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A REMARK ON THE HAUSDORFF DIMENSION OF CERTAIN NON-SELF-SIMILAR ATTRACTORS

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Abstract

Ellis and Branton introduced a class of non-self-similar sets; they gave an upper bound of Hausdorff dimension for such sets, and a conjecture of the lower bound for these sets. This paper gives a proof of this conjecture by using the lemma of Frostman.

Keywords Non-self-similar attractor, Hausdorff dimension, Ellis and Branton conjecture, Iterated function system
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§0. Introduction

A lot of the classical fractal sets are self-similar, built up by pieces wich are geometrically similar to the whole set, but in smaller scales[1,6,7,12].

Hutchinson defined in [5] the notion of self-similar set, which was also introduced by Moran[10] and Marion[8]. The dimension and structure of self-similar sets have been studied and generalized by many authors, such as Peyrière[11].

For the non-self-similar case, we have a few results in general. Ellis and Branton[3] have intruduced a class of non-self-similar sets by using the Markov attractors. They obtained an upper bound of Hausdorff dimension for such attractors, and they gave a conjecture for its lower bound of Hausdorff dimension.

This paper is organized in the following way : in the first section, we recall the notion of iterated function system, and state the results of Ellis and Branton[3], in second section, we give a new proof of their conjecture.

§1. Preliminaries

Let (X,d) be a compact metric space. We call the system $(X;T_1,\cdots,T_n)$ a hyperbolic iterated function system, if there exists a constant $s\in(0,\ 1)$ such that for all $1\le i\le n$ and $x,y\in X$,

$$d(T_i(x),T_i(y))\le sd(x,y). \tag{1.1}$$

For those systems, a subset $A\subset X$ is called an attractor for the system, provided

(1) A is closed, and $A\neq\emptyset$,

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(2) $T_i(A) \subset A$ for all $1 \le i \le n$,

(3) A is minimal with respect to 1 and 2.

With this definition, we have then[3]

$$A = T_1(A) \cup T_2(A) \cup \cdots \cup T_n(A). \tag{1.2}$$

A hyperbolic iterated function system $(X; T_1, \cdots, T_n)$ is said to be disjoint, if the collection of sub-sets $\{T_i(A)\}_{1 \le i \le n}$ is a partition of A.

Remark 1.1. (1) Let $(X; T_1, \cdots, T_n)$ be a hyperbolic iterated function system and A be an attractor of X. Then

$$A = T_1(A) \cup T_2(A) \cup \cdots \cup T_n(A). \tag{1.3}$$

We note $T_1(A) \cup T_2(A) \cup \cdots \cup T_n(A) := \psi(A)$. Then for all closed non-empty subset $E \subset X$, $\lim\limits_{k \to +\infty} \psi(E) = A$ in the Hausdorff metric.

(2) (Existence and unicity of the attractor): Let $(X; T_1, \cdots, T_n)$ be a hyperbolic iterated function system, then there exists one and only one attractor A, and A is compact.

1.1 Markov Attractors

Definition 1.1. *Let $M = (m_{ij})$ be an $n \times n$ matrix. M is called a Markov transition matrix, if $m_{ij} = 0$ or 1, for all $1 \le i, j \le n$.*

Definition 1.2. *An $n \times n$ matrix M is said to be positive, if there exists $k \ge 1$ such that $M^k > 0$.*

Definition 1.3. *Let $M = (m_{ij})$ be an $n \times n$ Markov transitive matrix. A word $(i_1 i_2 \cdots i_k \cdots)$ (finite or infinite) is called M-admissible if we have, for all $j \ge 1$, $m_{i_j i_{j+1}} = 1$.*

Lemma 1.1. *Let M be an $n \times n$ positive matrix* and *note $||M||$ the spectral radius of M. Then there exist $\beta \ge \alpha > 0$ such that, for any $k \ge 1$, we have*[3]

$$\alpha \lambda^k \le \sum_{i,j=1}^{n} (M^k)_{ij} \le \beta \lambda^k.$$

Definition 1.4. *Let $(X; T_1, \cdots, T_n)$ be a hyperbolic iterated function system, with the attractor A. Let M be an $n \times n$ Markov transitive matrix. We say that a point $a \in A$ is M-attractive, if there exists an infinite word $(i_1 i_2 \cdots i_k \cdots)$ M-admissible, such that*

$$a = \lim_{j \to \infty} T_{i_1}(T_{i_2}(\cdots(T_{i_j}(x))\cdots)$$

for all $x \in X$.

Let A_M be the set of M-attractivs points of A; A_M is called Markov attractor of system $(X; T_1, \cdots, T_n)$ associated to M.

1.2 Estimation of the Hausdorff Dimension of Markov Attractor

Let $(X; T_1, \cdots, T_n)$ be a disjoined hyperbolic iterated functions system, and d be the metric on X. Suppose that there exist some constants $0 < t \le t_i \le s_i \le s < 0$, such that for any $1 \le i \le n$, we have

$$t_i d(x, y) \le d(T_i(x), T_i(y)) \le s_i d(x, y). \tag{1.4}$$

1.2.1 Notations

A	Attractor of the disjoined hyperbolic iterated functions system $(X; T_1, \cdots, T_n)$,
$M = (m_{ij})_{1\le i,j\le n}$	An $n \times n$ primitive Markove transitive matrix,
$G(k) = (i_1 i_2 \cdots i_k)$	Sub-set of all mots $M-$ admissible of the length k,
$G^* = \bigcup\limits_{1\le k<\infty} G(k)$	The sub-set of all $M-$ admissible words with finite length,
$\sum_M$	The set of all M-admissible infinite words,
A_M	The Markov attractor of the system $(X; T_1, \cdots, T_n)$ associeted to M.

And we define also $t_{i_1 i_2 \cdots i_k} \in \mathbf{R}$ by

$$t_{i_1 i_2 \cdots i_k} = t_{i_1} t_{i_2} \cdots t_{i_k},$$

and two operations of words: if $I = (i_1 i_2 \cdots i_{k_1}),\ J = (j_1 j_2 \cdots j_{k_2})$, then

$$IJ = (i_1 i_2 \cdots i_{k_1} j_1 j_2 \cdots j_{k_2}),$$
$$I \wedge J = \{(i_1 i_2 \cdots i_p);\ p = \max k, \quad \text{such that } i_r = j_r,\ \forall 1 \le r \le p\}.$$

Write

$$S = \begin{pmatrix} s_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & s_n \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} t_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t_n \end{pmatrix}.$$

We recall that (s_i) and (t_i) are the constants in (1.4).

Let $P = MS^v$ and $Q = MT^u$, where $v,\ u > 0$ are two positive constants such that $\|P\| = \|Q\| = 1$.

Corollary 1.1. *With the above hypothesis, there exist* $\alpha_2 \ge \alpha_1 > 0, \quad \beta_2 \ge \beta_1 > 0$, *such that*

$$\alpha_1 \le \sum_{I\in G(k)} s_I^v \le \alpha_2, \quad \beta_1 \le \sum_{I\in G(k)} t_I^u \le \beta_2$$

for all $k \ge 1$.

Theorem 1.1 (**Ellis-Branton**, cf. [3]).

$$\dim A_M \le v. \tag{1.5}$$

Conjecture of Ellis-Branton 1.1 (cf. [3]).

$$\dim A_M \ge u. \tag{1.6}$$

§2. Proof of the Conjecture of Ellis-Branton

In this section, we will prove the conjecture of Ellis Branton; first we state some lemmas.

Lemma 2.1. *Define*

$$\delta(T_I(A), T_J(A)) = \inf\{d(x,y); x \in T_i(A),\ y \in T_j(A),\ (i,j) \in I \times J\},$$

and $c = \inf\limits_{1\le i\ne j\le n} \delta(T_i(A), T_j(A))$. *Then*

$$\delta(T_I(A), T_J(A)) \ge c t_{I\wedge J}$$

for any $I,\ J \in G(k)$ *and* $I \ne J$.

Proof. Let $I = (i_1 \cdots i_p \cdots i_k)$ and $J = (i_1 \cdots i_p j_{p+1} \cdots j_k)$ with $i_{p+1} \neq j_{p+1}$. Then for all $x,\ y \in A$ we get

$$\begin{aligned} d(T_I(x), T_J(y)) &\geq t_{I\wedge J} d(T_{i_{p+1}\cdots i_k}(x), T_{j_{p+1}\cdots j_k}(y)) \\ &= t_{I\wedge J} d(T_{i_{p+1}}(x'), T_{j_{p+1}}(y')), \end{aligned}$$

where $x' = T_{i_{p+2}\cdots i_k}(x) \in A$, and $y' = T_{j_{p+2}\cdots j_k}(y) \in A$, which implies that

$$d(T_{i_{p+1}}(x'), T_{j_{p+1}}(y')) \geq c,$$

thus

$$d(T_I(x), T_J(y)) \geq c t_{I\wedge J}.$$

This completes the proof of the lemma.

Lemma 2.2. *Let $(i_1 i_2 \cdots i_m \cdots)$ be a word in $\sum_M$. Then*

$$T_{(i_1 i_2 \cdots i_m \cdots)|_{k+1}}(A) \subset T_{(i_1 i_2 \cdots i_m \cdots)|_k}(A),$$

and $\big\{ \bigcup\limits_{I\in G(k)} T_I(A) \big\}_{k\geq 1}$ is a decreasing sequence, and moreover

$$A_M = \bigcap_{1\leq k<+\infty} \bigcup_{I\in G(k)} T_I(A).$$

Proof.

$$\begin{aligned} T_{(i_1 i_2 \cdots i_m \cdots)|_{k+1}}(A) &= T_{(i_1 i_2 \cdots i_m \cdots)|_k}(T_{i_{k+1}}(A)) \\ &\subset T_{(i_1 i_2 \cdots i_m \cdots)|_k}(A). \end{aligned}$$

Hence $\big\{ \bigcup\limits_{I\in G(k)} T_I(A) \big\}_{k\geq 1}$ is a decreasing sequence. Clearly we have

$$A_M \subset \bigcap_{1\leq k<+\infty} \bigcup_{I\in G(k)} T_I(A).$$

On the other hand, let $x \in \bigcap\limits_{1\leq k<+\infty} \bigcup\limits_{I\in G(k)} T_I(A)$. Then $x \in \bigcup\limits_{I\in G(k)} T_I(A)$ for all $k \in N^*$. Thus there exists a unique $I_k \in G(k)$ such that

$$x \in T_{I_k}(A).$$

Further, if $I_{k+1} \in G(k+1)$, $x \in T_{I_{k+1}}(A)$, then we have $I_{k+1}|_k = I_k$. Otherwise, if $I_{k+1}|_k \neq I_k$, we shall have

$$T_{I_{k+1}|_k}(A) \cap T_{I_k}(A) = \emptyset,$$

which means that

$$T_{I_{k+1}}(A) \cap T_{I_k}(A) = \emptyset.$$

This is a contraction. So there exists a unique sequence $T_{k 1\leq k<\infty}$, satisfying $I_m \wedge I_l = I_{\min(m,l)}$, $\forall m, l$, such that

$$x \in \bigcap_{1\leq k<\infty} T_{I_k}(A),$$

where $x = \lim\limits_k T_{I_k}(y)$, for any $y \in A$. Thus $x \in A_M$.

Thus we complete the proof of the lemma.

Theorem 2.3 (Conjecture of Ellis-Branton 1.8).

$$\dim A_M \geq u.$$

Proof. By Corollary 1.1, there exists a sequence of real numbers $\{\gamma_k\}_{k\geq 1}$ such that

$$\frac{1}{\beta_2} \leq \gamma_k \leq \frac{1}{\beta_1}$$

and

$$\gamma_k \sum_{I\in G(k)} t_I^u = 1.$$

So we can define a sequence of the Borel probability $\{\mu_k\}_{k\geq 1}$, such that the support of μ_k is always contained in A, and

$$\mu_k(T_I(A)) = \gamma_k (t_I)^u,$$

where $I \in G(k)$.

Since supp $\mu_k \subset \bigcup\limits_{I\in G(k)} T_I(A)$ and $\mu_k(A) = 1$, there exists a sub-sequence of integers $\{m_k\}_{k\geq 1}$, such that

- $\{\gamma_{m_k}\}_{k\geq 1}$ converges to a real number γ. Easily, we have
$$\frac{1}{\beta_2} \leq \gamma \leq \frac{1}{\beta_1}.$$
- $\{\mu_{m_k}\}_{k\geq 1}$ converges to a Borel probability measure μ, satisfying that
$$\mathrm{supp}\mu \subset A_M, \text{ and } \mu(A_M) = 1.$$

Let O be a ball small enough, satisfaying that $O \cap A_M \neq \emptyset$. Suppose that $x \in O \cap A_M$. Then there exists an infinite word $i_1 i_2 \cdots i_m \cdots \in \sum_M$ such that $x \in T_{(i_1 i_2 \cdots i_m \cdots)|_k}(A)$, for all $k \geq 1$.

So there exists $k \in N^*$ such that

$$ct_{i_1 i_2 \cdots i_{k+1}} \leq |O| < ct_{i_1 i_2 \cdots i_k}.$$

Take $I = (i_1 i_2 \cdots i_k)$ and $J \in G(k)$ with $I \neq J$. By Lemma 2.1, we have then

$$\delta(T_I(A), T_J(A)) \geq ct_{I\wedge J} > ct_I,$$

since $O \cap T_I(A) \neq \emptyset$, and $|O| \leq ct_I$, $O \cap T_J(A) = \emptyset$.

If $l \geq k$, we get

$$\begin{aligned}
\mu(O) &= \mu(O \cap A_M) \\
&= \mu\Big\{O \bigcap \Big(\bigcup_{J\in G(l)} T_J(A)\Big)\Big\} = \mu\Big\{\bigcup_{J\in G(l)} (O \cap T_J(A))\Big\} \\
&= \mu\Big\{\bigcup_{J\wedge I = I} (O \cap T_J(A))\Big\} = \mu\Big\{O \bigcap \Big(\bigcup_{J\wedge I=I} T_J(A)\Big)\Big\} \\
&= \mu(O \cap T_I(A)) \leq \mu\Big\{\bigcup_{J\wedge I=I} T_J(A)\Big\}.
\end{aligned}$$

Let m_j be fixed, $l \geq m_j$. Then

$$\mu_{m_j}\Big\{\bigcup_{J\in G(l)} T_J(A)\Big\} \leq \mu_{m_j}\Big\{\bigcup_{J\in G(m_j)} T_J(A)\Big\},$$

where $J \wedge I = I$.

We have

$$\mu_{m_j}\Big\{\bigcup_{J\in G(m_j)} T_J(A)\Big\} = \gamma_{m_j} \sum_{J\in G(m_j)} t_J^u \le \gamma_{m_j} t_I^u \sum_{L\in G(m_j-k)} t_L^u$$

$$= \frac{\gamma_{m_j} t_I^u}{\gamma_{m_j-k}}\Big(\gamma_{m_j-k} \sum_{L\in G(m_j-k)} t_L^u\Big) = \frac{\gamma_{m_j}}{\gamma_{m_j-k}} t_I^u$$

$$\le \frac{\beta_2}{\beta_1} t_I^u \le \frac{\beta_2}{\beta_1}\frac{1}{(ct)^u}(c^u t_{i_1 i_2\cdots i_{k+1}}^u),$$

where $J \wedge I = I$.

Then we obtain

$$\mu_{m_j}(O) \le \frac{\beta_2}{\beta_1}\frac{1}{(ct)^u}|O|^u, \quad \mu(O) \le \frac{\beta_2}{\beta_1}\frac{1}{(ct)^u}|O|^u.$$

So by the lemma of Frostman[4], we have

$$\dim A_M \ge u.$$

Then we obtain Theorem 2.3.

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