (2.4)-(2.7) *has the global solution in time, which is uniformly bounded* *with respect to time in H^1-norm. Moreover, $v(x,t)$ possesses positive lower and upper boundes, independent of t.*

Since the global existence and uniqueness with $v(x,t)>0$ $(x\in[0,1],\ t\geq 0)$ have been proved in [3] and [1], we only need to obtain the bounded estimates of the solution which will finish the proof of Theorem 2.1.

In view of $(2.1)_1$ and (2.7), it is easy to obtain that

$$\begin{aligned}\int_0^1 v(x,t)dx &= \int_0^1 v_0(x)dx+\int_0^t u_1(s)ds\\ &= 1+\int_0^t u_1(s)ds \equiv X(t).\end{aligned}$$

Now, we make the transformation of the unknown variable to get the piston boundary condition into the fixed boundary condition. Define

$$U(x,t)=\frac{u_1(t)}{X(t)}\int_0^x v(x,t)dx,$$

and let

$$w(x,t)=X(t)(u(t,x)-U(x,t)) \quad\text{and}\quad m(x,t)=v(x,t)/X(t).$$

In order to simplify the treatment of the condition $m>0$ we introduce the change of variable

$$m=e^n,$$

with which the piston problem takes the form

$$\begin{cases} n_t-\dfrac{e^n}{X(t)^2}w_x=0,\\ w_t+\dfrac{(ae^{-\gamma n})_x}{X(t)^{\gamma-1}}=\dfrac{1}{X(t)}(\mu e^{-n}w_x)_x-X(t)\dfrac{du_1(t)}{dt}\displaystyle\int_0^x e^n(\xi,t)d\xi\end{cases} \tag{2.8}$$

and

$$\int_0^1 e^n(x,t)dx=1,\quad w(0,t)=w(1,t)=0, \tag{2.9}$$

n,w are imposed with corresponding initial data.

Define

$$\varepsilon \overset{\text{def.}}{=} \sup_{t\geq 0}\left\{|u_1(t)|+\left|\frac{du_1(t)}{dt}\right|\right\}$$

and assume $\varepsilon>0$ for convenience.

In this section, C will denote a generic constant which may depend only on X_0,μ,a and γ.

After a straightforward calculation, and using (2.8), we arrive at

$$\begin{aligned}&\left\{\frac12 w^2+aX^{3-\gamma}(t)\Big[e^n-1-\frac{1}{(1-\gamma)}(e^{(1-\gamma)n}-1)\Big]\right\}_t\\ &=-aX^{1-\gamma}(e^{-\gamma n}w)_x+aX^{1-\gamma}w_x+\mu X^{-1}(e^{-n}w_x)w\\ &\quad -X\frac{du_1}{dt}w\int_0^x e^n d\xi+a(3-\gamma)X^{2-\gamma}u_1\Big[e^n-1-\frac{1}{(1-\gamma)}(e^{(1-\gamma)n}-1)\Big]\quad (t\geq 0).\end{aligned}$$

Integrating the above equation over $[0,1]$, with the help of (2.9), one can get

$$\Big\{\int_0^1\Big[\frac{1}{2}w^2+\frac{aX^{3-\gamma}}{\gamma-1}(e^{(1-\gamma)n}-1)\Big]dx\Big\}_t+\int_0^1\frac{\mu}{X}e^{-n}w_x^2dx$$
$$\le C\Big[\varepsilon\int_0^1|w|dx+\varepsilon\Big|\int_0^1(e^{(1-\gamma)n}-1)dx\Big|\Big],\quad t\ge 0. \tag{2.10}$$

Multiplying $(2.8)_2$ with $\frac{n_x}{X(t)}$ and using $(2.8)_1$, (2.9), we obtain

$$\Big\{\int_0^1\Big[\frac{\mu}{2}n_x^2-\frac{wn_x}{X(t)}\Big]dx\Big\}_t+\int_0^1\frac{\gamma ae^{-\gamma n}n_x^2}{X(t)^\gamma}dx$$
$$\le\frac{\varepsilon}{X(t)^2}\int_0^1|w||n_x|dx+\frac{1}{X(t)^3}\int_0^1e^{-n}w_x^2dx+C\varepsilon\int_0^1|n_x|dx,\quad t\ge 0. \tag{2.11}$$

Let

$$M=\max\Big\{\|n(\cdot,0)\|_{L^\infty},\ \frac{16X_0}{\mu}\sqrt{\frac{E(0)}{7}}\Big\}. \tag{2.12}$$

The definition of $E(0)$ will be given below.

Define T^* as

$$T^*=\sup\{t|\ \|n(\cdot,t)\|_{L^\infty}\le 2M,\quad t\ge 0\}. \tag{2.13}$$

We will prove that $T^*=+\infty$ provided $\varepsilon>0$ is suitably small.

At first, (2.9) implies that there exists $z(t)\in[0,1]$ for any $t\ge 0$ such that

$$n(z(t),t)=0. \tag{2.14}$$

(2.9) and Hölder inequality imply that

$$|w(x,t)|=|w(x,t)-w(0,t)|=\Big|\int_0^x w_x(\xi,t)d\xi\Big|$$
$$\le\Big(\int_0^1e^{-n}w_x^2dx\Big)^{1/2},\quad x\in[0,1],\quad t\ge 0. \tag{2.15}$$

Similarly, with the help of (2.14), it holds that

$$|n(x,t)|\le\Big(\int_0^1n_x^2dx\Big)^{1/2},\quad t\ge 0,\quad x\in[0,1]. \tag{2.16}$$

If $T^*<+\infty$, we have the following estimates.

(2.14) implies

$$|e^{(1-\gamma)n}(x,t)-1|=|e^{(1-\gamma)n}(x,t)-e^{(1-\gamma)n}(z(t),t)|$$
$$\le e^{2\gamma M}\Big(\int_0^1n_x^2dx\Big)^{1/2},\quad 0\le t\le T^*. \tag{2.17}$$

It follows from the above inequalities and Cauchy inequality that for any $\alpha>0$,

$$\Big\{\int_0^1\Big[\frac{1}{2}w^2+\frac{aX^{3-\gamma}}{\gamma-1}(e^{(1-\gamma)n}-1)+\frac{\mu\alpha}{2}n_x^2-\frac{\alpha wn_x}{X(t)}\Big]dx\Big\}_t$$
$$+\Big[\frac{\mu}{4X(t)}-\frac{3\alpha}{2}\frac{X(t)^\gamma e^{2\gamma M}}{\gamma aX(t)^4}\varepsilon^2-\frac{\alpha}{X(t)^3}\Big]\int_0^1e^{-n}w_x^2dx$$
$$+\frac{\alpha\gamma a}{2X(t)^\gamma}e^{-2\gamma M}\int_0^1n_x^2dx$$
$$\le C(1+\frac{1}{\alpha})\varepsilon^2\varepsilon^{6\gamma M},\quad 0\le t\le T^*. \tag{2.18}$$

Let

$$E(t) \overset{\text{def.}}{=} \int_0^1 \Big[\frac{1}{2}w^2 + \frac{aX^{3-\gamma}}{\gamma-1}(e^{(1-\gamma)n}-1) + \frac{\mu\alpha}{2}n_x^2 - \frac{\alpha w n_x}{X(t)}\Big](x,t)dx.$$

If ε is so small that

$$\frac{3e^{2\gamma M}\varepsilon^2}{2\gamma a} \le \frac{1}{X_0^{\gamma-1}} \le \frac{1}{X(t)^{\gamma-1}},$$

which implies

$$\frac{3e^{2\gamma M}\varepsilon^2 X(t)^{\gamma-4}}{2\gamma a} \le \frac{1}{X(t)^3},$$

then we can take $\alpha = \frac{\mu}{16X_0^2}$. Noting that $X_0 \ge 1$ due to $X_0^{-1} \le X_0$, it is easy to check that

$$\begin{aligned}&\int_0^1 \Big[\frac{1}{4}w^2 + \frac{aX^{3-\gamma}}{\gamma-1}(e^{(1-\gamma)n}-1) + \frac{7\mu^2}{256X_0^2}n_x^2\Big](x,t)dx \\ &\le E(t) \le \int_0^1 \Big[\frac{3}{4}w^2 + \frac{aX^{3-\gamma}}{\gamma-1}(e^{(1-\gamma)n}-1) + \frac{9\mu^2}{256X_0^2}n_x^2\Big](x,t)dx, \quad 0 \le t \le T^*,\end{aligned} \tag{2.19}$$

and

$$E(t)_t + \frac{\mu}{8X_0}\int_0^1 e^{-n}w_x^2 + \frac{\gamma a\mu}{32X_0^{2+\gamma}}e^{-2\gamma M}\int_0^1 n_x^2 dx \le C\varepsilon^2 e^{6\gamma M}, \quad 0 \le t \le T^*. \tag{2.20}$$

(2.17) implies that

$$\int_0^1 e^{(1-\gamma)n}dx \le \frac{1}{2\varepsilon}e^{-2\gamma M}\int_0^1 n_x^2 dx + \frac{\varepsilon}{2}e^{6\gamma M}, \quad 0 \le t \le T^*. \tag{2.21}$$

Since $e^n - \frac{1}{(1-\gamma)}(e^{(1-\gamma)n}-1) - 1 > 0$ for any n, it follows from (2.15), (2.19), (2.20) and (2.21) that if ε is so small that $\varepsilon \le K_1 e^{-2\gamma M}$ (K_1 is a positive constant not dependent on ε and μ), then

$$E(t)_t + \beta_1\varepsilon E(t) \le C\varepsilon^2 e^{6\gamma M}, \quad 0 \le t \le T^*, \tag{2.22}$$

where $\beta_1 > 0$ is a positive constant. It reads from (2.22) that

$$E(t) \le E(0) + \frac{C\varepsilon}{\beta_1}e^{6\gamma M}, \quad 0 \le t \le T^*, \tag{2.23}$$

which together with (2.16) and (2.19) gives

$$\|n(x,t)\|_{L^\infty} \le \Big[\frac{256X_0^2}{7\mu^2}\Big(E(0) + \frac{C\varepsilon}{\beta_1}e^{6\gamma M}\Big)\Big]^{1/2}, \quad 0 \le t \le T^*. \tag{2.24}$$

Since $\frac{16X_0}{\mu}\sqrt{\frac{E(0)}{7}} \le M$, it is known from (2.24) that if ε is suitably small, then

$$\|n(\cdot,T^*)\|_{L^\infty} < 2M.$$

This conflicts with the definition of T^*. Hence $T^* = +\infty$, which implies that $\|n(\cdot,t)\|_{L^\infty} \le 2M$ for $t \ge 0$. At last, it is not difficult to obtain

$$\frac{d}{dt}\Big(E(t) + \frac{\beta_2}{2}\int_0^1 w_x^2 dx\Big) + \nu\Big(E(t) + \frac{\beta_2}{2}\int_0^1 w_x^2 dx\Big) \le C, \quad t \ge 0$$

if we choose β_2 and ν small. This completes the proof of Theorem 2.1.

§3. Periodic Solutions

In this section, we suppose that the piston path$X(t)$ and so $u_1(t)=\frac{dX}{dt}$ be periodic with respect to time with periodic T, and $X_0^{-1}\le X(t)\le X_0$, for $t\in[0,1]$ and $X_0>0$. The mian result is

Theorem 3.1. *For any given $\delta>0$, there exists at least one periodic solution (n,w) for* (2.8), (2.9) *with the period T in the set $\{(n,w)|\,|n(x,t)|\le\delta,\ |w(x,t)|\le\delta,\ x\in[0,1],\ t\in[0,T]\}$, pervided*

$$\varepsilon\overset{\text{def.}}{=}\sup_{0\le t\le T}\Big\{|u_1(t)|+\Big|\frac{du_1(t)}{dt}\Big|\Big\}$$

is suitably small. Moreover, it holds that

$$\begin{cases} n\in C(0,T,H^1),\ n_t\in C(0,T,L^2)\cap L^2(0,T,H^1),\\ w\in C(0,T,H^1)\cap L^2(0,T,H^2),\ w_t\in L^2(0,T,L^2).\end{cases}$$

The proof of this theorem is divided into two parts:

(i) discretization with respect to the space variable,

(ii) energy estimstes to apply Leray-Schauder fixed point theory.

Let us discretize as follows

$$\begin{cases}\Delta x=1/N,\ n_i(t)=n(t,(i-1/2)\Delta x),\quad i=1,2,\cdots,N,\\ w_i(t)=w(t,i\Delta x),\quad i=1,2,\cdots,N-1,\\ n=(n_1,n_2,\cdots,n_N),\quad w=(w_1,\cdots,w_{N-1}).\end{cases}\tag{3.2}$$

Consider the system of nonlinear ordinary differential equations

$$\begin{cases} n_{it}-\dfrac{e^{-n_1}}{X(t)^2}\dfrac{w_i-w_{i-1}}{\Delta x}=0,\quad i=1,2,\cdots,N,\\ w_{it}+\dfrac{a(e^{-\gamma n_{i+1}}-e^{-\gamma n_i})}{X(t)^{\gamma-1}\Delta x}-\dfrac{\mu}{X(t)\Delta x}\Big(e^{-n_{i+1}}\dfrac{w_{i+1}-w_i}{\Delta x}\\ -e^{-n_i}\dfrac{w_i-w_{i-1}}{\Delta x}\Big)=q_i,\quad i=1,2,\cdots,N-1,\end{cases}\tag{3.3}$$

where

$$q_i=X(t)\frac{du_1}{dt}\sum_{k=1}^{i}\exp(n_k)\Delta x,\tag{3.4}$$

the boundary condition

$$w_0(t)=w_N(t)=0\tag{3.5}$$

is imposed and the condition

$$\sum_{i=1}^{N}\exp(n_i(t))\Delta x=1,\quad t\in[0,1]\tag{3.6}$$

is required.

At first, we establish the following lemma.

Lemma 3.1. *For any given $\delta>0$, there exists a periodic solution (n,w) for* (3.3)-(3.6) *with period T in set $\{(n,w)|\,|n|_\infty\le\delta,\quad|w|_\infty\le\delta\}$, provided*

$$\varepsilon=\sup_{0\le t\le T}\Big\{|u_1(t)|+\Big|\frac{du_1(t)}{dt}\Big|\Big\}$$

is suitably small, where

$$|n|_\infty = \max\{|n_1|, |n_2|, \cdots, |n_N|\},$$
$$|w|_\infty = \max\{|w_1|, |w_2|, \cdots, |w_{N-1}|\}.$$

Since $\varepsilon = 0$, i.e., $u_1(t) \equiv 0$ $(0 \le t \le T)$ is a triaval case, we assume $\varepsilon > 0$ for convenience.

For the proof of Lemma 3.1, the following lemma is needed, which can be found in [4].

Lemma 3.2. *Let* $h(t) = (h_1(t), h_2(t), \cdots, h_N(t))$ *and* $d(t) = (d_1(t), d_2(t), \cdots, d_{N-1}(t))$ *be smooth and periodic in* t *with period* T*. Assume*

$$\sum_{i=1}^{N} \int_0^T h_i(s)ds = 0. \tag{3.7}$$

Then the following system of linear differential equations (3.8)-(3.10) *for* (n, w) *has a unique periodic solution* (n, w) *with the same period* T*.*

$$n_{it} - \frac{w_i - w_{i-1}}{\Delta x} = h_i, \quad i = 1, 2, \cdots, N, \tag{3.8}$$

$$w_{it} - \frac{a(n_{i+1} - n_i)}{\Delta x} - \frac{\mu}{\Delta x}\Big(\frac{w_{i+1} - w_i}{\Delta x} - \frac{w_i - w_{i-1}}{\Delta x}\Big) = d_i, \quad i = 1, 2, \cdots, N-1, \tag{3.9}$$

with $w_0(t) = w_N(t) = 0$ *and*

$$\int_0^T \sum_{i=1}^{N} \exp(n_i(s))\Delta x dx = T. \tag{3.10}$$

We will use Leray-Schauder fixed point theorem to prove Lemma 3.1. For this purpose, we denote $\overline{X} = \frac{1}{T}\int_0^T X(t)dt$, the mean positive of the piston, and define

$$\begin{cases} X(t,\lambda) = \overline{X} + \lambda(X(t) - \overline{X}), \quad 0 \le \lambda \le 1, \\ u_1(t,\lambda) = \dfrac{dX(t,\lambda)}{dt}, \\ f_i(t,\lambda) = X(t,\lambda)\dfrac{du_1(t,\lambda)}{dt}\displaystyle\sum_{k=1}^{i}\exp(n_k(t))\Delta x. \end{cases} \tag{3.11}$$

Rewrite the system (3.3) in the form

$$\begin{cases} n_{it} - \dfrac{w_i - w_{i-1}}{\Delta x} = h_i, \quad i = 1, 2, \cdots, N, \\ w_{it} + \dfrac{a(n_{i+1} - n_i)}{\Delta x} - \dfrac{\mu}{\Delta x}\Big(\dfrac{w_{i+1} - w_i}{\Delta x} - \dfrac{w_i - w_{i-1}}{\Delta x}\Big) = g_i, \quad i = 1, 2, \cdots, N-1, \end{cases}$$

where

$$h_i = \Big(\frac{e^{-n_i}}{X(t,\lambda)^2} - 1\Big)\frac{w_i - w_{i-1}}{\Delta x} - \bar{h},$$

$$\begin{aligned} g_i = {} & \frac{\mu}{\Delta x}\Big(\Big(\frac{e^{-n_{i+1}}}{X(t,\lambda)} - 1\Big)\frac{w_{i+1} - w_i}{\Delta x} - \Big(\frac{e^{-n_i}}{X(t,\lambda)} - 1\Big)\frac{w_i - w_{i-1}}{\Delta x}\Big) \\ & - \frac{a(e^{-\gamma n_{i+1}} - e^{-\gamma n_i})}{\Delta x(X(t,\lambda))^{\gamma-1}} - \frac{a(n_{i+1} - n_i)}{\Delta x} - f_i(t,\lambda), \end{aligned}$$

$$\bar{h} = \frac{1}{TN}\int_0^T \sum_{i=1}^{N} \frac{e^{-n_i}}{X(t,\lambda)^2}\frac{w_i - w_{i-1}}{\Delta x}dt. \tag{3.13}$$

Here we also suppose

$$w_0(t) = w_N(t) = 0 \tag{3.14}$$

and

$$\sum_{i=1}^{N} \exp(n_i(t))\Delta x = 1, \quad 0 \le t \le T. \tag{3.15}$$

The periodic solution of the original problem (3.3)-(3.5) is equivalent to the periodic solution of (3.12)-(3.15) in the case of $\lambda = 1$. Since $\bar{h}$ is defined in (3.13) and so the right hand side of (3.12) satisfies the condition (3.7), we can apply Lemma 3.2 to the "linear inhomogenous" system (3.12)-(3.15). Let us denote the solution operator of (3.8)-(3.10) by L^{-1}. Then the problem (3.12)-(3.15) is equivalent to the equation

$$(n, w) = F(n, w, \lambda) = L^{-1}(h(n, w, \lambda), g(n, w, \lambda)). \tag{3.16}$$

To obtain a fixed point of mapping F for $\lambda = 1$, we apply Leray-Schauder Theorem in the form[5].

Theorem (Leray-Schauder). *Let K be a non-empty bounded open convex set in a Banach space B, and F be a continuous mapping from $\overline{K} \times [0, 1]$ into B. Suppose*

(i) *F is compact,*

(ii) *There exists a unique point $x_0 \in K$ such that $F(x_0, 0) = x_0$,*

(iii) *F is Frechét differentiable at $(x_0, 0)$ and $I - F_x(x_0, 0)$ has the inverse in $L(B, B)$,*

(iv) *$F(x, \lambda) \ne x$ for any $x \in \partial K$ and $\lambda \in [0, 1)$.*

Then there exists a fixed point $x_1 \in \overline{K}$ such that $F(x_1, 1) = x_1$.

For any given $\delta > 0$, let

$$K = \{(n, w)(t) = (n_1(t), \cdots, n_N(t), w_1(t), \cdots, w_{N-1}(t)),$$

bounded continuous and periodic in t with period T, with the norm $\max\limits_{0\le t\le T}\{|n|_\infty,\ |w|_\infty \le \delta\}$,

$$B = \{(n, w)(t)|(n, w) \text{ bounded continuous and periodic in } t \text{ with period } T\}.$$

In the following we derive the a priori estimate for the solution of (3.16) to guarantee (iv). In this process we can also see that $x_0 = (0, 0)$ for $\lambda = 0$ is our case and the remaining conditions are easy to see. If (n, w) is the periodic solution for (3.12)-(3.15) with $\lambda \in [0, 1)$, then it satisfies the equation

$$\begin{cases} n_{it} - \dfrac{e^{-n_i}}{X^2}\dfrac{w_i - w_{i-1}}{\Delta x} = 0, \quad i = 1, 2, \cdots, N-1, \\ w_{it} + \dfrac{a(e^{-\gamma n_{i+1}} - e^{-\gamma n_i})}{X^{\gamma-1}\Delta x} - \dfrac{\mu}{X\Delta x}\Big(e^{-n_{i+1}}\dfrac{w_{i+1} - w_i}{\Delta x} \\ - e^{-n_i}\dfrac{w_i - w_{i-1}}{\Delta x}\Big) = f_i, \quad i = 1, 2, \cdots, N-1, \end{cases} \tag{3.17}$$

where f_i is given by (3.11),

$$X = X(t, \lambda), \quad u_1 = u_1(t, \lambda).$$

Let

$$\varepsilon(\lambda) = \sup_{0\le t\le T}\Big\{|u_1(t, \lambda)| + \Big|\frac{du_1(t, \lambda)}{dt}\Big|\Big\}.$$

Then

$$\varepsilon(\lambda) \le \varepsilon, \quad \lambda \in [0, 1), \quad \varepsilon(0) = 0.$$

From (3.17), (3.14) and (3.15) we have

$$\begin{aligned}&\sum_{i=1}^{N}\Big[\frac{1}{2}w_i^2+aX^{3-\gamma}\Big(e^{n_i}-1-\frac{1}{(1-\gamma)}(e^{(1-\gamma)n_i}-1)\Big)\Big]_t\Delta x\\&+\sum_{i=1}^{N}\frac{\mu e^{-n_i}}{X}\Big(\frac{w_i-w_{i-1}}{\Delta x}\Big)^2\Delta x\\&=\sum_{i=1}^{N}\Big\{f_iw_i+a(3-\gamma)X^{2-\gamma}u_1\Big[\frac{1}{\gamma-1}(e^{(1-\gamma)n_i}-1)\Big]\Big\}\Delta x.\end{aligned}\tag{3.18}$$

(3.17) implies that

$$\frac{w_i}{X}+\frac{a(e^{-\gamma n_{i+1}}-e^{-\gamma n_i})}{X^\gamma\Delta x}-\frac{\mu}{\Delta x}(n_{i+1}-n_i)_t=\frac{f_i}{X}.$$

Multiplying the above $(n_{i+1}-n_i)$ and summing up with respect to i, we have

$$\begin{aligned}&\sum_{i=1}^{N-1}\Big[\frac{\mu}{2}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2-\frac{w_i}{X}\frac{n_{i+1}-n_i}{\Delta x}\Big]_t\Delta x\\&-\sum_{i=1}^{N-1}\frac{a}{X^\gamma}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)\cdot\frac{e^{-\gamma n_{i+1}}-e^{-\gamma n_i}}{\Delta x}\Delta x\\&=\sum_{i=1}^{N-1}\Big[\Big(\frac{u_1w_i}{X^2}-\frac{f_i}{X}\Big)\frac{n_{i+1}-n_i}{\Delta x}+\frac{e^{-n_i}}{X^3}\Big(\frac{w_i-w_{i-1}}{\Delta x}\Big)^2\Big]\Delta x.\end{aligned}\tag{3.19}$$

This, combined with (3.18), yields the following lemma.

Lemma 3.3. *For any given $\delta>0$, if*

$$\varepsilon=\sup_{0\le t\le T}\Big\{|u_1(t)|+\Big|\frac{du_1(t)}{dt}\Big|\Big\}$$

is suitably small, then $|n(t)|_\infty\le\delta$ implies $|n(t)|_\infty<\delta$ and $|w(t)|_\infty<\delta$ for the periodic solution (n,w) of the operator equation (3.16) with $\lambda\in[0,1)$.

Proof. Let

$$\begin{aligned}Q&\overset{\text{def.}}{=}-\sum_{i=1}^{N-1}\frac{n_{i+1}-n_i}{\Delta x}\frac{e^{-\gamma n_{i+1}}-e^{-\gamma n_i}}{\Delta x}\Delta x\\&=\sum_{i=1}^{N-1}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2\int_0^1\exp(-\gamma(n_i+\theta(n_{i+1}-n_i)))d\theta\Delta x.\end{aligned}$$

Using this notation we estimate the first term on the right hand side of (3.19), i.e.,

$$\begin{aligned}&\Big|\sum_{i=1}^{N-1}\frac{f_i}{X}\frac{n_{i+1}-n_i}{\Delta x}\Delta x\Big|\\&\le X_0\varepsilon(\lambda)\sum_{i=1}^{N-1}\Big|\frac{n_{i+1}-n_i}{\Delta x}\Big|\Big(\int_0^1\exp(-\gamma(n_i+\theta(n_{i+1}-n_i)))d\theta\Big)^{1/2}\\&\cdot\Big(\int_0^1\exp(-\gamma(i+\theta(n_{i+1}-n_i)))d\theta\Big)^{1/2}\Delta x\\&\le X_0\varepsilon(\lambda)Q^{1/2}\Big(\sum_{i=1}^{N-1}\Delta x\Big/\int_0^1\exp(-\gamma(n_i+\theta(n_{i+1}-n_i)))d\theta\Big)^{1/2},\end{aligned}\tag{3.20}$$

$$\sum_{i=1}^{N-1}\Delta x\Big/\Big(\int_0^1\exp(-\gamma(n_i+\theta(n_{i+1}-n_i)))d\theta\Big)$$
$$\le\sum_{i=1}^{N-1}\Big(\int_0^1\exp(\gamma(n_i+\theta(n_{i+1}-n_i)))d\theta\Big)\Delta x$$
$$\le\frac{1}{2}\exp(\gamma-1)\delta\cdot\sum_{i=1}^{N-1}(\exp(n_{i+1})+\exp(n_i))\Delta x$$
$$\le\exp(\gamma-1)\delta. \tag{3.21}$$

Then we deduce that

$$\Big|\sum_{i+1}^{N-1}\frac{f_i}{X}\frac{n_{i+1}-n_i}{\Delta x}\Delta x\Big|\le\frac{a}{6X_0^\gamma}Q+\Lambda[\varepsilon(\lambda)]^2\exp(\gamma-1)\delta, \tag{3.22}$$

whereafter, Λ denotes a generic constant which may depend only on X_0,μ,a and γ.

Since

$$|w_i|\le\Big(\sum_k e^k\Delta x\Big)^{1/2}\Big(\sum_k\Big|\frac{w_k-w_{k-1}}{\Delta x}\Big|e^{-n_k}\Delta x\Big)^{1/2},$$

we have

$$\sum|f_iw_i|\Delta x\le|f|_\infty|w|_\infty\le\frac{\mu}{2X_0}\sum e^{-n_i}\Big(\frac{w_i-w_{i-1}}{\Delta x}\Big)^2\Delta x+\Lambda(\varepsilon(\Lambda))^2 \tag{3.24}$$

and

$$\Big|\sum\frac{u_iw_i}{X^2}\frac{n_{i+1}-n_i}{\Delta x}\Big|$$
$$\le\frac{a}{6X_0^\gamma}Q+\Lambda(\varepsilon(\lambda))^2\exp(\gamma-1)\delta\sum e^{-n_i}\Big(\frac{w_i-w_{i-1}}{\Delta x}\Big)^2\Delta x. \tag{3.25}$$

It can be shown from (3.15) that

$$|n|_\infty\le Q^{1/2}\exp[(\gamma-1)\delta/2]. \tag{3.26}$$

Due to $\gamma>1$ and $\sum\limits_{i=1}^N e^{n_i}\Delta x=1$, it holds that

$$\sum_{i=1}^N e^{(1-\gamma)n_i}\Delta x\ge 1 \tag{3.27}$$

and there exists $k\in\{1,2,\cdots,N\}$ such that

$$e^{n_k}\ge 1,$$

namely $r^{(1-\gamma)n_k}\le 1$. Thus, it follows from (3.21) and (3.27) that

$$\Big|\sum_{i=1}^N(e^{(1-\gamma)n_i}-1)\Delta x\Big|=\sum_{i=1}^N(e^{(1-\gamma)n_i}-1)\Delta x\le\sum_{i=1}^N(e^{(1-\gamma)n_i}-e^{(1-\gamma)n_k})\Delta x.$$

We estimate $|e^{(1-\gamma)n_i}-e^{(1-\gamma)n_k}|$ as follows:

$$|e^{(1-\gamma)n_i}-e^{(1-\gamma)n_k}|\le\sum_{i=1}^{N-1}|e^{(1-\gamma)n_{i+1}}-e^{(1-\gamma)n_i}|\le Q^{1/2}\exp(\gamma-1)\delta,$$

where the similar approach as used in the (3.20)-(3.20) is used. So, we have

$$\Big|\sum_{i=1}^N(e^{(1-\gamma)n_i}-1)\Big|\Delta x\le Q^{1/2}\exp(\gamma-1)\delta. \tag{3.28}$$

The above estimstes imply the following two inequalities

$$\begin{aligned}&\sum_{i=1}^{N}\Big[\frac{1}{2}w_i^2+aX^{3-\gamma}\Big(e^{n_i}-1-\frac{1}{(1-\gamma)}(e^{(1-\gamma)n_i}-1)\Big)\Big]_t\Delta x\\&+\frac{\mu}{2X_0}\sum_{i=1}^{N}e^{-n_i}\Big(\frac{w_i-w_{i-1}}{\Delta x}\Big)^2\Delta x\\&\le\Lambda[\varepsilon(\lambda)]^2+\Lambda\varepsilon(\lambda)Q^{1/2}\exp[(\gamma-1)\delta]\end{aligned}\tag{3.29}$$

and

$$\begin{aligned}&\sum_{i=1}^{N-1}\Big[\frac{\mu}{2}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2-\frac{w_i}{X}\frac{n_{i+1}-n_i}{\Delta x}\Big]_t\Delta x+\frac{aQ}{2X_0^\gamma}\\&\le\Lambda[\varepsilon(\lambda)]^2\exp(\gamma-1)\delta+\Lambda[\varepsilon(\lambda)]^2\exp[(\gamma-1)\delta]\sum_{i=1}^{N-1}e^{-n_i}\Big(\frac{w_i-w_{i-1}}{\Delta x}\Big)^2\Delta x.\end{aligned}\tag{3.30}$$

Multiplying (3.30) with a positive number σ, which will be chosen latter, we get the following estimate, with the help of (3.29).

$$\begin{aligned}&\sum_{i=1}^{N}\Big[\frac{1}{2}w_i^2+aX^{3-\gamma}\Big(e^{n_i}-1-\frac{1}{(1-\gamma)}(e^{(1-\gamma)n_i}-1)\Big)\\&+\frac{\mu\sigma}{2}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2-\frac{\sigma w_i}{X}\frac{n_{i+1}-n_i}{\Delta x}\Big]_t\Delta x\\&+\Big(\frac{\mu}{2X_0}-\Lambda\sigma\varepsilon^2\exp(\gamma-1)\delta\Big)\sum_{i=1}^{N}e^{-n_i}\Big(\frac{w_i-w_{i-1}}{\Delta x}\Big)^2\Delta x+\frac{a\sigma}{4X_0^\gamma}Q\\&\le\Lambda\exp[2(\gamma-1)\delta][\varepsilon(\lambda)]^2\le\lambda\exp[2(\gamma-1)]\varepsilon^2.\end{aligned}\tag{3.31}$$

For given $\delta>0$, if ε is suitably small such that $\varepsilon^2\le K_2\exp(1-\gamma)\delta$, where $K_2>0$ is a constant not depending on δ and ε, then we can choose $\sigma>0$ suitably such that it holds that

$$\frac{\mu}{2X_0}-\Lambda\sigma\varepsilon^2\exp(\gamma-1)\delta\ge a_0>0\tag{3.32}$$

for a positive constant a_0, and

$$\begin{aligned}a_1w_i^2+a_2\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2&\le\frac{1}{2}w_i^2+\frac{\mu\sigma}{2}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2-\frac{\sigma w_i}{X}\frac{n_{i+1}-n_i}{\Delta x}\\&\le a_1'w_i^2+a_2'\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2,\end{aligned}\tag{3.33}$$

where a_i, a_i' $(i=1,2,)$ are positive constants. Since $\varepsilon(0)=0$, (3.33)-(3.35) imply (ii), (3.28) yields

$$\begin{aligned}\Big|\sum\frac{1}{\gamma-1}(e^{(1-\gamma)n_i}-1)\Big|&=\sum\frac{1}{\gamma-1}(e^{(1-\gamma)n_i}-1)\\&\le\frac{Q}{2\varepsilon}+\Lambda\varepsilon\exp[2(\gamma-1)\delta].\end{aligned}\tag{3.34}$$

It is easy to check

$$\sum_{i=1}^{N-1}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2\Delta x\le Q\exp[2(\gamma-1)\delta].\tag{3.35}$$

With a similar approach as in Section 2, we can get from (3.31)-(3.35) that

$$\frac{dA(t)}{dt} + \varepsilon A(t) \le \Lambda\varepsilon^2 \exp[2(\gamma-1)\delta], \quad 0 \le t \le T, \tag{3.36}$$

if ε is suitably small, where

$$\begin{aligned} A(t) = \sum_{i=1}^{N-1} \Big[\frac{1}{2}w_i^2 + aX^{3-\gamma}\Big[e^{n_i} - 1 - \frac{1}{(1-\gamma)}(e^{(1-\gamma)n_i} - 1)\Big] \\ + \frac{\mu\sigma}{2}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2 - \frac{\sigma w_i}{X}\frac{n_{i+1}-n_i}{\Delta x}\Big]\Delta x. \end{aligned}$$

Together with the fact that $A(t)$ is periodic in t with period T, (3.36) implies

$$\int_0^T A(t)dt \le \Lambda T\varepsilon \exp[2(\gamma-1)\delta]. \tag{3.37}$$

With the help of the following Lemma 3.4, (3.31) and (3.37), we obtain

$$\max_{0\le t\le T} A(t) \le \Lambda(\varepsilon + \varepsilon^2 T)\exp[2(\gamma-1)\delta]. \tag{3.38}$$

Lemma 3.4. *Let $A(t) \ge 0$ and $z(t)$ be periodic with period T. If $\frac{dA(t)}{dt} \le z(t)$, then it holds that*

$$\max_{0\le t\le T} A(t) \le \frac{1}{T}\int_0^T A(t)dt + \int_0^t |z(t)|dt.$$

The proof of this lemma can be found in [4]. By virtue of (3.26), (3.35) and (3.27), one gets

$$\begin{aligned} |n|_\infty &\le \Big\{\sum_{i=1}^{N-1}\Big[\frac{n_{i+1}-n_i}{\Delta x}\Big]^2\Delta x\Big\}^{1/2} \cdot \exp(\gamma - 1/2)\delta \\ &\le \Big(\frac{1}{a_2}A(t)\Big)^{1/2} \\ &\le \Big[\frac{\Lambda}{a_2}(\varepsilon+\varepsilon^2 T)\Big]^{1/2}\exp(2\gamma - 3/2)\delta, \end{aligned}$$

with which we have that $|n(t)|_\infty < \delta$ $(0 \le t \le T)$ if ε is suitably small.

We turn to the estimate of $|w|_\infty$ next. Multiplying $(3.12)_2$ with $\frac{1}{\Delta x}\big(\frac{w_{i+1}-w_i}{\Delta x} - \frac{w_i - w_{i-1}}{\Delta x}\big)$, we obtain

$$\begin{aligned} &\sum_{i=1}^{N-1}\frac{1}{2}\Big(\frac{w_{i+1}-w_i}{\Delta x}\Big)_t^2\Delta x + \frac{\mu}{2X_0}e^{-\delta}\sum_{i=1}^{N-1}\Big\{\frac{1}{\Delta x}\Big[\frac{w_{i+1}-w_i}{\Delta x} - \frac{w_i-w_{i-1}}{\Delta x}\Big]\Big\}^2\Delta x \\ &\le \Lambda e^{\delta}\max_i\Big|\frac{w_{i+1}-w_i}{\Delta x}\Big|^2\sum_{i=1}^{N-1}\Big(\frac{e^{-n_{i+1}} - e^{-n_i}}{\Delta x}\Big)^2\Delta x + \Lambda\varepsilon^2 e^{\delta} \\ &\quad + \Lambda e^{\delta}\sum_{i=1}^{N_1}\Big(\frac{e^{-\gamma n_{i+1}} - e^{-\gamma n_i}}{\Delta x}\Big)^2\Delta x \\ &\le \Lambda e^{3\delta}\max_i\Big|\frac{w_{i+1}-w_i}{\Delta x}\Big|^2\sum_{i=1}^{N-1}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2\Delta x + \Lambda\varepsilon^2 e^{\delta} \\ &\quad + \Lambda\exp[(2\gamma+1)\delta]\cdot\sum_{i=1}^{N-1}\Big(\frac{n_{i+1}-n_i}{\Delta x}\Big)^2\Delta x. \end{aligned} \tag{3.39}$$

Suppose that $l, j \in \{1, 2, \cdots, N-1\}$ satisfy

$$\Big[\frac{w_{l-1}-w_l}{\Delta x}\Big]^2 \le \sum_{i=1}^{N-1}\Big[\frac{w_{i+1}-w_i}{\Delta x}\Big]^2 \Delta x$$

and

$$\max_i \Big[\frac{w_{i+1}-w_i}{\Delta x}\Big]^2 = \Big[\frac{w_{j+1}-w_j}{\Delta x}\Big]^2.$$

Therefore

$$\begin{aligned}
&\max_i \Big[\frac{w_{i+1}-w_i}{\Delta x}\Big]^2 \\
&\le \Big[\frac{w_{j+1}-w_j}{\Delta x}\Big]^2 - \Big[\frac{w_{l+1}-w_l}{\Delta x}\Big]^2 + \sum_{i=1}^{N-1}\Big[\frac{w_{i+1}-w_i}{\Delta x}\Big]^2\Delta x \\
&\le \sum_{i=1}^{N_1}\Big|\Big[\frac{w_{i+1}-w_i}{\Delta x}\Big]^2 - \Big[\frac{w_i-w_{i-1}}{\Delta x}\Big]^2\Big| + \sum_{i=1}^{N-1}\Big[\frac{w_{i+1}-w_i}{\Delta x}\Big]^2\Delta x \\
&\le \sum_{i=1}^{N-1}\Big|\frac{w_{i+1}-w_i}{\Delta x} - \frac{w_i-w_{i-1}}{\Delta x}\Big|\Big|\frac{w_{i+1}-w_i}{\Delta x} + \frac{w_i-w_{i-1}}{\Delta x}\Big| \\
&\quad + \sum_{i=1}^{N-1}\Big[\frac{w_{i+1}-w_i}{\Delta x}\Big]^2\Delta x. \qquad (3.40)
\end{aligned}$$

Thus, (3.33), (3.37), (3.38), (3.39) and (3.40), with the help of Cauchy inequality, imply that

$$\begin{aligned}
&\sum_{i=1}^{N-1}\frac{1}{2}\Big(\frac{w_{i+1}-w_i}{\Delta x}\Big)_t^2\Delta x + \frac{\mu e^{-\delta}}{4X_0}\sum_{i=1}^{N-1}\Big\{\frac{1}{\Delta x}\Big(\frac{w_{i+1}-w_i}{\Delta x} - \frac{w_i-w_{i-1}}{\Delta x}\Big)\Big\}^2\Delta x \\
&\le \Lambda\varepsilon^2\exp\delta + \Lambda(\varepsilon+\varepsilon^2 T)\exp(4\gamma-1)\delta \\
&\quad + \Lambda(\varepsilon+\varepsilon^2 T)^2\exp[(4\gamma+3)\delta]\sum_{i=1}^{N-1}\Big(\frac{w_i-w_{i-1}}{\Delta x}\Big)^2\Delta x. \qquad (3.41)
\end{aligned}$$

Integrating (3.31) over $[0, T)$, one gets

$$\int_0^T\sum_{i=1}^{N-1}\Big(\frac{w_{i+1}-w_i}{\Delta x}\Big)^2\Delta x dt \le \Lambda\exp[(2\gamma-1)\delta]\varepsilon^2 T. \qquad (3.42)$$

It turns out from (3.41), (3.42) and Lemma 3.4 that

$$\begin{aligned}
&\max_{0\le t\le T}\sum_{i=1}^{N-1}\Big(\frac{w_{i+1}-w_i}{\Delta x}\Big)^2\Delta x \\
&\le \Lambda\varepsilon^2\exp(2\gamma-1)\delta + \Lambda\varepsilon^2 T\exp\delta \\
&\quad + \Lambda(\varepsilon+\varepsilon^2T)^2\varepsilon^2T^2\exp[(6\gamma+2)\delta] + \Lambda(\varepsilon+\varepsilon^2T)T\exp[(4\gamma-1)\delta].
\end{aligned}$$

Since

$$|w(t)|_\infty^2 \le \Big(\sum_{i=1}^{N-1}\Big(\frac{w_{i+1}-w_i}{\Delta x}\Big)^2\Delta x\Big)^{1/2} \text{ due to } w_0(t) = w_N(t) = 0,$$

we can choose ε so small that

$$\max_{0\le t\le T}|w(t)|_\infty < \delta.$$

This completes the proof of Lemma 3.2.

Lemma 3.3 and Leray-Schauder Theorem imply that there exists a solution $(n, w) \in \overline{K}$ of (3.16) in the case of $\lambda = 1$, i.e., the system (3.3)-(3.6) has a periodic solution $(n, w) \in \overline{K}$. All the estimates in the proof of Lemma 3.2 are valid for the obtained periodic solution if we use ε instead of $\varepsilon(\lambda)$. Because all of these estimates do not depend on N, we can take the limit as $N \to +\infty$ along a subsequence to obtain Theorem 3.1 as in [4]. The remaining proof of Theorem 3.1 is the same as in [4], we omit it.

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