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SPLIT INCLUSION AND METRICALLY NUCLEAR MAP**

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Abstract

The author relates the split inclusion property to the metrically nuclearty of the natural embedding ϕ_1 and proves the following result.

Let $A \subset B$ be an inclusion of factors, ω a faithful normal state of B such that $\omega = (\cdot\Omega, \Omega)$. If $\phi_1 : A \to L^1(B)$ defined by $\phi_1(x) = (\cdot\Omega, J_B x\Omega), \forall x \in A$, is the natural embedding, then (A, B) is a split inclusion if and only if ϕ_1 is a metrically nuclear map.

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§1. Introduction

When M is a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ω, the following natural embeddings associated with Ω are of particular physical interests:

$$\begin{array}{ccc} x \in L^\infty(M) & \xrightarrow{\phi_1} & (\cdot\Omega, Jx\Omega) \in L^1(M) \\ \phi_2 \searrow & & \\ & \triangle^{\frac{1}{4}} x\Omega \in L^2(M) & \end{array}$$

where $\triangle, J$ are the modular operator and conjugation associated with M and Ω.

Definition 1.1. *A W^*-inclusion (A, B) means that A, B are von Neumann algebras such that $A \subset B$. A W^*-inclusion $A \subset B$ is called split if there exists a type I factor N such that $A \subset N \subset B$.*

D. Buchholz, C. D'Antoni and R. Longo[1] related the split property of an inclusion $A \subset B$ of von Neumann algebras to the nucleality properties of the natural embeddings $\phi_p : L^\infty(A) \to L^p(B)$ $(p = 1, 2)$.

Definition 1.2. *Let N, M be von Neamann algebras. We shall say that a completely positive normal map*

$$\phi : N \to L^p(M)$$

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is extendible if, whenever $\tilde{N}$ *is a von Neumann algebra containing* N*, there exists a completely positive normal map* $\tilde{\phi} : \tilde{N} \to L^p(M)$ *that extends* ϕ*:*

$$\begin{array}{ccc} \widetilde{N} & & \\ \uparrow & \searrow \tilde{\phi} & \\ N & \underset{\phi}{\to} & L^p(M) \end{array}$$

In [1] the following theorem was proved.

Proposition 1.1. *Let* $A \subset B$ *be an inclusion of factors and* ω *a faithful normal state of* B*. The following are equivalent:*

(1) *The restriction to* A *of the embedding* $\phi_1 : B \to L^1(B)$ *is extendible;*

(2) *The restriction to* A *of the embedding* $\phi_2 : B \to L^2(B)$ *is extendible;*

(3) $A \subset B$ *is a split inclusion.*

But this criterion is not easy for applications. They also proved that the inclusion of factors $A \subset B$ is split if $\phi_p : L^\infty(A) \to L^p(B)$ is nuclear and only if ϕ_p is compact (p=1,2).

It is an interesting question whether the split inclusion property could be characterized by some kind of nuclearity properties of ϕ_p. In this paper we will present the affirmative answer for $p = 1$. We show that if $A \subset B$ is an inclusion of factors as above, then it is a split inclusion if and only if ϕ_1 is a metrically nuclear map.

§2. Preliminary

For the later use we will give some notations of operator spaces and metrically nuclear maps.

Definition 2.1. *Let* X *be a vector space over the complex numbers with a norm* $\|\cdot\|_n$ *on each* $M_n(X)$ *for* $n \in N$*. The space* X *is called an operator space if it satisfies*

(1) $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$,

(2) $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\|$

for all $x \in M_n(X), y \in M_m(Y)$ *and* $\alpha, \beta \in M_n(C)$*. We assume that all operator spaces are norm complete.*

Let X and Y be operator spaces and $\phi : X \to Y$ a linear map. Then there is a natural map $\phi_n : M_n(X) \to M_n(Y)$ defined by

$$\phi_n([x_{ij}]) = [\phi(x_{ij})]$$

for all $[x_{ij}] \in M_n(X)$.

We call ϕ completely bounded if its cb-norm

$$\|\phi\|_{cb} = \sup\{\|\phi_n\|, n \in N\} < \infty.$$

The map is called a completely contraction if $\|\phi\|_{cb} \leq 1$. The set of all completely bounded maps from X to Y is denoted by $CB(X, Y)$.

Definition 2.2. *Given operator spaces* X *and* Y*, and* $u \in M_n(X \otimes Y)$*, we define*

$$\|u\|_\wedge = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\|\},$$

where the infimum is taken for all possible representations $u = \alpha(x \otimes y)\beta$ *with* $x \in M_k(X), y \in M_l(Y)$ *and* $\alpha \in M_{n,kl}(C), \beta \in M_{kl,n}(C)$*. We write* $X \otimes_\wedge Y$ *for the tensor product of* X

and Y with above matrix norm and $X \otimes^\wedge Y$ for the completion which is called an operator projective tensor product for operator spaces.

Let X and Y be operator subspaces of $B(H)$ and $B(K)$. Viewing the algebraic tensor product $X \otimes Y$ as a subspace of $B(H \otimes K)$, we have an operator matrix norm, i.e., the spatial tensor norm ν on $X \otimes Y$. Let $X \otimes_\vee Y$ denote the tensor product of X and Y with the spatial tensor norm ν, and the completion $X \otimes^\vee Y$ with respect to the norm is called a spatial tensor product.

Definition 2.3. *Let X and Y be operator spaces. We define the metrically nuclear mapping space, denoted by $N(X,Y)$, to be the image of the natural complete contraction*

$$\Phi\colon X^* \overset{\wedge}{\otimes} Y \longrightarrow X^* \overset{\vee}{\otimes} Y \hookrightarrow CB(X,Y)$$

with the matrix quotient norm denoted by $\{\nu_n\}$, i.e., we define

$$N(X,Y) \cong X^* \overset{\wedge}{\otimes} Y/\ker\Phi.$$

It follows that $N(X,Y)$ is an operator space, and that elements in $M_n(N(X,Y))$ can be regarded as maps from X into $M_n(Y)$, which are called metrically nuclear maps.

Let M_∞, T_∞ and H_∞ be the space of bounded operators, trace class operators and Hilbert-Schmidt operators on the separable Hilbert space ℓ^2, respectively. For any $a \in H_\infty^*$, the dual of H_∞, and $b \in H_\infty$, we may define a completely bounded map $\theta_{a,b}\colon M_\infty \to T_\infty$ by

$$\theta_{a,b}(T) = aTb.$$

It is a simple matter to check that $\|\theta_{a,b}\|_{cb} \leq \|a\|\|b\|$.

Theorem 2.1.[3] *A linear map $\phi\colon X \to M_n(Y)$ lies in $M_n(N(X,Y))$ if and only if there are completely bounded maps*

$$\sigma\colon X \to M_\infty, \theta_{a,b}\colon M_\infty \to M_n(T_\infty) \text{ and } \tau\colon T_\infty \to Y$$

such that

$$\phi = \tau_n \circ \theta_{a,b} \circ \sigma.$$

Then we have

$$\nu_n(\phi) = \inf\{\|\tau\|_{cb}\|a\|\|b\|\|\sigma\|_{cb}\},$$

where the infimum is taken for all such completely bounded maps.

§3. Main Results

Let M be a von Neumann algebra and ω a faithful normal state of M such that $\omega(\cdot) = (\cdot\Omega, \Omega)$. We shall denote by

$$\phi_p = \phi_{p,\omega} : M \to L^p(M) \ \ (p = 1, 2)$$

the natural embeddings given by

$$\phi_1(x) = (\cdot\Omega, Jx\Omega), \quad x \in M,$$
$$\phi_2(x) = \triangle^{\frac{1}{4}} x\Omega, \quad x \in M.$$

In this section we will present the main results in this paper. We need some lemmas.

Lemma 3.1. *Let $\phi : A \to X$ be a metrically nuclear map from a von Neumann algebra A into a Banach space X. If ϕ is continuous from A, equipped with the σ-weakly topology, into X, equipped with the weak topology, then ϕ can be expressed as*

$$\phi(a) = \alpha(f(a) \otimes \xi)\beta,$$

where $f = \{f_{ij}\} \in M_\infty(A_*), \xi = \{\xi_{ij}\} \in K_\infty(X)$, α, β are $1 \times \infty^2$ and $\infty^2 \times 1$ matrices.

Proof. Given a completely bounded map $\phi \in CB(A, X)$, we have that $\phi \in N(A, X)$ with $\nu_1(\phi) < 1$ if and only if $\phi = \Phi(u)$ for some $u \in A^* \otimes^\wedge X$ satisfying $\|u\|_\wedge < 1$. That is ,

$$\phi(a) = \alpha(f(a) \otimes \xi)\beta,$$

where $f = \{f_{ij}\} \in M_\infty(A^*), \xi = \{\xi_{ij}\} \in K_\infty(X), \alpha, \beta$ are $1 \times \infty^2$ and $\infty^2 \times 1$ matrices, $\|\alpha\|\|f\|\|\xi\|\|\beta\| < 1$ (cf. [2, p. 174]). Suppose that $f_{ij} = f_{ij}^{(n)} + f_{ij}^{(s)}$ is the decomposition of f_{ij} into its normal and singular parts[5]. If $f^{(n)} = \{f_{ij}^{(n)}\}$, $f^{(s)} = \{f_{ij}^{(s)}\}$, then

$$\phi^{(n)} = \alpha(f^{(n)}(\cdot) \otimes \xi)\beta, \quad \phi^{(s)} = \alpha(f^{(s)}(\cdot) \otimes \xi)\beta$$

are the normal part and the singular part of ϕ such that $\phi = \phi^{(n)} + \phi^{(s)}$. Since $\phi^{(s)}$ is normal and singular, $\phi^{(s)} = 0$. So $\phi = \phi^{(n)}$.

Let $\mathfrak{H}$ be an infinite dimensional separable Hilbert space. We denote by $L^p(\mathfrak{H})$, $p \geq 1$, the Schatten ideals of $B(\mathfrak{H})$.

Lemma 3.2.[1,p.238] *Let M be a von Neumann algebra and $\phi\colon M \to L^p(\mathfrak{H}), p \geq 1$, a completely positive normal map. Then there exists a normal representation π of M in $B(\mathfrak{H})$ and $T \in L^{2p}(\mathfrak{H})$ such that*

$$\phi(x) = T^*\pi(x)T, \qquad \forall x \in M.$$

Lemma 3.3. *Let $A \subset B$ be an inclusion of factors, $\omega = (\cdot\Omega, \Omega)$ a faithful normal state of B. If $\Psi : L^1(A) \to L^1(B)$ is defined by*

$$\langle \cdot\Omega, J_A x\Omega\rangle \to \langle \cdot\Omega, J_B x\Omega\rangle, \ x \in A,$$

where J_A, J_B are the modular conjugations, then Ψ is a completely contraction.

Proof. From Tomita-Takesaki theory, if $\triangle_A, \triangle_B$ are the modular operators, then

$$S_A = J_A \triangle_A^{\frac{1}{2}} = \triangle_A^{-\frac{1}{2}} J_A,$$
$$J_A x\Omega = \triangle_A^{\frac{1}{2}} S_A x\Omega = \triangle_A^{\frac{1}{2}} x^*\Omega,$$
$$J_B x\Omega = \triangle_B^{\frac{1}{2}} x^*\Omega, \quad \forall x \in A.$$

Suppose that A_0 is the space of all $a \in A$ with compact spectrum with respect to the modular group of A, Ω and put $\mathcal{D} = A_0\Omega$.

We define

$$T_{\frac{1}{2}} = \triangle_B^{\frac{1}{2}} \triangle_A^{-\frac{1}{2}},$$

which is densely defined on $[A\Omega]$ with domain including $\mathcal{D}$.

$$T_{\frac{1}{2}}\xi = J_B S_B S_A J_A \xi = J_B J_A \xi, \ \ \forall \xi \in \mathcal{D},$$
$$\|T_{\frac{1}{2}}\| \leq 1.$$

Therefore

$$\|\triangle_B^{\frac{1}{2}}x^*\Omega\| \le \|\triangle_A^{\frac{1}{2}}x^*\Omega\|,$$
$$\|J_Bx\Omega\| \le \|J_Ax\Omega\|, \quad \forall x \in A,$$

Ψ is a contraction.

If we consider the tensor product $M_n \otimes A$ in stead of A, we can prove that Ψ is a completely contraction.

We consider a mapping

$$\Psi^{(n)} : (\langle \cdot\Omega, J_Ax_{ij}\Omega\rangle) \to (\langle \cdot\Omega, J_Bx_{ij}\Omega\rangle), \quad 1 \le i,j \le n, \quad x_{ij} \in A,$$

where $(\langle \Omega, J_Ax_{ij}\Omega\rangle)$ is a functional defined on $M_n \otimes A$ for which the dual is given by

$$\begin{aligned}\langle (\langle \cdot\Omega, J_Ax_{ij}\rangle), (a_{ij})\rangle &= \langle (a_{ij}\Omega), (\triangle_A^{\frac{1}{2}}x_{ij}^*\Omega)\rangle \\ &= \sum_{i,j=1}^{n} \langle a_{ij}\Omega, \triangle_A^{\frac{1}{2}}x_{ij}^*\Omega\rangle, \quad \forall (a_{ij}) \in M_n \otimes A.\end{aligned}$$

Define $T_{\frac{1}{2}} = \triangle_B^{\frac{1}{2}}\triangle_A^{-\frac{1}{2}} \otimes I_n$ on $[(M_n \otimes A_0)\Omega]$ such that

$$T_{\frac{1}{2}}\xi = ((J_BS_BS_AJ_A) \otimes I_n)\xi = (J_BJ_A \otimes I_n)\xi, \quad \forall \xi \in M_n \otimes \mathcal{D}.$$

Therefore

$$\|T_{\frac{1}{2}}\| \le 1,$$

that is,

$$\|(J_Bx_{ij}\Omega)\| \le \|(J_Ax_{ij}\Omega)\|.$$

Ψ is a completely contraction.

Theorem 3.1. *Let $A \subset B$ be an inclusion of factors, ω a faithful normal state of B such that $\omega = (\cdot\Omega, \Omega)$. If $\phi_1 : A \to L^1(B)$ defined by $\phi_1(x) = (\cdot\Omega, J_Bx\Omega), \forall x \in A$, is the natural embedding, then (A, B) is a split inclusion if and only if ϕ_1 is a metrically nuclear map.*

Proof. If ϕ_1 is metrically nuclear, we will prove that (A, B) is a split inclusion. Since $A \subset B$ is a split inclusion if and only if ϕ_1 is extendible, the only thing we have to prove is that ϕ_1 is extendible.

By Lemma 3.1,

$$\phi_1(a) = \alpha\Big(\sum f_{ij}(a) \otimes g_{kl}\Big)\beta,$$

where $f = \{f_{ij}\} \in M_\infty(A_*), g = \{g_{kl}\} \in K_\infty(B_*), \alpha, \beta$ are $1 \times \infty^2$ and $\infty^2 \times 1$.

We define

$$\lambda = \alpha\Big(\sum f_{ij} \otimes g_{kl}\Big)\beta.$$

Then λ is a normal functional on $A \otimes B$. If $\tilde{A}$ is a von Neumann algebra such that $A \subset \tilde{A}$, we can extend λ as a normal functional $\tilde{\lambda}$ on $\tilde{A} \otimes B$. So

$$\tilde{\phi}_1 : \tilde{a} \in \tilde{A} \to \tilde{\lambda}(\tilde{a} \otimes \cdot) \in L^1(B)$$

is completely positive, normal and extends ϕ_1. Thus ϕ_1 is extendible.

Conversely, if $A \subset B$ is a split inclusion, there exists a type I factors F such that $A \subset F \subset B$.

Then we have a diagram as follows:

$$\begin{array}{ccc} A & \xrightarrow{\phi_1} & L^1(B) \\ & \psi_1 \searrow \quad \swarrow E & \\ & L^1(F) & \end{array}$$

If $F = B(\mathcal{H})$, where $\mathcal{H}$ is a infinite dimensional separable Hilbert space, $L^1(F) = L^1(\mathcal{H})$, the set of all trace operators on $\mathcal{H}$.

By use of Lemma 3.2,

$$\psi_1 = T^*\pi(x)T, \quad \forall x \in A,$$

where $T \in L^2(\mathcal{H})$, and π is a normal representation of A in $B(\mathcal{H})$.

From Lemma 3.3, the embedding $L^1(F) \to L^1(B)$ defined by

$$E : (\cdot\Omega, J_F x\Omega) \to (\cdot\Omega, J_B x\Omega), \forall x \in F,$$

is a completely contraction so that

$$\phi_1 = E(T^*\pi(x)T) = E \circ \theta_{T^*,T} \circ \pi.$$

Using Theorem 2.1, we see that ϕ_1 is metrically nuclear.

In a forthcoming paper we will deal with the case $p \neq 1$.

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