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EMBEDDING FLOWS AND SMOOTH CONJUGACY***

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Abstract

The authors use the functional equations for embedding vector fields to study smooth embedding flows of one-dimensional diffeomorphisms. The existence and uniqueness for smooth embedding flows and vector fields are proved. As an application of embedding flows, some classification results about local and global diffeomorphisms under smooth conjugacy are given.

Keywords Embedding flow, Embedding vector field, Smooth conjugacy, Smooth linearization
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§1. Introduction

In this paper we will use the embedding equations to consider smooth embedding flows and vector fields for diffeomorphisms of the real line $\mathbb{R}$. As an application of embedding flows, we will give some classification results for local and global diffeomorphisms under smooth conjugacy.

The embedding flow problem originates from the discussion for the relation between flows with discrete time and flows with continuous time. For the embedding problem of 1-dimensional diffeomorphisms, there have been many results (see [3, 5, 7–9, 14, 17]). For higher dimensional systems, Palis[12] pointed out that diffeomorphisms which admit embedding flows with some smoothness are "few" in the Baire sense.

Let f be a diffeomorphism on a smooth manifold M. A smooth flow of M, $\{f^t\}$ $(t \in \mathbb{R})$, is said to be an embedding flow of f if $f^1 = f$. The corresponding vector field, $V(x) = \frac{\partial}{\partial t} f^t(x)\big|_{t=0}$, is called an embedding vector field of f.

In this paper, we only consider embedding flows of local and global diffeomorphisms on $\mathbb{R}$. The strategies for global diffeomorphisms of $\mathbb{R}$ are as follows. Let f be an orientation preserving diffeomorphism of $\mathbb{R}$. Then $\mathbb{R}$ can be divided into several intervals by fixed points of f. Typically, let $a_- < a_0 < a_+$ be three consecutive fixed points of f.

Strategy 1. Consider f on the open interval (a_-, a_+) on which f has a unique fixed point a_0. In this case, embedding flows can be separately studied on the two sides of the fixed point, namely, on $(a_-, a_0]$ and on $[a_0, a_+)$ (see [7, 8]). It is well known that the convergence

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of distortions of diffeomorphisms plays a fundamental role in embedding problem (see [7, 8, 17]). We will prove this convergence when f is C^r, $r \geq 2$. This is the main content of §2 of this paper.

Strategy 2. Consider f on the closed interval $[a_0, a_+]$. After obtaining smooth embedding flows on $[a_0, a_+)$ and $(a_0, a_+]$ in Strategy 1, in order to connect these two embedding flows to obtain a global embedding flow on $[a_0, a_+]$, a necessary "connecting condition" should be imposed. This is given in §4 by introducing a more general concept—time difference functions—as the "connecting invariants" of smooth conjugacy.

Let us now begin with the local diffeomorphisms. Denote $\mathbb{R}_+ = [0, \infty)$. Let, for $1 \leq r \leq \infty$,

$$D_+^r(0) = \{f : f \text{ is an orientation preserving } C^r \text{ self-diffeomorphism of } \mathbb{R}_+ \text{ such that 0 is the unique fixed point of } f\}.$$

For a given $f \in D_+^r(0)$, we always write $\lambda = f'(0)$ and

$$\gamma = \begin{cases} 1, & \text{if } \lambda = 1, \\ (\log \lambda)/(\lambda - 1), & \text{if } \lambda \neq 1. \end{cases}$$

As f has a unique fixed point 0, there is a "direction index" $\sigma = +$ or $-$ such that

$$\lim_{n \to \sigma\infty} f^n(x) = 0 \quad \text{on } \mathbb{R}_+.$$

The distortions of f are

$$q_n(x, y) = \frac{(f^n)'(x)}{(f^n)'(y)}, \; x, \; y \in \mathbb{R}_+.$$

The sufficient part of (i) and (ii) of the following theorem are derived from the works of [7-9].

Theorem 1.1. *Let $f \in D_+^1(0)$. Then*

(i) *f has a C^1 embedding flow iff $q_n(x, y)$ converges to some $q(x, y)$ uniformly for x, y in any compact set of $(0, \infty)$ when $n \to \sigma\infty$.*

(ii) *If f has a C^1 embedding flow, then the C^1 embedding flow of f is unique (denoted by $\{f^t\}$). Furthermore, the corresponding embedding vector field $V = V^f$ is given by*

$$V(x) = \gamma \int_x^{f(x)} q(s, x)\, ds = \gamma \lim_{n \to \sigma\infty} \frac{f^{n+1}(x) - f^n(x)}{(f^n)'(x)}, \; x \in \mathbb{R}_+.$$

For the necessary part of (i), we observe that if f has a C^1 embedding flow $\{f^t\}$ on $\mathbb{R}_+$, then for any compact set K of $(0, \infty)$ and x, $y \in K$ there exists $t = t(y, x) \in \mathbb{R}$ such that $x = f^t(y)$. Thus

$$\frac{(f^n)'(x)}{(f^n)'(y)} = \frac{(f^n)'(f^t(y))}{(f^n)'(y)} = \frac{(f^{n+t})'(y)}{(f^n)'(y)(f^t)'(y)} = \frac{(f^t)'(f^n(y))}{(f^t)'(y)}$$

tends to $\lambda^t/(f^t)'(y)$ uniformly as $n \to \sigma\infty$.

There are examples in [9] which show that not all C^1 diffeomorphisms of $\mathbb{R}$ have C^1 embedding flows. In this paper we will consider the following basic problem.

Problem. *Does any $f \in D_+^r(0)$ $(r \geq 2)$ have a C^r embedding flow?*

Let $f \in D_+^r(0)$. If $r \geq 2$, then the function

$$V(x) = \gamma \lim_{n \to \sigma\infty} \frac{f^{n+1}(x) - f^n(x)}{(f^n)'(x)}, \; x \in \mathbb{R}_+ \tag{1.1}$$

is well defined. The reason is that f is C^r $(r \geq 2)$ implies that the function $\log f'$ (when $\sigma = +$) or $\log(f^{-1})'$ (when $\sigma = -$) is of bounded variation on any finite interval of $\mathbb{R}_+$. Thus the distortions have a limit $q(x, y) = \lim_{n\to\infty} q_n(x, y)$, which converges uniformly on any compact set of $(0, \infty)$ (see [8]). Therefore, if V is of some smoothness, then one can use V as a vector field to generate an embedding flow which is also of some smoothness. Combining Theorem 1.1 with the strategies explained above, one can easily understand the results in [5].

In this paper we will use the ideas in [3, 17] to study the smoothness of embedding vector fields. We will not directly study the smoothness of $V(x)$ given in (1.1) because it follows from the reduction from the embedding equations to ordinary differential equations in [17] (see also Theorem 2.1 in the next section) that we need only study the smoothness of the limiting function of the sequence

$$p_n(x) = \frac{(f^n)''(x)}{(f^n)'(x)}, \ x \in \mathbb{R}_+.$$

An obvious relation between p_n and q_n is $q_n(x, y) = \exp\left(\int_y^x p_n(s)ds\right)$.

After proving a technical result (Lemma 2.1), we will obtain the following result (Theorem 2.2).

Let $f \in D_+^r(0)$, $r \geq 2$. Then f has a unique C^1 embedding vector field $V = V^f$ on $\mathbb{R}_+$. Moreover, V is C^{r-1} on $(0, \infty)$, and $V''(0)$ always exists.

As a result, we know that any $f \in D_+^r(0)$ $(r \geq 2)$ has a unique C^1 embedding flow on $\mathbb{R}_+$ which is C^r on $(0, \infty)$. For hyperbolic fixed points, i.e., $\lambda \neq 1$, we can obtain further smoothness of V (Theorem 2.2). As a result, in §2 we will prove that a C^r $(r \geq 2)$ diffeomorphism of $\mathbb{R}$ can be C^r linearizable near hyperbolic fixed points. This result is well known for the cases $r = \infty$, ω (see [14, 20]).

As an application of embedding flows, in §3 and §4 we will consider smooth conjugacy of local and global 1-dimensional diffeomorphisms, respectively. In §3 a result of Firmo[4] for C^1 normal forms of diffeomorphisms near non-hyperbolic fixed points is generalized (Theorem 3.2). In §4 we will use embedding flows to define a unique "time difference function" (TDF, for short) for a diffeomorphism f of an interval which has only the endpoints as fixed points. The existence of a global C^1 embedding flow on such an interval is equivalent to that f has a zero TDF. It will be shown in §4 that TDFs are just the "connecting" invariants between fixed points under smooth conjugacy (Theorem 4.3). Such invariants have been discovered in the works of [2, 16]. However, some new results about smooth classification of diffeomorphisms are obtained (see Theorems 4.1 and 4.3).

For the studying of smooth conjugacy of diffeomorphisms, our main idea is to "embed" a single global diffeomorphism into several local differential flows. Moser[10] also used this idea to discuss monotone twist maps of the plane. Following this idea, in [18] we will give the structure of centralizers and iterated radicals for all C^r $(r \geq 2)$ Morse-Smale diffeomorphisms on the circle. The corresponding results for C^∞ Morse-Smale diffeomorphisms on higher dimensional manifolds can be found in [13]. In [19] we will use this method to obtain a Rigidity Lemma (see, e.g., [11]) for bifurcation of families of diffeomorphisms so that it can be applied to degenerate saddle-node bifurcation of C^2 families of diffeomorphisms.

§2. Embedding Vector Fields and Embedding Flows

In this section, we consider the existence and smoothness of (local) embedding vector fields and flows for diffeomorphisms of $\mathbb{R}$ near isolated fixed points.

If V is an embedding vector field for a diffeomorphism f of M, then V satisfies the following functional equation (see [3, 17])

$$V(f^t(x)) = (f^t)'(x)V(x),\ x \in M,\ t \in \mathbb{R}.$$

Taking $t = 1$ we know that V satisfies the following embedding equation

$$V(f(x)) = f'(x)V(x),\ x \in M. \tag{2.1}$$

For $f \in D^r_+(0)$ $(r \geq 2)$, in [17] we have used the iterates of f to reduce the functional equation (2.1) to a linear ODE. More exactly, we have

Theorem 2.1.[17] *Let $f \in D^r_+(0)$ $(r \geq 2)$. Suppose that the sequence $\{p_n(x)\}$ converges to some $p(x)$ uniformly for x in any compact set of $(0, \infty)$ when $n \to \sigma\infty$. Then a C^1 function $V : \mathbb{R}_+ \to \mathbb{R}$ is an embedding vector field of f on $\mathbb{R}_+$ iff V satisfies the following ODE*

$$V'(x) + p(x)V(x) = \log \lambda,\ x \in (0, \infty), \tag{2.2a}$$

$$\int_c^{f(c)} \frac{ds}{V(s)} = 1, \tag{2.2b}$$

$$V(0) = 0, \tag{2.2c}$$

where $c > 0$ is any fixed number.

In [17], the convergence of $\{p_n(x)\}$ is proved for some cases. Now we give the following technical lemma.

Lemma 2.1. *Let $f \in D^r_+(0)$ $(r \geq 2)$. We have*

(i) *For any integer $s : 0 \leq s \leq (r-2)$, the sequence $\{p_n^{(s)}(x)\}$ converges uniformly for x in any compact set of $(0, \infty)$ when $n \to \infty$.*

(ii) *If $\lambda \neq 1$, then for any integer $s : 0 \leq s \leq (r-2)$ the sequence $\{p_n^{(s)}(x)\}$ converges uniformly for x in any compact set of $[0, \infty)$ when $n \to \sigma\infty$.*

Proof. Without loss of generality, assume that $\sigma = +$, i.e., $f(x) < x$ on $(0, \infty)$. When $\sigma = -$, we can consider f^{-1} because $V^f = -V^{f^{-1}}$.

At first we have the following equalities

$$p_n(x) = \frac{(f^n)''(x)}{(f^n)'(x)} \equiv \sum_{k=0}^{n-1} F(f^k x)(f^k)'(x),\ n \geq 1, \tag{2.3}$$

where $F(x) = f''(x)/f'(x)$ and $f^k x = f^k(x)$. Let

$$P_n(x) = \sum_{k=0}^{n} (f^k)'(x),\ n \geq 0.$$

For any $a \geq 0$, let $M(a) = \max_{x \in [0,a]} |F(x)|$. Then $M(a)$ is continuous. For any fixed $a > 0$, let $I_a = [fa, a] \subset (0, \infty)$. Then $\bigcup_{k \in \mathbf{Z}} f^k(I_a) = \bigcup_{k \in \mathbf{Z}} [f^{k+1}a, f^k a] = (0, \infty)$.

For $x,\ y \in I_a$ and $k \ge 0$, we have

$$\begin{aligned}&|\log(f^k)'(x) - \log(f^k)'(y)|\\ =&\Big|\sum_{i=0}^{k-1}(\log f'(f^i x) - \log f'(f^i y))\Big| = \Big|\sum_{i=0}^{k-1} F(\theta_{i,k})(f^i x - f^i y)\Big|\\ \le& M(a)\sum_{i=0}^{k-1}|f^i x - f^i y| \le M(a)\sum_{i=0}^{k-1}(f^i a - f^{i+1}a) = M(a)(a - f^k a) \le aM(a). \quad (2.4)\end{aligned}$$

Let $N = N(a) = \exp(aM(a))$. Then (2.4) implies

$$N^{-1}(f^k)'(y) \le (f^k)'(x) \le N(f^k)'(y),\ x,\ y \in I_a,\ k \ge 0. \quad (2.5)$$

Integrating (2.5) for y on I_a and noticing that $\int_{fa}^{a}(f^k)'(y)dy = f^k a - f^{k+1}a$, we get

$$N^{-1}\frac{f^k a - f^{k+1}a}{a - fa} \le (f^k)'(x) \le N\frac{f^k a - f^{k+1}a}{a - fa},\ x \in I_a,\ k \ge 0. \quad (2.6)$$

For any $n \ge m \ge 0$, (2.6) implies

$$N^{-1}\frac{f^{m+1}a - f^{n+1}a}{a - fa} \le P_n(x) - P_m(x) \le N\frac{f^{m+1}a - f^{n+1}a}{a - fa},\ x \in I_a. \quad (2.7)$$

As $\lim\limits_{n\to\infty} f^n a = 0$, (2.7) shows that $\{P_n(x)\}$ converges uniformly on I_a. As $\bigcup\limits_{k\in\mathbf{Z}} f^k(I_a) = (0,\infty)$, $\{P_n(x)\}$ converges uniformly for x in any compact set of $(0,\infty)$. By (2.3) then so does $\{p_n(x)\}$.

Now if $r > 2$, we can prove that $\{p_n'(x)\}$ is also uniformly convergent on compact sets of $(0,\infty)$. Differentiating (2.3) with respect to x, we obtain

$$\begin{aligned}p_n'(x) &= \sum_{k=0}^{n-1}[F'(f^k x)((f^k)'(x))^2 + F(f^k x)(f^k)''(x)]\\ &= \sum_{k=0}^{n-1}[F'(f^k x)((f^k)'(x))^2 + F(f^k x)p_k(x)(f^k)'(x)]. \quad (2.8)\end{aligned}$$

On any compact set $S \subset (0,\infty)$, $\{P_n(x)\}$, $\{p_n(x)\}$ are uniformly convergent, thus it follows from (2.8) that $\{p_n'(x)\}$ is also uniformly convergent on S. Inductively it is not difficult to prove that (i) holds.

In order to prove (ii), assume that $\lambda \in (0,1)$. For any $\lambda < \mu < 1$ and $a > 0$, there is some $L = L(a,\mu) > 0$ such that $(f^n)'(x) \le L\mu^n,\ x \in [0,a],\ n \ge 0$. By the equalities (2.3), we know that $\{p_n(x)\}$ is convergent uniformly on $[0,a]$ (therefore on any compact set of $\mathbb{R}_+$). If $r > 2$, the convergence for $\{p_n^{(s)}(x)\}$, $1 \le s \le r-2$, can be similarly proved.

Theorem 2.2 (Embedding Vector Fields). *Let $f \in D_+^r(0)$, $r \ge 2$. Then*

(i) *f has a unique C^1 embedding vector field $V = V^f$ on $\mathbb{R}_+$. Moreover, V is C^{r-1} on $(0,\infty)$, and $V''(0)$ always exists.*

(ii) *If $\lambda \ne 1$, then V is C^{r-1} on $\mathbb{R}_+$ and $V^{(r)}(0)$ exists when $r < \infty$.*

Proof. We only consider the case $\sigma = +$.

(i) By Lemma 2.1(i) and Theorem 1.1 we know that f has an embedding vector field V given by (1.1).

As $\{p_n^{(s)}(x)\}$ converges uniformly on any compact set of $(0,\infty)$ for $0 \le s \le r-2$, $p(x) = \lim\limits_{n\to\infty} p_n(x)$ is C^{r-2} on $(0,\infty)$. Therefore Equation (2.2a) has a unique C^{r-1} solution V_c on

$(0, \infty)$ such that V_c satisfies (2.2b). It is easy to check that V_c does not depend upon c. In fact $V_c = V$.

Now we are going to prove that V is C^1 on $\mathbb{R}_+$ and $V''(0)$ always exists even when $r = 2$. From (2.5) we get $N(a)^{-1} \le q_n(x, y) \le N(a),\ x,\ y \in I_a,\ n \ge 0$. Taking the limit we have

$$N(a)^{-1} \le q(x, y) = \lim_{n\to\infty} q_n(x, y) \le N(a),\ x,\ y \in I_a. \tag{2.9}$$

It follows from (1.1) and (2.9) that

$$\gamma N(x)(fx - x) \le V(x) = \gamma \int_x^{fx} q(s, x)ds \le \gamma N(x)^{-1}(fx - x) < 0, \quad x > 0. \tag{2.10}$$

As $\lim\limits_{x\to 0} N(x) = 1$, by (2.10) we get $\lim\limits_{x\to 0} V(x) = 0 = V(0)$. Once again by (2.10) we have

$$\lim_{x\to 0} \frac{V(x)}{x} = \lim \frac{\gamma(fx - x)}{x} = \gamma(\lambda - 1) = \log\lambda = V'(0).$$

This means that V is differentiable on $\mathbb{R}_+$. For the continuity of V' at $x = 0$, we need only to prove that $V''(0)$ exists. Taking $m = 0$ in (2.7) we get (by noticing that $P_0(x) \equiv 1$)

$$N(a)^{-1}\frac{fa - f^{n+1}a}{a - fa} \le P_n(x) - 1 \le N(a)\frac{fa - f^{n+1}a}{a - fa},\ x \in I_a,\ n \ge 0.$$

The limit gives

$$\frac{aN(a) - (N(a) - 1)fa}{N(a)(a - fa)} \le P(x) = \lim_{n\to\infty} P_n(x) \le \frac{a + (N(a) - 1)fa}{a - fa},\ x \in I_a. \tag{2.11}$$

When $x \to 0$, $N(x) \to 1$ and $f(x) = O(x)$. Thus (2.11) implies (by taking $a = x$)

$$P(x) \sim \frac{x}{x - f(x)} \quad \text{when } x \to 0.$$

Combining this with (2.10) we have $\lim\limits_{x\to 0} \frac{P(x)V(x)}{x} = -\gamma$.

Using (2.3) we get

$$\begin{aligned} p(x) &= \sum_{k=0}^{\infty} F(f^k x)(f^k)'(x) \\ &= F(0)P(x) + \sum_{k=0}^{\infty}(F(f^k x) - F(0))(f^k)'(x) \\ &= (F(0) + O(K(x)))P(x), \end{aligned}$$

where $K(x) = \max\limits_{t\in[0,x]} |F(t) - F(0)| \to 0$ as $x \to 0$. Thus $p(x)V(x)/x \to -\gamma F(0)$. Now we use (2.2) to obtain

$$V''(0) = \lim_{x\to 0} \frac{V'(x) - V'(0)}{x} = \lim_x \frac{V'(x) - \log\lambda}{x} = -\lim_x \frac{p(x)V(x)}{x} = \gamma F(0).$$

(ii) It follows from Lemma 2.1(ii) that $p(x)$ is C^{r-2} on $\mathbb{R}_+$. Now Equation (2.2) implies that V is C^{r-1} on $\mathbb{R}_+$. Suppose $r < \infty$. We need to prove that $V^{(r)}(0)$ exists. Differentiating (2.2) $(r - 2)$ times, we have

$$V^{(r-1)}(x) = -p^{(r-2)}(x)V(x) - W(x),\ x > 0,$$

where

$$W(x) = \sum_{i=1}^{r-2} \binom{r-2}{i} p^{(r-2-i)}(x)V^{(i)}(x)$$

is C^1. Thus

$$\begin{aligned} V^{(r)}(0) &= \lim_{x\to 0}(V^{(r-1)}(x) - V^{(r-1)}(0))/x \\ &= \lim_{x\to 0} -p^{(r-2)}(x)V(x)/x - W'(0) \\ &= -p^{(r-2)}(0)V'(0) - W'(0). \end{aligned}$$

In order to emphasize the importance of formula (2.10), we write down it as

Proposition 2.1. *Let $f \in D^r_+(0)$ $(r \geq 2)$. Then the C^1 embedding vector field V satisfies*

$$\lim_{x\to 0} \frac{V(x)}{f(x) - x} = \gamma. \tag{2.12}$$

Remark 2.1. (i) If $\lambda = 1$ then $p_n(0) = nf''(0)$. Thus $p(0) = \infty$ and Equation (2.2a) is singular at $x = 0$.

(ii) From Proposition 2.1, it is easy to see that f and its embedding vector field V have the same orders at 0 (see [5]). The same formulas as in [5, Lemma 2] are also easily derived. Thus the restriction on the orders in the technical Lemma 2 of [5], and therefore in all results of [5], can be dropped.

Conjecture 2.1. *Let $f \in D^r_+(0)$ $(2 \leq r \leq \infty)$ with $\lambda = f'(0) = 1$. Then $V = V^f$ is C^{r-1} on $\mathbb{R}_+$ and $V^{(r)}(0)$ exists if $r < \infty$.*

For the case $r = \omega$, the results in [1] show that for most analytic (near $z = 0$) $f(z) = z + az^2 + \cdots$ $(a \neq 0)$, V^f is far from being analytic. For the case $r = \infty$ and $f^{(m)}(0) \neq 0$ for some integer $m \geq 2$, the conjecture is true (see [15, Theorem 4]). Lemma 2.1(i) also shows that it is true for the case $r = 2$.

As a corollary of Theorems 1.1 and 2.2 we have

Theorem 2.3 (Embedding Flows). *Let $f \in D^r_+(0)$, $2 \leq r \leq \infty$. Then*

(i) *f has a unique C^1 embedding flow $\{f^t\}$ on $\mathbb{R}_+$. Moreover, it is C^r on $(0, \infty)$.*

(ii) *If $\lambda \neq 1$, then f has a unique C^r embedding flow on $\mathbb{R}_+$.*

Proof. The uniqueness is a corollary of Theorem 1.1. Conclusion (i) directly follows from Theorem 2.2(i). For (ii), we will prove that there is a C^r diffeomorphism $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(0) = 0$, $h'(0) = 1$ and

$$h'(x)V(x) = \mu h(x) \quad (\mu = V'(0) = \log \lambda), \tag{2.13}$$

where $V = V^f$. The proof is similar to that in [20]. Let

$$W(x) = \begin{cases} (V(x) - \mu x)/x^2, & x > 0, \\ V''(0)/2, & x = 0. \end{cases}$$

Then $V(x) \equiv \mu x + x^2 W(x)$. By the property that $V^{(r)}(0)$ exists if $r < \infty$, it is not difficult to prove that $W(x)$ is C^{r-2} and $xW(x)$ is C^{r-1} on $\mathbb{R}_+$.

Let $h(x) = x(1 + H(x))$. Then (2.13) becomes

$$H'(x) + u(x)H(x) = -u(x), \tag{2.14}$$

where $u(x) = W(x)/(\mu + xW(x))$ is C^{r-2}. The solution of (2.14) satisfying $H(0) = 0$ is $H(x) = \exp(-\int_0^x u(t)dt) - 1$, which is C^{r-1}. As

$$h'(x) = \frac{\mu h(x)}{V(x)} = \frac{\mu(1 + H(x))}{\mu + xW(x)}$$

is also C^{r-1}, we obtain the required C^r solution h of (2.13).

Now it follows from (2.13) that the embedding flow of f is given by

$$f^t(x) \equiv h^{-1}(\lambda^t h(x)), \ t, \ x \in \mathbb{R}_+. \tag{2.15}$$

It is C^r. Thus Theorem 2.3 is proved.

Let $t = 1$ in (2.15). It means that $f = f^1$ can be C^r linearized at 0. In general, we have

Theorem 2.4. *Any C^r $(r \geq 2)$ diffeomorphism of $\mathbb{R}$ can be C^r linearized at hyperbolic fixed points.*

Proof. Theorem 2.4 is well-known for $r = \infty, \ \omega$ (see [14, 20]). Now assume that $r < \infty$. Without loss of generality, we may assume that $f : \mathbb{R} \to \mathbb{R}$ is a global C^r diffeomorphism with one hyperbolic fixed point $x = 0$. We need only consider the case that f is orientation reversing. By the C^1 linearization theorem (see [10]), there is a C^1 diffeomorphism h_1 of $\mathbb{R}$ such that $h_1(0) = 0, \ h_1'(0) = 1$ and

$$h_1(f(x)) \equiv \lambda h_1(x). \tag{2.16}$$

As f^2 is orientation preserving, there is a C^r diffeomorphism h_2 of $\mathbb{R}$ such that $h_2(0) = 0, \ h_2'(0) = 1$ and

$$h_2(f^2(x)) \equiv \lambda^2 h_2(x). \tag{2.17}$$

Let $h = h_1 \circ h_2^{-1}$. It follows from (2.16) and (2.17) that h is a C^1 conjugacy between linear, hyperbolic diffeomorphism $F(x) \equiv \lambda^2 x$ and itself. Thus h is linear: $h(x) \equiv h'(0)x \equiv x$. Therefore $h_1 = h_2$ is C^r.

For the case $r = 1$, the corresponding result is not true.

Example 2.1. Consider the following differential equation

$$\frac{dx}{dt} = V_\mu(x) \equiv (\log \lambda)\Big(1 - \frac{\mu}{\log |x|}\Big)x, \quad |x| < 1, \tag{2.18}$$

where $\lambda > 0, \ \neq 1$ and $\mu \in \mathbb{R}$. Obviously, V_μ is C^1 on $|x| < 1$. The time function of V_μ is

$$T_\mu(x) = \int^x \frac{ds}{V_\mu(s)} = \frac{\log S_\mu(x)}{\log \lambda},$$

where $S_\mu(x) = x(\mu - \log |x|)^\mu, \ |x| \ll 1$.

Equation (2.18) generates a C^1 flow

$$f_\mu^t(x) = T_\mu^{-1}(T_\mu(x) + t) = S_\mu^{-1}(\lambda^t S_\mu(x)), \ |x| \ll 1.$$

The time one map $f_\mu = f_\mu^1(x) = S_\mu^{-1}(\lambda S_\mu(x))$ is a C^1 local diffeomorphism of $\mathbb{R}$ and has a hyperbolic fixed point 0 for any $\mu \in \mathbb{R}$. In fact $f_\mu'(0) \equiv \lambda$ for any μ.

We will prove that $f_\mu, \ f_\nu$ are not C^1 conjugate near 0 if $\mu \neq \nu$. Especially, any f_μ $(\mu \neq 0)$ cannot be C^1 linearized at 0.

Assume that there exists a C^1 diffeomorphism h such that $h(0) = 0, \ h'(0) > 0$ and $h \circ f_\mu = f_\nu \circ h$. By the uniqueness of C^1 embedding flows, we have $h(f_\mu^t(x)) \equiv f_\nu^t(h(x))$.

Let $H = S_\nu \circ h \circ S_\mu^{-1}$. Using the expressions for $\{f_\mu^t\}$, we obtain

$$H(\lambda^t x) = \lambda^t H(x), \ |x| \ll 1, \ t \in \mathbb{R}. \tag{2.19}$$

Since H is C^1 on $x \neq 0$, the derivative of (2.19) with respect to x shows that $H'(x)$ is a constant (denoted by $c > 0$) on $x > 0$. Thus we have

$$S_\nu(h(x)) = cS_\mu(x), \ 0 \leq x \ll 1. \tag{2.20}$$

However, as $x \to 0$, $S_\nu(h(x))$ is of order $x(-\log|x|)^\nu$ and $S_\mu(x)$ is of order $x(-\log|x|)^\mu$. Thus (2.20) is a contradiction.

Concerning the Strategy 1 in §1, it follows from the proof of Theorem 2.2 that we have

Theorem 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be an orientation preserving C^r diffeomorphism such that 0 is the unique fixed point of f, where $2 \le r \le \infty$. Then*

(i) *$V = V^f$ is C^1 on $\mathbb{R}$;*

(ii) *$V''(0)$ always exists; and*

(iii) *V is C^{r-1} on $\mathbb{R}\backslash\{0\}$. Moreover, if $\lambda \ne 1$, then V is C^{r-1} on $\mathbb{R}$ and $V^{(r)}(0)$ exists if $r < \infty$.*

In the following, we will use the embedding vector field V to give the embedding flow. For any $c > 0$, define a "time function"

$$T_c(x) = \int_c^x \frac{ds}{V(s)},\ x > 0.$$

Obviously, for any $c > 0$, $T_c : (0, \infty) \to \mathbb{R}$ is a C^r diffeomorphism. Now the C^1 embedding flow of f is

$$f^t(x) = \begin{cases} 0, & \text{if } x = 0,\ t \in \mathbb{R}, \\ T_c^{-1}(T_c(x) + t), & \text{if } x > 0,\ t \in \mathbb{R}. \end{cases} \tag{2.21}$$

§3. Conjugacy of Local Diffeomorphisms

In this section, we will consider (orientation preserving) smooth conjugacy for local diffeomorphisms of $\mathbb{R}$.

Let f, g be diffeomorphisms. We say that f and g are C^s conjugate if there is a C^s diffeomorphism h such that $h \circ f = g \circ h$. Such an h is called a conjugacy (between f and g).

The following lemma is a direct result of the uniqueness for C^1 embedding flows.

Lemma 3.1. *Let f, $g \in D^r_+(0)$. If h is a C^1 conjugacy of f and g, then* (i) *$h \circ f^t = g^t \circ h$; and* (ii) *$h'(x)V^f(x) = V^g(h(x))$.*

Definition 3.1. *Let $f \in D^r_+(0)$, $r \ge 2$. We say that f is not flat (with respect to id) at 0 if there are constants $\mu > 1$, $a_\mu \ne 0$ such that $f(x) = x + a_\mu x^\mu + o(x^\mu)$ as $x \to 0$. The constant μ is called the order of f (at 0).*

For a C^∞ diffeomorphism of $\mathbb{R}$ being not flat, Takens[19] gives the following normal form under C^∞ conjugacy.

Theorem 3.1.[15, Theorem 2] *Let f be a C^∞ diffeomorphism of $\mathbb{R}$ such that f has a unique fixed point 0 and $f^2(x) = x + x^k F(x)$, where $F(0) \ne 0$ and $k \ge 2$ is an integer. Then there is a C^∞ orientation preserving diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ f \circ h^{-1}(x) = \varepsilon x + \delta x^k + \alpha x^{2k-1}$, where $\varepsilon = \mathrm{sign} f'(0)$, $\delta = \pm 1$ and α are constants depending upon f.*

The following is a generalization of [4, Theorem 1.1] for C^1 normal forms.

Theorem 3.2. *Suppose that $f = x + a_\mu x^\mu + o(x^\mu)$, $g = x + b_\nu x^\nu + o(x^\nu) \in D^r_+(0)$, $2 \le r \le \infty$, are not flat at $x = 0$. Then*

(i) *f, g are C^1 conjugate on $\mathbb{R}_+$ iff $\nu = \mu$ and $a_\mu b_\mu > 0$.*

(ii) *Suppose that f, g are C^1 conjugate. If h is a topological conjugacy such that h is C^1 on $(0, \infty)$, then h is a C^1 diffeomorphism of $\mathbb{R}_+$ and is C^r on $(0, \infty)$.*

(iii) *Suppose that* $f,\ g$ *are* C^1 *conjugate. Then for any* $w,\ z>0$*, there is a unique* C^1 *conjugacy* h *satisfying* $h(w)=z$.

Proof. We only give the proof for (i). By (2.12), in this case we have $V^f(x)\sim(f(x)-x)$ as $x\to 0$. Now suppose that $h(x)=cx+o(x)$ $(c=h'(0)>0)$ is a C^1 conjugacy. By Lemma 3.1, we have

$$h'(x)V^f(x)\equiv V^g(h(x)). \tag{3.1}$$

When $x\to 0$,

$$h'(x)V^f(x)\sim ca_\mu x^\mu,\quad V^g(h(x))\sim b_\nu c^\nu x^\nu,$$

thus $\nu=\mu$ and $a_\mu/b_\mu=c^{\mu-1}>0$. This proves the necessity. The above proof also shows that all C^1 conjugacies have the same derivative at 0 (see also [4]), i.e., $h'(0)=\left(\frac{a_\mu}{b_\mu}\right)^{\frac{1}{\mu-1}}$.

For sufficiency, suppose that $\nu=\mu$, $a_\mu b_\mu>0$. For any $w,\ z>0$, the singular ODE (3.1) has a unique solution h on $(0,\infty)$ and satisfies $h(w)=z$. As $V^f,\ V^g$ are C^{r-1}, h is C^r on $(0,\infty)$. Obviously, $h(x)\to 0$ as $x\to 0$.

We need to prove that $\lim\limits_{x\to 0}h'(x)$ exists and is positive. By (3.1) we have

$$h'(x)=\frac{b_\mu h(x)^\mu}{a_\mu x^\mu}\alpha(x),\ x>0, \tag{3.2}$$

where $\alpha(x)\to 1$ as $x\to 0$. By L'Hospital Theorem, we use (3.2) to obtain

$$\lim_{x\to 0}\frac{h(x)^{1-\mu}}{x^{1-\mu}}=\lim_{x}\frac{h(x)^{-\mu}}{x^{-\mu}}h'(x)=\frac{b_\mu}{a_\mu}.$$

Thus $\lim\limits_{x\to 0}\frac{h(x)}{x}=\left(\frac{b_\mu}{a_\mu}\right)^{\frac{1}{1-\mu}}$. Once again by (3.2) we get $\lim\limits_{x\to 0}h'(x)=\left(\frac{a_\mu}{b_\mu}\right)^{\frac{1}{\mu-1}}>0$. This shows that h is a C^1 diffeomorphism on $\mathbb{R}_+$.

By expressions (2.21) for embedding flows, the diffeomorphism $h=h_{z,w}$ in Theorem 3.2(iii) is given by

$$h_{z,w}(x)=\begin{cases}0, & x=0,\\ \widehat{T}_z^{-1}\circ T_w(x), & x>0,\end{cases} \tag{3.3}$$

where T_w, $\widehat{T}_z$ are the time functions of $f,\ g$ respectively.

Theorem 3.2 shows that the C^1 normal forms for $f\in D_+^r(0)$ $(r\geq 2)$ being not flat at 0 are $f_{\delta\mu}(x)=x+\delta x^\mu$, where $\delta=\pm 1$ and $\mu\in\{2,3,\cdots,r-1\}\cup[r,\infty)$.

For hyperbolic fixed points, the following result can be easily derived from Theorem 2.4.

Theorem 3.3. *Let* $f,\ g$ *be* C^r $(r\geq 2)$ *diffeomorphisms of* $\mathbb{R}$ *such that* $f(0)=g(0)=0$. *If* $f'(0)\neq\pm 1,\ g'(0)\neq\pm 1$*, then* $f,\ g$ *are* C^r *conjugate near* 0 *iff* $f'(0)=g'(0)$.

§4. Conjugacy of Global Diffeomorphisms

For global diffeomorphisms of $\mathbb{R}$, we need only consider diffeomorphisms of the interval which fix the endpoints of the interval (see Strategy 2 of §1). We will define, for such a diffeomorphism, a unique "time difference function" (TDF). It will be shown that TDFs are just the "connecting" invariants between fixed points.

Let $I=[a,b]\subset\mathbb{R}$ be an interval. Denote $I^\alpha=[a,b)$, $I^\omega=(a,b]$ and $I^o=(a,b)$. For

$1 \le r \le \infty$, let

$$\begin{aligned} D^r(I) = \{f : I \to I : & f \text{ is an orientation preserving } C^r \text{ diffeomorphism of } I; \\ & f \text{ has only two fixed points } \ x = a,\ b. \text{ Moreover, if } r = 1, \text{ then} \\ & (f^n)'(x)/(f^n)'(y) \text{ converges uniformly for } \ x,\ y \ \text{ in any compact subset of} \\ & I^o \text{ when } \ n \to +\infty,\ n \to -\infty\}. \end{aligned}$$

For a given $f \in D^r(I)$, by the results in §2 we know that f has a unique embedding vector field V^α (resp. V^ω) on the interval I^α (resp. I^ω). They can be given by formulas similar to (1.1) and are C^{r-1} on I^o. In general, they are not equal.

Let $\{f_\sigma^t\}$, $\{T_c^\sigma\}$ be the corresponding C^1 embedding flows and time functions, where $c \in I^o$ and $\sigma = \alpha,\ \omega$. Obviously, the time functions $\{T_c^\sigma\}$ are C^r diffeomorphisms from I^o onto $\mathbb{R}$. Thus for any $c \in I^o$ we can define a C^r self-diffeomorphism of $\mathbb{R}$ by

$$T_c(t) = T_c^\alpha((T_c^\omega)^{-1}(t)),\ t \in \mathbb{R}.$$

The following two lemmas are obvious.

Lemma 4.1. *Let $f \in D^r(I)$. For any $c,\ d,\ x \in I^o$, $\sigma = \alpha,\ \omega$, we have* (i) $T_c^\sigma(x) = T_d^\sigma(x) + T_c^\sigma(d)$; (ii) $T_c^\sigma(d) = -T_d^\sigma(c)$; *and* (iii) $T_c^\sigma(f(x)) = T_c^\sigma(x) + 1$.

Lemma 4.2. *Let $f \in D^r(I)$. Then* (i) $T_c(t+1) \equiv T_c(t) + 1$; *and* (ii) $T_c(t) \equiv T_d(t + T_d^\omega(c)) + T_c^\alpha(d)$.

For any $c \in I^o$, define

$$\psi_c(t) = T_c(t) - t - t_c,\ t \in \mathbb{R}, \tag{4.1}$$

where $t_c = \int_0^1 (T_c(t) - t)dt$.

From Lemma 4.2 and (4.1), one can check that, for any $c,\ d \in I^o$,

$$\psi_c(t) \equiv \psi_d(t + T_d^\omega(c)), \tag{4.2}$$

$$t_c = t_d + T_d^\omega(c) - T_d^\alpha(c). \tag{4.3}$$

Let $P^r = \{\psi : \mathbb{R} \to \mathbb{R} \text{ is } C^r \text{ and } \psi(t+1) \equiv \psi(t);\ \psi'(t) > -1; \text{ and } \int_0^1 \psi(t)dt = 0\}$. Define an equivalence $\sim$ on P^r by $\psi_1 \sim \psi_2$ iff there exists t_0 such that $\psi_1(t) \equiv \psi_2(t + t_0)$. Thus $\psi_c \in P^r$ and $\psi_c \sim \psi_d$ for any $c,\ d \in I^o$.

Definition 4.1. *For $f \in D^r(I)$, let $\psi^f = \psi_{c_0}$ and $t^f = t_{c_0}$, where $c_0 = (a+b)/2$. ψ^f is called the time difference function* (TDF) *of f.*

We will show that TDFs are the "connecting" invariants between fixed points for C^1 conjugacy. In essence, TDFs coincide with those objects defined in [2, 16]. However, TDFs are more convenient in some applications. For the realization of functions in P^r as invariants, we give the following result.

Theorem 4.1. *For any $0 < \lambda < 1 < \mu$ and any $\psi \in P^r$ $(1 \le r \le \infty)$, there exists $f \in D^r(I)$ such that $f'(a) = \lambda$, $f'(b) = \mu$ and $\psi^f \sim \psi$.*

Proof. For simplicity, assume $I = [0, 1]$. For a given $\psi \in P^r$, let $p(t) = t + \psi(t)$ and $q(t)$ be the inverse function of $p(t)$. Obviously, $q(t) - t$ is 1-periodic. Let

$$G(t) = -\exp(q(\log t/\log\lambda)\log\mu),\ t \in (0, \infty).$$

Then G has properties (i) G is C^r; (ii) $G'(t) > 0$; (iii) G maps $(0, \infty)$ onto $(-\infty, 0)$; (iv)

$G(t) \to 0$ as $t \to \infty$; and (v) $G(\lambda t) \equiv \mu G(t)$. Obviously,

$$G^{-1}(t) = \exp(p(\log(-t)/\log\mu)\log\lambda),\ t \in (-\infty, 0).$$

As $G(t)$ is not good near $t = 0$, we modify $G(t)$ by $C(t)$: $C(t)$ is a positive C^{r-1} function on $[0, \infty)$ satisfying (a) $C(t) = G'(t)$ if $t \gg 1$; (b) $C(0) = 1$; and (c) $\int_0^\infty C(t)dt = 1$.

Define a C^r diffeomorphism $H : (-\infty, 0] \mapsto (0, 1]$ by

$$H(t) = \begin{cases} 1, & t = 0, \\ \int_0^{G^{-1}(t)} C(s)ds, & t < 0. \end{cases}$$

Define another C^r diffeomorphism $K : [0, \infty) \mapsto [0, 1)$ by

$$K(t) = \int_0^t C(s)ds = \begin{cases} 0, & t = 0, \\ H(G(t)), & t > 0. \end{cases}$$

Let

$$f(x) = \begin{cases} 1, & x = 1 \\ K(\lambda K^{-1}(x)), & x \in [0, 1) \end{cases}$$
$$\equiv \begin{cases} 0, & x = 0, \\ H(\mu H^{-1}(x)), & x \in (0, 1]. \end{cases}$$

Obviously, $f \in D^r(I)$ and $f'(0) = \lambda,\ f'(1) = \mu$. Moreover we have the following equalities

$$T_c^\alpha(x) = \frac{1}{\log\lambda}\log\frac{K^{-1}(x)}{K^{-1}(c)}, \quad T_c^\omega(x) = \frac{1}{\log\mu}\log\frac{H^{-1}(x)}{H^{-1}(c)}, \quad 0 < c,\ x < 1.$$

Thus

$$\begin{aligned} T_c(t) &= T_c^\alpha \circ (T_c^\omega)^{-1}(t) \\ &= T_c^\alpha(H(\mu^t H^{-1}(c))) \\ &= \frac{1}{\log\lambda}\log\frac{K^{-1} \circ H(\mu^t H^{-1}(c))}{K^{-1}(c)} \\ &= \frac{1}{\log\lambda}[\log G^{-1}(\mu^t H^{-1}(c)) - \log K^{-1}(c)] \\ &= p\Big(t + \frac{\log(-H^{-1}(c))}{\log\mu}\Big) - \frac{\log K^{-1}(c)}{\log\lambda} \\ &= t + \psi\Big(t + \frac{\log(-H^{-1}(c))}{\log\mu}\Big) + \frac{\log(-H^{-1}(c))}{\log\mu} - \frac{\log K^{-1}(c)}{\log\lambda}. \end{aligned}$$

Therefore,

$$\psi_c^f(t) = \psi\Big(t + \frac{\log(-H^{-1}(c))}{\log\mu}\Big) \sim \psi(t).$$

We do not know if Theorem 4.1 also holds for the case $r = \omega$. Using TDFs, we can improve some results in [9, 17].

Corollary 4.1. (i) *$f \in D^r(I)$ has a C^1 embedding flows on I iff $\psi^f = 0$.*

(ii) *There exists $f \in D^\infty(I)$ such that $0 < f'(a) < 1 < f'(b)$ and f has no any C^1 embedding flow on I.*

Let now $f : \mathbb{R} \to \mathbb{R}$ be a C^r $(r \geq 1)$ orientation preserving diffeomorphism with isolated fixed points $a_m < \cdots < a_0 < \cdots < a_s$, where $|m|,\ s \leq \infty$. Moreover, if $r = 1$, we assume that $\log f'$ and $\log(f^{-1})'$ are of bounded variation on any bounded interval. Thus, on any bounded interval $[a_i, a_{i+1}]$, we can define a TDF $\psi_i \in P^r$, $i = m, \cdots, s-1$.

Corollary 4.2. *Let f be as above. We have*

(i) *f has a C^1 embedding flow on $\mathbb{R}$ iff $\psi_i = 0$ for all $i = m, \cdots, s-1$.*

(ii) *Assume that $\psi_i = 0$ for all i. Then the unique C^1 embedding flow $\{f^t\}$ is C^r on $\mathbb{R}\backslash\{a_m, \cdots, a_s\}$. If, in addition, all fixed points of f are hyperbolic, then $\{f^t\}$ is C^r on $\mathbb{R}$.*

Now we consider smooth conjugacy for diffeomorphisms in $D^r(I)$. For $f \in D^r(I)$, let $\{f_\sigma^t\}$ (resp. $\{T_c^\sigma\}$) be the embedding flows (resp. time functions) of f, $\sigma = \alpha,\ \omega$. For $g \in D^r(I)$, we use the notations $\{g_\sigma^t\}$, $\{\widehat{T}_d^\sigma\}$.

Theorem 4.2. *Let $f,\ g \in D^r(I)$. Suppose that h is a C^1 orientation preserving conjugacy between f and g. Then* (i) *$h \circ f_\sigma^t = g_\sigma^t \circ h,\ t \in \mathbb{R},\ \sigma = \alpha,\ \omega$; and* (ii) *$\psi^f \sim \psi^g$.*

Proof. (i) follows from the uniqueness of C^1 embedding flows. For (ii), let c be the center point of I and $d = h(c) \in I^o$. By (i) we have the following equalities

$$\begin{aligned} h(f_\sigma^t(c)) &\equiv g_\sigma^t(h(c)) = g_\sigma^t(d),\ \sigma = \alpha,\ \omega; \\ h(x) &\equiv (\widehat{T}_d^\sigma)^{-1}(T_c^\sigma(x)),\ x \in I^\sigma,\ \sigma = \alpha,\ \omega; \\ T_c(t) &\equiv \widehat{T}_d(t); \\ \psi^f(t) + t^f &\equiv \psi_d^g(t) + t_d^g. \end{aligned}$$

Let $t^\sigma = \widehat{T}_c^\sigma(d) = \widehat{T}_c^\sigma(h(c))$ (depending upon h). By the definition of TDFs and (4.2), (4.3), the last equality is equivalent to the following two conditions

$$\psi^f(t) \equiv \psi^g(t + t^\omega), \tag{4.4}$$

$$t^f = t^g + t^\omega - t^\alpha. \tag{4.5}$$

This shows that $\psi^f \sim \psi^g$.

Theorem 4.2 shows that TDFs are invariants of C^1 conjugacy. By Theorems 4.1 and 4.2, we can improve some results in [2].

Corollary 4.3. (i) *There exist $f,\ g \in D^\infty(I)$ such that*

$$0 < f'(a) = g'(a) < 1 < f'(b) = g'(b)$$

and they are not C^1 conjugate.

(ii) *For any $1 \le r \le \infty$, there exists $f \in D^r(I)$ such that f is not C^1 conjugate to any C^{r+1} diffeomorphism, where $\infty + 1 = \omega$.*

The following result shows that TDFs are just the connecting invariants between fixed points.

Theorem 4.3. *Let $f,\ g \in D^r(I)$. Then f is C^s $(1 \le s \le r)$ conjugate to g iff* (i) *$f,\ g$ are C^s conjugate on I^α, I^ω respectively; and* (ii) *$\psi^f \sim \psi^g$.*

Proof. We need only to prove the sufficiency. As $\psi^f \sim \psi^g$, there is τ^ω such that $\psi^f(t) \equiv \psi^g(t + \tau^\omega)$. Let $d = g_\omega^{\tau^\omega}(c) \in I^o$. As $f,\ g$ are C^s conjugate on I^ω, there is a unique C^s conjugacy h on I^ω satisfying $h(c) = d$.

Now we use (4.5) to define $\tau^\alpha = t^g - t^f + \tau^\omega$ and let $e = g_\alpha^{\tau^\alpha}(c) \in I^o$. Similarly, there is a unique C^s conjugacy $\hat{h}$ on I^α satisfying $\hat{h}(c) = e$.

We need only to prove that $\hat{h} = h$ on I^o. As $h,\ \hat{h}$ are C^s conjugacies of $f,\ g$ on I^ω, I^α, by formula (3.3) we have

$$\begin{aligned} h(x) &= (\widehat{T}_d^\omega)^{-1} \circ T_c^\omega(x),\ x \in I^o, \\ \hat{h}(x) &= (\widehat{T}_e^\alpha)^{-1} \circ T_c^\alpha(x),\ x \in I^o. \end{aligned}$$

At first we note that the equality implies $d = e$. Now $h = \hat{h}$ is equivalent to $T_c(0) = \psi^f(0) + t^f = 0$. Thus $\hat{h} = h$ is equivalent to the following equalities:

$$\widehat{T}_d^{\alpha}((\widehat{T}_d^{\omega})^{-1}(t)) \equiv T_c^{\alpha}((T_c^{\omega})^{-1}(t)),$$
$$\psi_d^g(t) + t_d^g \equiv \psi^f(t) + t^f,$$
$$\psi^g(t + \widehat{T}_c^{\omega}(d)) + t^g + \widehat{T}_c^{\omega}(d) - \widehat{T}_c^{\alpha}(d) \equiv \psi^f(t) + t^f,$$
$$\psi^g(t + \tau^{\omega}) + t^g + \tau^{\omega} - \tau^{\alpha} \equiv \psi^f(t) + t^f.$$

Obviously, the last one holds because of the choice for τ^{ω}, τ^{α}.

Corollary 4.4. *Suppose that f, $g \in D^r(I)$ are C^s $(1 \le s \le r)$ conjugate. Then f, g have a C^s conjugacy h satisfying $h(c) = d$ iff $\psi_c^f = \psi_d^g$.*

Following the ideas in this paper, one can also consider orientation reversing smooth conjugacy for local and global diffeomorphisms of $\mathbb{R}$. For global case, the inverse functions of $T_c(t)$, i.e., $\bar{T}_c(t) \equiv T_c^{\omega}((T_c^{\alpha})^{-1}(t))$, will be useful.

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