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INTERFACE PROBLEMS FOR ELLIPTIC DIFFERENTIAL EQUATIONS**

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Abstract

A new approach is given to analyse the regularity of solutions near singular points for the interface problems of second order elliptic partial differential equations. For general equations with nonsymmetric dominant terms and discontinuous piecewise smooth coefficients, it is proved that solutions in H^1 can be docomposed into two parts, one of which is a finite sum of particular solutions to the corresponding homogeneous equations with piecewise constant coefficients, and the other one of which is the regular part. Moreover a priori estimations are proven.

Keywords Elliptic equation, Interface problem, Singular point, Regularity, A priori estimation
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§1. Introduction

We consider the following second order elliptic partial differential equations in two independent variables

$$Lu = \frac{\partial}{\partial x_j}\Big(a_{i\,j}(x)\frac{\partial u}{\partial x_i}\Big) + b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x), \quad x \in \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $i, j = 1, 2$, and the summation convention is assumed. We assume that Ω is decomposed into a finite number of polygonal subdomains $\Omega^{(k)}$, such that $\bigcup \overline{\Omega^{(k)}} = \overline{\Omega}$, and $a_{i\,j} \in C^1(\overline{\Omega^{(k)}})$, $b_i \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$. The matrix $(a_{i\,j})$ is not necessarily symmetric, but the condition of ellipticity,

$$a_{i\,j}\xi_i\xi_j \ge \chi|\xi|^2, \quad \forall \xi_i, \xi_j \in \mathbb{R},$$

should be satisfied, where $\chi > 0$ is a constant. For simplicity we impose the Dirichlet boundary condition,

$$u|_{x\in\partial\Omega} = 0, \tag{1.2}$$

on (1.1), where $\partial\Omega$ is the boundary. If 0 is not an eigenvalue of the operator L, then the problem (1.1), (1.2) admits a weak solution $u \in H_0^1(\Omega)$ provided $f \in H^{-1}(\Omega)$ (see [3]). The problem considered in this paper is: for more regular f does the solution u possess higher regularity?

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The following points will be generally known as singular points: the crosspoints of interfaces, the turning points of interfaces, the crosspoints of interfaces with the boundary $\partial\Omega$, and the points on $\partial\Omega$ with interior angles greater than π. Let Σ be the set of singular points. We assume that Σ is finite. It is easy to prove that for each subdomain $\Omega^{(k)}$, $u \in H^2_{\rm loc}(\Omega^{(k)} \setminus \Sigma)$ if $f \in L^2(\Omega)$, and the regularity of u can be even higher if $a_{i\,j}$, b_i, c, f possess higher regularity. The problem is the behavior of u near the singular points.

This problem has been extensively studied. Mostly the method of separation variables, or the Mellin transform is applied. Especially for those domains possessing corner points or conical points we refer readers to the books [6, 4] and the survey [8]. For interface problems Kellogg[7] has studied the case of $a_{i\,j} = a'_{i\,j}p$, where $a'_{i\,j}$ is a smooth function, and p is a piecewise constant function, and the matrix $(a'_{i\,j})$ is symmetric, and Blumenfeld[2] has studied the case of $a_{i\,j} = \delta_{i\,j}p$, where p is a piecewise smooth function and $\delta_{i\,j}$ is the Kronecker symbol.

We make use of a different approach and study the general case of the interface problems in this paper. Our main result reads

Theorem 1.1. *We assume that $f \in L^2(\Omega)$. If $\tilde{x} \in \Sigma$, and $\tilde{\Omega}$ is a neighborhood of $\tilde{x}$ which contains one singular point $\tilde{x}$ only, then there is an integer M depending only on L such that*

$$u = v + w \tag{1.3}$$

in $\tilde{\Omega}$, where $v \in H^1(\tilde{\Omega})$ and v satisfies the equation

$$\frac{\partial}{\partial x_j}\Big(a_{i\,j}(\tilde{x})\frac{\partial u}{\partial x_i}\Big) = 0, \tag{1.4}$$

and the boundary condition (1.2) *if $\tilde{x} \in \partial\Omega$, where $a_{i\,j}(\tilde{x})$ are the constant coefficients frozen in $\tilde{x}$, and*

$$\|v\|_{1,\tilde{\Omega}} + \|w\|_{1,\tilde{\Omega}} + \Big\|\frac{D^2 w}{(|\log r| + 1)^M}\Big\|_{0,\tilde{\Omega}\cap\Omega^{(k)}} \le C(\|u\|_1 + \|f\|_0), \tag{1.5}$$

where D^2 refers to the second order derivatives and r is the distance of a point to $\tilde{x}$. Moreover we have (1.3) *with*

$$\|v\|_{1,\tilde{\Omega}} + \|w\|_{2,\tilde{\Omega}\cap\Omega^{(k)}} \le C(\|u\|_1 + \|(|\log r| + 1)^M f\|_0), \tag{1.6}$$

provided the right hand side of (1.6) *is finite.*

Throughout this paper C is always a generic constant, and the notations of Sobolev norms $\|\cdot\|_s$ and seminorms $|\cdot|_s$ are applied.

The rest part of this paper is organized as follows. In §2 we study the solution to the equation (1.4) near singular points. In §3 we construct a particular regular solution to the nonhomogeneous equations corresponding to (1.4). In §4 we prove Theorem 1.1. In what follows we assume that the singular point is an interior point. For those singular points on the boundary the argument is analogous, and in fact the result can be obtained by using a simpler approach[5].

§2. Homogeneous Equation with Constant Coefficients

Without loss of generality we assume that the domain is $\Omega = S(o, 1)$, a disk with center o and radius 1. Let the point o be the singular point. Then the domain Ω is divided into some sectors $S_m, m = 1, \cdots, m_0$, by some rays starting from the point o. We consider the equation

$$L_0 u = \frac{\partial}{\partial x_j}\Big(a_{i\,j}\frac{\partial u}{\partial x_i}\Big) = 0 \tag{2.1}$$

on Ω, where $a_{i\,j}$ are constants on each sector S_m. Denote by Γ_0 the boundary of Ω. We take a constant $\xi \in (0,1)$. Then we define subdomains $\Omega_0, \Omega_1, \cdots, \Omega_k, \cdots$, where $\Omega_k = \{\xi^k > r > \xi^{k+1}\}$, and (r,θ) are the polar coordinates. In addition, we denote $\xi^k\Omega = \{0 < r < \xi^k\}$, and $\Gamma_k = \{r = \xi^k\}$. Let H be the space $H^{\frac{1}{2}}(\Gamma_0)$. Define a mapping $T_k : x \to \xi^k x$. In the following for simplicity we say a function g defined on Γ_k belongs to H if $g \circ T_k \in H$. It is easy to verify that the following equalities hold for any function f,

$$|f \circ T_k|_{s,\Omega_0} = \xi^{(s-1)k}|f|_{s,\Omega_k}, \qquad s = 0,1,2,$$

provided the above norms are finite.

We take an arbitrary $g \in H$, and consider the boundary condition $u|_{\Gamma_0} = g$. Then the equation (2.1) admits a unique solution $u \in H^1(\Omega)$ satisfying this boundary condition. Let $\tilde{g} = u|_{\Gamma_1}$. Then by the trace theorem $\|\tilde{g}\|_H \le C\|u\|_1 \le C\|g\|_H$. Therefore $X : g \to \tilde{g}$ is a bounded operator from H to H.

Lemma 2.1. *X is a compact operator.*

Proof. Let $\{g^{(l)}\} \subset H$ be a bounded sequence, and the solution corresponding to it be $\{u^{(l)}\}$. We take two constants ξ_1, ξ_2 such that $\xi^2 < \xi_1 < \xi < \xi_2 < 1$, and define a domain $\Omega(\xi_1,\xi_2) = \{\xi_1 < r < \xi_2\}$. Then $u^{(l)}$ are uniformly bounded in $H^1(\Omega(\xi_1,\xi_2))$. We extract a weakly convergent subsequence, still denoted by $\{u^{(l)}\}$. Let u be the limit. Besides, using the technique of interior estimation[3], we obtain that $u^{(l)}$ are uniformly bounded in each $H^3(S_m \cap \Omega(\xi_1,\xi_2))$. Then the embedding theorem[1] implies that we can extract a subsequence such that it is convergent on $S_m \cap \Omega(\xi_1,\xi_2)$ with respect to the C^1-norm, the limit of which is still u. But $u \in H^1(\Omega(\xi_1,\xi_2))$, so the traces of u on the interface from the both sides are equal. Therefore $u \in C(\Omega(\xi_1,\xi_2))$. Particularly u is continuous on Γ_1. And u belongs to $H^1(\Gamma_1 \cap S_m)$, hence $u \in H^1(\Gamma_1)$. We have

$$\|\nabla(u - u^{(l)})\|^2_{0,\Gamma_1} = \sum_m \|\nabla(u - u^{(l)})\|^2_{0,\Gamma_1\cap S_m} \to 0 \quad (l \to \infty),$$

which means $\{u^{(l)}\}$ converges strongly in $H^1(\Gamma_1)$. Thus X is compact.

By the Riesz-Schauder Theorem, the spectrum of X consists of isolated eigenvalues and the point o. The null spaces $N((X-\lambda I)^p)$ for all eigenvalues are finite dimensional. We arrange the eigenvalues so that $|\lambda_1| \ge |\lambda_2| \ge \cdots$, and define two spectrum sets: $\{\lambda_1, \cdots, \lambda_N\}$, $\{\lambda_{N+1}, \cdots, 0\}$, where $|\lambda_N| > |\lambda_{N+1}|$. The space H is decomposed to two subspaces such that $H = H_1 \oplus H_2$ and the spectrum of X_{H_1} in H_1 is just $\{\lambda_1, \cdots, \lambda_N\}$, the spectrum of X_{H_2} in H_2 is $\{\lambda_{N+1}, \cdots, 0\}$. Since $\lim\limits_{k\to\infty} \|X^k_{H_2}\|^{\frac{1}{k}} = |\lambda_{N+1}|$, where $\|\cdot\|$ stands for the spectrum norm, we have

$$\|X^k_{H_2}\| \le (|\lambda_{N+1}| + \varepsilon)^k \tag{2.2}$$

for any $\varepsilon > 0$ and sufficiently large k. We require that $|\lambda_{N+1}| + \varepsilon < |\lambda_N|$.

For any $g \in H$, we have a unique decomposition $g = g_1 + g_2$, $g_1 \in H_1$, $g_2 \in H_2$. Let u_1, u_2 be the solutions corresponding to g_1, g_2 respectively.

Lemma 2.2. *If* $|\lambda_{N+1}| < \xi$, *then* $u_2 \in H^2(\xi\Omega \cap S_m)$.

Proof. For small $\varepsilon > 0$, we have $\xi^{-1}(|\lambda_{N+1}| + \varepsilon) < 1$. Applying interior estimation we have for $k \geq 1$ that

$$|u_2|^2_{2,\Omega_k\cap S_m} = \xi^{-2(k-1)}|u_2 \circ T_{k-1}|^2_{2,\Omega_1\cap S_m}$$
$$\leq C\xi^{-2(k-1)}\|u_2 \circ T_{k-1}\|^2_{0,\Omega\setminus\overline{\xi^3\Omega}}.$$

We consider the boundary value problem on $\Omega \setminus \overline{\xi^3\Omega}$ and obtain

$$\|u_2 \circ T_{k-1}\|^2_{0,\Omega\setminus\overline{\xi^3\Omega}} \leq C(\|X^{k-1}g_2\|^2_H + \|X^{k+2}g_2\|^2_H).$$

Hence for k large enough, we have

$$|u_2|^2_{2,\Omega_k\cap S_m} \leq C\xi^{-2(k-1)}((|\lambda_{N+1}| + \varepsilon)^{2(k-1)} + (|\lambda_{N+1}| + \varepsilon)^{2(k+2)})\|g_2\|^2_H. \tag{2.3}$$

Therefore $\sum\limits_{k=1}^{\infty} |u_2|^2_{2,\Omega_k\cap S_m}$ converges, which proves the assertion.

We turn now to the study of the solution u_1. Since the space H_1 can be decomposed into the direct sum of a finite number of null spaces $N((X - \lambda I)^p)$ with $\lambda = \lambda_1, \cdots, \lambda_N$, u_1 is the sum of a finite number of solutions, each one of which is related to an eigenvalue. We study one of them.

Lemma 2.3. *Let* $\{\lambda, g\}$ *be a pair of eigenvalue and eigenfunction. Then either* $\lambda = 1$, $g =$const. *or* $|\lambda| < 1$.

Proof. Let u be the solution with boundary data g. We have $Xg = \lambda g$, hence

$$|u|^2_{1,\Omega} = \sum_{k=0}^{\infty} |u|^2_{1,\Omega_k} = \sum_{k=0}^{\infty} |\lambda|^{2k}|u|^2_{1,\Omega_0}.$$

Since $u \in H^1(\Omega)$, this series converges. If $|u|_{1,\Omega_0} = 0$, then $u =$const on Ω_0. So $u \equiv$const on every Ω_k. By continuity $u \equiv$const on Ω, which gives $\lambda = 1$. If $|u|_{1,\Omega_0} \neq 0$, we have $|\lambda| < 1$.

Lemma 2.4. *Let* λ *be an eigenvalue. Then there exists a basis* $\{\bar{g}_1, \cdots, \bar{g}_s\}$ *of the eigenspace such that the solutions with boundary data* $\bar{g}_j$, $j = 1, \cdots, s$, *are* $r^{\alpha_j}\bar{g}_j$, *where*

$$\alpha_j = \frac{\log\lambda}{\log\xi} + i\beta_j, \tag{2.4}$$

where β_j *are real numbers, such that* $\xi^{i\beta_j} = 1$.

Proof. Let $\alpha = \frac{\log\lambda}{\log\xi}$, and g be an eigenfunction. Then we have

$$u|_{\Gamma_k} = X^k g = \lambda^k g = \xi^{\alpha k} g = r^\alpha g, \tag{2.5}$$

hence the assertion is valid on Γ_k.

Let $\bar{\xi} \in (0, 1)$ be another constant such that $\xi, \bar{\xi}$ are unreducible. Let $\overline{X}$ and $\{\bar{\lambda}_i\}$ be the operator and spectrum corresponding to $\bar{\xi}$. We expand g as the following:

$$g = \sum_1 c_j\bar{g}_j + \sum_2 c_j\bar{g}_j + \tilde{g},$$

where $\tilde{g} \in \tilde{H}_2$, $H = \tilde{H}_1 \oplus \tilde{H}_2$, $\tilde{H}_2$ is the subspace corresponding to a spectrum set consisting of all $|\bar{\lambda}_j| < |\bar{\xi}|^{\text{Re}\alpha}$, the $\bar{g}_j's$ in $\sum\limits_2$ are the eigenfunctions corresponding to $|\bar{\lambda}_j| = |\bar{\xi}|^{\text{Re}\alpha}$, and $\sum\limits_1$ is a finite sum of linearly independent $\bar{g}_j$ corresponding to $|\bar{\lambda}_j| > |\bar{\xi}|^{\text{Re}\alpha}$, or $|\bar{\lambda}_j| = |\bar{\xi}|^{\text{Re}\alpha}$, where $\bar{g}_j \in N((\overline{X} - \bar{\lambda}_j I)^p)$, but $\bar{g}_j$ is not an eigenfunction.

Let $r = \bar{\xi}^k$, $k = 1, 2, \cdots$. Then we get

$$u|_{r=\bar{\xi}^k} = \bar{X}^k g = \sum_1 c_j \bar{X}^k \bar{g}_j + \sum_2 c_j \bar{X}^k \bar{g}_j + \bar{X}^k \tilde{g}. \tag{2.6}$$

Multiply it by $(|\log r| + 1)^{-\frac{1}{2}} r^{-\mathrm{Re}\alpha}$ with $r = \bar{\xi}^k$, and then let $k \to \infty$. By (2.2) and the definition of $\sum_2$, the second and the third terms tend to zero. Let us consider the left hand side . We have (2.5) on Γ_k, but the points with $r = \bar{\xi}^k$ are not on Γ_k. However, we have

$$u|_{\Omega_k} \circ T_k = \lambda^k u|_{\Omega_0}, \tag{2.7}$$

so $u = O(r^{-\mathrm{Re}\alpha})$ as $r \to 0$. The left hand side also tends to zero. Therefore

$$\lim_{k\to\infty} (|\log r| + 1)^{-\frac{1}{2}} r^{-\mathrm{Re}\alpha} \Big|_{r=\xi^k} \sum_1 c_j \bar{X}^k \bar{g}_j = 0.$$

Each term tends to infinity as $k \to \infty$, and $\bar{g}_j$ are linearly independent, so $c_j = 0$. Therefore (2.6) is reduced to

$$\overline{X}^k g - \sum_2 c_j \overline{X}^k \bar{g}_j = \overline{X}^k \tilde{g}. \tag{2.8}$$

We define $F(r, \omega_1, \cdot) = \max_\theta |r^{-\alpha} u(r, \theta) - \sum_2 c_j \omega_j \bar{g}_j|$ for $\xi \le r \le 1$ and $|\omega_i| = 1$. If $F(\bar{r}, \omega_1, \cdots) = 0$ for a certain $\bar{r}$ and some complex numbers ω_i, then

$$u(\bar{r}, \theta) = \bar{r}^\alpha \sum_2 c_j \omega_j \bar{g}_j.$$

Let $\bar{u}_j$ be the solution corresponding to the boundary data $\bar{r}^\alpha c_j \omega_j \bar{g}_j$. Then $u = \sum_2 \bar{u}_j$ $(r \le \bar{r})$. Let $r = \xi$. Then $u(r, \theta) = \lambda g$. By (2.7), $\bar{u}_j(r, \theta)$ are eigenfunctions of the operator $\bar{X}$ corresponding to the eigenvalue $\bar{\lambda}_j$ for any r. We set $\bar{u}_j(r, \theta) = \lambda \bar{g}_j$, where $\bar{g}_j$ may be different from the above, but it is still an eigenfunction. Hence

$$g = \sum \bar{g}_j. \tag{2.9}$$

If $F(r, \omega_1, \cdots) > 0$ always holds, then by continuity $F \ge \delta$ for a positive constant δ. We multiply (2.8) by $r^{-\mathrm{Re}\alpha}(|\log r| + 1)^{\frac{1}{2}}$ and let $k \to \infty$. Then the right hand side tends to zero as $k \to \infty$. For each k, there is a θ such that

$$r^{-\mathrm{Re}\alpha}(|\log r| + 1)^{\frac{1}{2}} |\bar{X}^k g - \sum_2 c_j \bar{X}^k \bar{g}_j| \ge \delta(|\log r| + 1)^{\frac{1}{2}},$$

which tends to infinity and leads to a contradiction. Therefore (2.9) holds.

We may assume that all $\bar{g}_j' s$ in (2.9) correspond to different eigenvalues, otherwise some $\bar{g}_j' s$ can merge into one. If there are s terms in (2.9), then we have

$$\begin{aligned} u(1, \theta) &= \bar{g}_1 + \cdots + \bar{g}_s, \\ u(\bar{\xi}, \theta) &= \bar{\lambda}_1 \bar{g}_1 + \cdots + \bar{\lambda}_s \bar{g}_s, \\ &\cdots\cdots \\ u(\bar{\xi}^{s-1}, \theta) &= \bar{\lambda}_1^{s-1} \bar{g}_1 + \cdots + \bar{\lambda}_s^{s-1} \bar{g}_s. \end{aligned}$$

The determinant of coefficients is just the Vandermonde determinant, so we take the inverse and obtain $\bar{g}_j = \sum d_{j\,l} u(\bar{\xi}^l, \theta)$, $\quad j = 1, \cdots, s$. By (2.7) $u(\bar{\xi}^l, \theta)$ are the eigenfunctions of X corresponding to λ, so is $\bar{g}_j$. Let $u^{(j)}$ be the solution with boundary data $\bar{g}_j$. Then for $r = \xi^k \bar{\xi}^{\bar{k}} < 1$ with $k, \bar{k}$ integers (not necessaryly positive) we have $u^{(j)} = \lambda^k \bar{\lambda}_j^{\bar{k}} \bar{g}_j$. Such

r is dense in $[0,1]$, and $u^{(j)}$ is continuous, hence $u^{(j)}=R(r)\bar{g}_j$, where $|R(r)|=r^{\mathrm{Re}\alpha}$. Let $R(r)=r^{\alpha}e^{i\varphi(r)}$, where $\varphi(r)$ is a real continuous function. Since $k\varphi(r)=\varphi(r^k)$ for all positive integer k, we have $\varphi=$const $\cdot\log r$. Therefore

$$u^{(j)}=r^{\alpha+i\beta_j}\bar{g}_j, \tag{2.10}$$

where β_j is a real number. Because $u^{(j)}(\xi,\theta)=\lambda\bar{g}_j$, we can take the logarithm function appropriately such that $\xi^{i\beta_j}=1$. $\mathrm{span}(\bar{g}_1,\cdots,\bar{g}_s)$ belongs to the eigenspace of λ of the operator X. If they are not identical, we can take one eigenfunction $g\notin\mathrm{span}(\bar{g}_1,\cdots,\bar{g}_s)$ and repeat the above procedure, and obtain another subspace containing g. Then we take the summation of these two spaces. Since the dimension is finite, we get a basis $\{\bar{g}_1,\cdots,\bar{g}_s\}$ by some steps finally.

Remark. It is easy to see that $\beta_j=\frac{2k\pi}{\log\xi}$, k is an integer. We can take $\tilde{\xi}=\xi^{\frac{1}{K}}$, where K is the least common multiple of all k's. Then we set $\tilde{\lambda}=\lambda^{\frac{1}{K}}e^{\frac{2k\pi i}{K}}$. $\tilde{\lambda}$ is the eigenvalue of the corresponding operator $\bar{X}$. The formula (2.4) is reduced to $\alpha_j=\frac{\log\tilde{\lambda}}{\log\tilde{\xi}}$. The exponents α_j corresponding to $\tilde{\lambda}$ are the same. For notational convenience, we will donote this particular $\tilde{\xi}$ by ξ, the corresponding operator by X, and the basis of eigenfunctions by $\{g_1,\cdots,g_s\}$. We have the solution

$$u_j=r^{\alpha}g_j, \qquad j=1,\cdots,s, \tag{2.11}$$

with

$$\alpha=\frac{\log\lambda}{\log\xi}. \tag{2.12}$$

The elementary divisor may not be linear. Then we get some other particular solutions.

Lemma 2.5. *If the elementary divisor is quadratic for an eigenvalue* λ, $h\in N((X-\lambda I)^2)$, *and* $(X-\lambda I)h=g$, *where* g *is an eigenfunction, then the solution with boundary data* h *is*

$$u=r^{\alpha}\Big(h+\frac{1}{\lambda}\frac{\log r}{\log\xi}g\Big), \tag{2.13}$$

where α *is determined by* (2.12).

Proof. We have

$$X^kh=\lambda^kh+k\lambda^{k-1}g, \qquad k=1,2,\cdots,$$

hence (2.13) is valid for $r=\xi^k$.

Multiplying the solution u by $\lambda^{-k}/k|\log\xi|$ on Ω_k and letting $k\to\infty$, we study the limit of $v_k=\frac{\lambda^{-k}}{k|\log\xi|}u\circ T_k$ on Ω_0. The limit on Γ_0 is $\frac{1}{|\log\xi|\lambda}g$, and the limit on Γ_1 is $\frac{1}{|\log\xi|}g$. Since v_k is a solution, by uniqueness, the limit on Ω_0 is $\frac{1}{|\log\xi|\lambda}r^{\alpha}g$. Next, we multiply the solution by $\frac{r^{-\alpha}}{|\log r|}$ on Ω_k, and let $w_k=\Big(\frac{r^{-\alpha}}{|\log r|}u\Big)\circ T_k$. Since

$$\frac{r^{-\alpha}/|\log r|}{\lambda^{-k}/k|\log\xi|}=\Big(\frac{r}{\xi^k}\Big)^{-\alpha}\frac{|\log\xi^k|}{|\log r|},$$

we have

$$w_k=r^{-\alpha}\frac{|\log\xi^k|}{|\log\xi^kr|}v_k\to\frac{1}{|\log\xi|\lambda}g.$$

Therefore

$$\lim_{r\to0}\frac{r^{-\alpha}}{|\log r|}u=\frac{1}{|\log\xi|\lambda}g. \tag{2.14}$$

Let $\bar{\xi} \in (0,1)$ be another constant such that $\xi, \bar{\xi}$ are unreducible like Lemma 2.4. Then by analogy with Lemma 2.4 we have $\bar{X}^k h = \sum\limits_{j=1}^{s} \bar{X}^k \bar{h}_j$, where $\bar{h}_j \in N((\bar{X} - \bar{\lambda}_j I)^2)$ with $|\bar{\lambda}_j| = |\bar{\xi}|^{\mathrm{Re}\alpha}$. Let $\bar{\lambda}_j = \bar{\xi}^{\alpha + i\beta_j}$. We claim that $\beta_j = 0$. Otherwise we would have

$$u(\bar{\xi}^k, \theta) = \sum_{j=1}^{s} \left\{ (\bar{\xi}^{\alpha+i\beta_j})^k \bar{h}_j + k(\bar{\xi}^{\alpha+i\beta_j})^{k-1} \bar{g}_j \right\},$$

where $\bar{g}_j$ are eigenfunctions. The functions $\bar{g}_j$ are linearly independent, and $\bar{\xi}^{i\beta_j k}$ has no limit as $k \to \infty$, which contradicts (2.14). Therefore $\bar{\lambda}_1 = \cdots = \bar{\lambda}_s$, namely $s = 1$, hence $h = \bar{h}_1$. Following the same lines of the proof of Lemma 2.4, we get (2.13).

If the degree of the elementary divisors is even higher, we can argue in an analogous way. Finally we have

Theorem 2.1. *The solution u_1 is a finite sum of particular solutions* (2.11), (2.13)*, and so on.*

§3. Nonhomogeneous Equation with Constant Coefficients

We need some preliminaries for the studying of nonhomogeneous equations. To begin with, we consider one example which is useful later on,

$$L_0 u = 1, \qquad u|_{r=1} = 0. \tag{3.1}$$

Clearly there exists a unique weak solution. Let $\xi \in (0,1)$ as the previous section, and $u|_{\Gamma_1} = g$. Then

$$u|_{\Gamma_k} = X^{k-1} g + \xi^2 X^{k-2} g + \cdots + \xi^{2(k-1)} g, \qquad k = 1, 2, \cdots.$$

Let $u|_{\Omega_1} = \tilde{u}$ and $u^{(0)}$ be the solution to the equation (2.1) and boundary data g. Then we get

$$u|_{\Omega_k} \circ T_k = u^{(0)}|_{\Omega_{k-1}} \circ T_{k-1} + \xi^2 u^{(0)}|_{\Omega_{k-2}} \circ T_{k-2} + \cdots + \xi^{2(k-1)} \tilde{u} \circ T_1.$$

It can be verified that the solution to (3.1) consists of a finite number of particular solutions to the homogeneous equation and a function in $H^2(\xi\Omega \cap S_m)$ for $1 \le m \le m_0$.

Next, let us consider the problem on the space $\mathbb{R}^2$. The sectors S_m are extended to $r = \infty$, and then $\mathbb{R}^2$ is divided into m_0 sectors. We define a space

$$Z^1(\mathbb{R}^2) = \Big\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^2); \nabla u \in L^2(\mathbb{R}^2), \int_{r<1} u\, dx = 0 \Big\}.$$

Then equipped with the norm $\|\nabla u\|_0$ it is a Hilbert space. We assume that the right hand side $f \in L^2(\mathbb{R}^2)$ and $\mathrm{supp} f \subset S(o,1)$. Consider the equation

$$L_0 u = f, \tag{3.2}$$

and define the corresponding sesquilinear form

$$a_0(u,v) = \int_{\mathbb{R}^2} a_{ij} \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}}\, dx.$$

First we assume that

$$\int_{r<1} f\, dx = 0. \tag{3.3}$$

The weak formulation of (3.2) is: find $u \in Z^1(\mathbb{R}^2)$ such that

$$a_0(u,v) = (f,v), \qquad \forall v \in Z^1(\mathbb{R}^2).$$

By the Lax-Milgram theorem, there exists a unique solution u.

Lemma 3.1. *If $b \in (0,1)$, then*

$$|u|_1 \le C\|r^{\frac{b}{2}} f\|_0. \tag{3.4}$$

Proof. By Hölder inequality

$$|a_0(u,u)| = |(f,u)| \le C\|r^{\frac{b}{2}} f\|_0 \|u\|_{0,4} \le C\|r^{\frac{b}{2}} f\|_0 |u|_1.$$

On the other hand $|a_0(u,u)| \ge \chi |u|_1^2$, which gives (3.4).

Next let us get rid of the condition (3.3). For a special case, $f = \kappa(x)$, where

$$\kappa = \begin{cases} 1, & r < 1, \\ 0, & r > 1, \end{cases}$$

we define

$$h = \begin{cases} 0, & r > 1, \\ 1, & 1 > r > \eta, \\ 1 - \frac{1}{\eta^2}, & r < \eta. \end{cases}$$

Then $\int h\,dx = 0$, hence the equation (3.2) with $f = h$ admits a unique solution, denoted by $\tilde{u}$. Let $q = \int_0^r h\,dr^2$. Then

$$a_0(\tilde{u}, v) = \int hvr\,dr\,d\theta = -\frac{1}{2}\int q\frac{\partial v}{\partial r^2}\,dr^2\,d\theta = -\frac{1}{2}\int \frac{q}{r}\frac{\partial v}{\partial r} r\,dr\,d\theta.$$

Thus $|\tilde{u}|_1 \le \frac{1}{2}\|\frac{q}{r}\|_0 \le C(1-\eta)$. Let $\eta \to 1$, $\frac{\tilde{u}}{1-\eta}$ converges weakly in $Z^1(\mathbb{R}^2)$. Let w be the limit. Define $\tilde{q} = -2r\kappa(x)$. Then w satisfies

$$a_0(w,v) = -\frac{1}{2}\int \tilde{q}\frac{\partial v}{\partial r} r\,dr\,d\theta = \frac{1}{2}\int v\frac{\partial \tilde{q} r}{\partial r}\,dr\,d\theta = \int \bar{q} v\,dx,$$

where $\bar{q} = \delta(r-1) - 2\kappa(x)$. In the disc $S(o,\xi)$ we can use the result for the problem (3.1) and Theorem 2.1 to conclude that w consists of a finite number of particular solutions to the homogeneous equation and a function in $H^2(\xi\Omega \cap S_m)$ for $1 \le m \le m_0$. Applying the embedding theorem[1] we get that w belongs to a Hölder space $C^{0,\mu}$ with a positive number μ. The function $w(x) - w(0)$ vanishes at $x = 0$ and satisfies the same equation, which is still denoted by w for simplicity.

Let us define $u = -\int_0^r w(r,\theta)\frac{dr}{r}$ and derive the equation satisfied by u. We fix an arbitrary point $(\bar{r},\theta)$ and take some points $r_1, r_2, \cdots, r_n, \cdots$ on the interval $[0,\bar{r}]$ such that they are equidistributed. Let $\Delta r = r_2 - r_1$, and $w_n(r,\theta) = w(\frac{r_n}{\bar{r}}r, \theta)$. Then

$$I = -\sum \frac{w(r_n,\theta)\Delta r}{r_n} = -\sum \frac{w_n(\bar{r},\theta)\Delta r}{r_n}.$$

Let $\bar{w}_\Delta = -\sum \frac{w_n(\bar{r},\theta)\Delta r}{r_n}$. Then we apply the differential operator L_0 to $\bar{w}_\Delta$ and obtain

$$L_0 \bar{w}_\Delta = -\sum \frac{1}{r_n}\frac{r_n^2}{\bar{r}^2}\bar{q}\Big(\frac{r_n}{\bar{r}}r, \theta\Big)\,\Delta r.$$

Let $\Delta r \to 0$, and we define $\bar{w}$ as the limit of $\bar{w}_\Delta$. Then $\bar{w}(\bar{r},\theta) = u(\bar{r},\theta)$. $\bar{w}$ satisfies

$$L_0 \bar{w} = -\int_0^{\bar{r}} \frac{t}{\bar{r}^2}\bar{q}\Big(\frac{t}{\bar{r}}x\Big)\,dt = -\int_0^r \frac{s}{r^2}\bar{q}\Big(\frac{s}{r}x\Big)\,ds.$$

Therefore $\bar{w}$ is in fact independent of $(\bar{r}, \theta)$, consequently $\bar{w} \equiv u$. We obtain $L_0 u = \kappa(x)$. So u is the desired solution which is in $H^1_{\text{loc}}(\mathbb{R}^2)$. Applying the previous results we can get the structure of u near the point o.

For general f we have

Lemma 3.2. *If $f \in L^2(\mathbb{R}^2)$ and $\text{supp} f \subset S(o,1)$, then the equation $L_0 u = f$ on $\mathbb{R}^2$ admits a solution $u \in H^1_{\text{loc}}(\mathbb{R}^2)$ such that on any bounded domain D,*

$$|u|_{1,D} \le C(D)\|f\|_0, \tag{3.5}$$

where $C(D)$ is a constant depending on D.

Proof. We define

$$f_0 = f - \frac{1}{\pi}\int_{r<1} f\,dx.$$

Then $\int f_0\,dx = 0$. We have

$$\Big|\frac{1}{\pi}\int_{r<1} f\,dx\Big| \le \frac{1}{\sqrt{\pi}}\|f\|_0, \qquad \|f_0\|_0 \le \|f\|_0.$$

The solution u is decomposed into two parts, $u = u_0 + \tilde{u}$. u_0 corresponds to f_0 and satisfies

$$|u_0|_1 \le C\|f_0\|_0 \le C\|f\|_0.$$

$\tilde{u}$ corresponds to the constant, which has been already studied.

We are now in a position to study the main result of this section. In $S(o,1)$ we construct a particular solution u to the equation $L_0 u = f$, $f \in L^2$, such that u possesses the desired regularity. We take $\xi \in (0,1)$ and $\Omega = S(o,1)$ as before.

Lemma 3.3. *There is a particular solution u such that for all $l \ge 1$,*

$$\begin{aligned}|u|^2_{2,\Omega_l\cap S_m} \le C\bigg(&\xi^{-bl}\|r^{\frac{b}{2}}f\|^2_{0,\xi^{l-1}\Omega} + \sum_{k=1}^{l-2}\left(\frac{|\lambda_{N+1}|+\varepsilon}{\xi}\right)^{l-k}\|f\|^2_{0,\Omega_k}\\ &+ l^{2M-2}\sum_{k=1}^{l-2}\|f\|^2_{0,\Omega_k}\bigg),\end{aligned} \tag{3.6}$$

where b is a positive constant, M is a positive integer both depending on L_0, and $|\lambda_{N+1}|+\varepsilon < \xi$. Moreover there is another particular solution u such that

$$\begin{aligned}|u|^2_{2,\Omega_l\cap S_m} \le \bigg(&\xi^{-bl}\|r^{\frac{b}{2}}f\|^2_{0,\xi^{l-1}\Omega} + \sum_{k=1}^{l-2}\left(\frac{|\lambda_{N+1}|+\varepsilon}{\xi}\right)^{l-k}\|f\|^2_{0,\Omega_k}\\ &+ l^{-3}\|(|\log r|+1)^M f\|^2_{0,\xi^{l-1}\Omega}\bigg),\end{aligned} \tag{3.7}$$

provided the last norm is finite.

Proof. Let

$$f_k = \begin{cases} f, & x \in \Omega_k,\\ 0, & \text{otherwise}.\end{cases}$$

Then $f = \sum f_k$. Let u_k be the solution in Lemma 3.2 corresponding to f_k. Because u_k satisfies homogeneous equation on $\xi^{k+1}\Omega$, according to §2 we have the decomposition $u_k = u_k^{(1)} + u_k^{(2)}$ with $u_k^{(1)}|_{\Gamma_{k+1}} \in H_1$ and $u_k^{(2)}|_{\Gamma_{k+1}} \in H_2$. We extend $u_k^{(1)}$ analytically to Ω, which is still denoted by $u_k^{(1)}$. Let $u=\sum_{k=1}^{\infty}(u_k-u_k^{(1)})$. We estimate u on Ω_l.

$$u=\sum_{k\geq l-1}u_k+\sum_{k<l-1}u_k^{(2)}-\sum_{k\geq l-1}u^{(1)}.$$

For the first term we set $\tilde{u}=\big(\sum_{k\geq l-1}u_k\big)\circ T_{l-1}$. Then $\tilde{u}$ satisfies

$$L_0\tilde{u}=\xi^{2(l-1)}\Big(\sum_{k\geq l-1}f_k\Big)\circ T_{l-1}.$$

Let

$$\sigma=\frac{1}{\pi}\int_{r<1}\xi^{2(l-1)}\Big(\sum_{k\geq l-1}f_k\Big)\circ T_{l-1}\,dx.$$

Then we define a solution $\tilde{u}^{(1)}$ satisfying $L_0\tilde{u}^{(1)}=\sigma\kappa(x)$. By Lemma 3.2, $|\tilde{u}^{(1)}|_{1,\Omega\setminus\overline{\xi^3\Omega}}\leq C|\sigma|$. Applying interior estimates we get $|\tilde{u}^{(1)}|_{2,\Omega_1\cap S_m}\leq C|\sigma|$. Then we have

$$\begin{aligned}|\tilde{u}^{(1)}\circ T_{1-l}|_{2,\Omega_l\cap S_m}&\leq C\xi^{-(l-1)}|\sigma|=C\xi^{-(l-1)}\Big|\int_{\xi^{l-1}\Omega}f\,dx\Big|\\&\leq C\xi^{-(l-1)}\Big(\int_{\xi^{l-1}\Omega}r^{-b}\,dx\Big)^{\frac{1}{2}}\Big(\int_{\xi^{l-1}\Omega}r^bf^2\,dx\Big)^{\frac{1}{2}}\\&\leq C\xi^{-\frac{bl}{2}}\|r^{\frac{b}{2}}f\|_{0,\xi^{l-1}\Omega}.\end{aligned}$$

Let $\tilde{u}^{(2)}=\tilde{u}-\tilde{u}^{(1)}$. Then applying Lemma 3.1 we get

$$|\tilde{u}^{(2)}|_1\leq C\xi^{2(l-1)}\Big\|r^{\frac{b}{2}}\Big(\sum_{k\geq l-1}f_k\Big)\circ T_{l-1}\Big\|_0+C|\sigma|.$$

Here

$$\Big\|r^{\frac{b}{2}}\Big(\sum_{k\geq l-1}f_k\Big)\circ T_{l-1}\Big\|_0=\xi^{-(l-1)}\xi^{-\frac{b(l-1)}{2}}\|r^{\frac{b}{2}}f\|_{0,\xi^{l-1}\Omega}.$$

Following the same lines as the estimate of $\tilde{u}^{(1)}$, we get

$$|\tilde{u}^{(2)}\circ T_{1-l}|_{2,\Omega_l\cap S_m}\leq C\xi^{-\frac{bl}{2}}\|r^{\frac{b}{2}}f\|_{0,\xi^{l-1}\Omega}.$$

Therefore

$$\Big|\sum_{k\geq l-1}u_k\Big|_{2,\Omega_l\cap S_m}\leq C\xi^{-\frac{bl}{2}}\|r^{\frac{b}{2}}f\|_{0,\xi^{l-1}\Omega}. \tag{3.8}$$

For the second term with $l\geq 3$ let $u_k\circ T_k|_{\Gamma_1}=g$. Then $g=g_1+g_2$, $g_1\in H_1$ and $g_2\in H_2$. Let $\tilde{u}=u_k\circ T_k$. Then $\tilde{u}$ satisfies $L_0\tilde{u}=\xi^{2k}f_k\circ T_k$. By Lemma 3.2

$$|\tilde{u}|_{1,\Omega}\leq C\|\xi^{2k}f_k\circ T_k\|_0=C\xi^k\|f_k\|_0,$$

hence $\|g_2\|_H\leq C\|g\|_H\leq C\xi^k\|f_k\|_0$. Applying X^{l-k-2} to g_2 gives $u_k^{(2)}\circ T_k|_{\Gamma_{l-k-1}}$. Therefore for K_0 large enough we have

$$\|(u_k^{(2)}\circ T_k)|_{\Gamma_{l-k-1}}\|_H\leq C(|\lambda_{N+1}|+\varepsilon)^{l-k-2}\xi^k\|f_k\|_0$$

provided $l-k\geq K_0$. We regard $u_k^{(2)}\circ T_k|_{\Gamma_{l-k-1}}$ as the boundary data and obtain

$$|u_k^{(2)}\circ T_{l-1}|_{1,\Omega}\leq C(|\lambda_{N+1}|+\varepsilon)^{l-k-2}\xi^k\|f_k\|_0.$$

The interior estimate gives

$$|u_k^{(2)}\circ T_{l-1}|_{2,\Omega_1\cap S_m}\leq C(|\lambda_{N+1}|+\varepsilon)^{l-k-2}\xi^k\|f_k\|_0,$$

hence $|u_k^{(2)}|_{2,\Omega_l\cap S_m} \le C(|\lambda_{N+1}|+\varepsilon)^{l-k-2}\xi^{k-l+1}\|f_k\|_0$. If $l-k<K_0$, we notice the fact that X is a bounded operator and get

$$|u_k^{(2)}|_{2,\Omega_l\cap S_m} \le C\xi^{k-l+1}\|f_k\|_0.$$

The triangle inequality leads to

$$\Big|\sum_{k<l-1} u_k^{(2)}\Big|_{2,\Omega_l\cap S_m} \le \sum_{k<l-1} |u_k^{(2)}|_{2,\Omega_l\cap S_m} \le C\Big(\sum_{k=1}^{l-K_0}\Big(\frac{|\lambda_{N+1}|+\varepsilon}{\xi}\Big)^{l-k}\|f_k\|_0 + \sum_{k=l-K_0+1}^{l-2}\xi^{k-l+1}\|f_k\|_0\Big).$$

By the assumption $|\lambda_{N+1}|+\varepsilon<\xi$, and there are only K_0-2 terms in the second summation, so we can take the constant C appropriately such that

$$\Big|\sum_{k<l-1} u_k^{(2)}\Big|_{2,\Omega_l\cap S_m} \le C\sum_{k=1}^{l-2}\Big(\frac{|\lambda_{N+1}|+\varepsilon}{\xi}\Big)^{l-k}\|f_k\|_0.$$

Applying the Schwarz inequality we get

$$\Big|\sum_{k<l-1} u_k^{(2)}\Big|_{2,\Omega_l\cap S_m} \le C\Big(\sum_{k=1}^{l-2}\Big(\frac{|\lambda_{N+1}|+\varepsilon}{\xi}\Big)^{l-k}\|f_k\|_0^2\Big)^{\frac{1}{2}}. \tag{3.9}$$

To estimate the third term we need to use the domain Ω_k for negative k which is also defined by $\{\xi^k > r > \xi^{k+1}\}$. We notice that $u^{(1)}\circ T_k = \sum w_{kj}$, which is a finite sum of particular solutions w_{kj} in the form of (2.11), (2.13), etc. First we consider the term $w_{kj} = \beta r^\alpha g$, where $\mathrm{Re}\alpha < 1$ and g is an eigenfunction. Following the argument for the second term we get $|\beta| \le C\xi^k\|f_k\|_0$. Thus $|D^2 w_{kj}| \le C\xi^k\|f_k\|_0 r^{\mathrm{Re}\alpha-2}$, and

$$|w_{kj}|_{2,\Omega_{l-k}\cap S_m} \le C\xi^k\xi^{(l-k)(\mathrm{Re}\alpha-1)}\|f_k\|_0.$$

Hence $|w_{kj}\circ T_{-k}|_{2,\Omega_l\cap S_m} \le C\xi^{(l-k)(\mathrm{Re}\alpha-1)}\|f_k\|_0$. The triangle inequality and the Schwarz inequality lead to

$$\Big|\sum_{k\ge l-1} w_{kj}\circ T_{-k}\Big|_{2,\Omega_l\cap S_m} \le C\sum_{k\ge l-1}\xi^{(l-k)(\mathrm{Re}\alpha-1)}\|f_k\|_0 \le C\Big(\sum_{k\ge l-1}\xi^{2(k-l)(1-\mathrm{Re}\alpha)-bk}\Big)^{\frac{1}{2}}\Big(\sum_{k\ge l-1}\xi^{bk}\|f_k\|_0^2\Big)^{\frac{1}{2}},$$

where $b>0$ is sufficiently small such that $2(1-\mathrm{Re}\ \alpha)-b>0$. Then we have

$$\Big|\sum_{k\ge l-1} w_{kj}\circ T_{-k}\Big|_{2,\Omega_l\cap S_m} \le C\xi^{-\frac{bl}{2}}\|r^{\frac{b}{2}}f\|_{0,\xi^{l-1}\Omega}. \tag{3.10}$$

For the particular solution (2.13) the argument is the same except there is a logarithm factor and we have

$$|w_{kj}\circ T_{-k}|_{2,\Omega_l\cap S_m} \le C\xi^{(l-k)(\mathrm{Re}\alpha-1)}(k-l+2)\|f_k\|_0.$$

From this inequality we also get (3.10).

$\mathrm{Re}\alpha<1$ is equivalent to $|\lambda|>\xi$. So it remains to consider the case of $|\lambda|=\xi$. For the particular solution (2.11) we have

$$|w_{kj}\circ T_{-k}|_{2,\Omega_l\cap S_m} \le C\|f_k\|_0.$$

Let $M \geq 2$ be an integer. Then by the Schwarz inequality

$$\Big| \sum_{k \geq l-1} w_{kj} \circ T_{-k} \Big|_{2,\Omega_l \cap S_m} \leq C l^{-\frac{3}{2}} \|(|\log r| + 1)^M f\|_{0,\xi^{l-1}\Omega}. \tag{3.11}$$

For the particular solution (2.13) the argument is the same except the integer M should be larger.

Combining the estimations (3.8), (3.9), (3.10) and (3.11) we obtain (3.7).

We can change the definition of H_1, H_2 so that the eigenvalue λ with $|\lambda| = \xi$ belongs to the spectrum set of H_1. Then (3.6) is verified for an appropriate M.

Lemma 3.4. *There is a particular solution u such that*

$$\|u\|_{1,\xi\Omega} + \Big\| \frac{D^2 u}{(|\log r| + 1)^M} \Big\|_{0,\xi\Omega \cap S_m} \leq C \|f\|_0. \tag{3.12}$$

Moreover there is another solution u such that

$$\|u\|_{1,\xi\Omega} + |u|_{2,\xi\Omega \cap S_m} \leq C \|(|\log r| + 1)^M f\|_0, \tag{3.13}$$

where M is determined by Lemma 3.3.

Proof. Let us take the summation of the first two terms of (3.6) or (3.7).

$$\begin{aligned}
&C \sum_{l=1}^{\infty} \left(\xi^{-bl} \|r^{\frac{b}{2}} f\|^2_{0,\xi^{l-1}\Omega} + \sum_{k=1}^{l-2} \left(\frac{|\lambda_{N+1}| + \varepsilon}{\xi} \right)^{l-k} \|f\|^2_{0,\Omega_k} \right) \\
&= C \left(\sum_{l=1}^{\infty} \xi^{-bl} \sum_{k=l-1}^{\infty} \|r^{\frac{b}{2}} f\|^2_{0,\Omega_k} + \sum_{l=1}^{\infty} \sum_{k=1}^{l-2} \left(\frac{|\lambda_{N+1}| + \varepsilon}{\xi} \right)^{l-k} \|f\|^2_{0,\Omega_k} \right) \\
&= C \left(\sum_{k=0}^{\infty} \sum_{l=1}^{k+1} \xi^{-bl} \|r^{\frac{b}{2}} f\|^2_{0,\Omega_k} + \sum_{k=1}^{\infty} \sum_{l=k+2}^{\infty} \left(\frac{|\lambda_{N+1}| + \varepsilon}{\xi} \right)^{l-k} \|f\|^2_{0,\Omega_k} \right) \\
&\leq C \left(\sum_{k=0}^{\infty} \xi^{-b(k+1)} \|r^{\frac{b}{2}} f\|^2_{0,\Omega_k} + \sum_{k=1}^{\infty} \|f\|^2_{0,\Omega_k} \right) \leq C \|f\|^2_0.
\end{aligned}$$

Multiplying (3.6) by l^{-M}, and then taking the summation, we get

$$\Big\| \frac{D^2 u}{(|\log r| + 1)^M} \Big\|^2_{0,\xi\Omega \cap S_m} \leq C \|f\|^2_0 + C \sum_{l=1}^{\infty} \sum_{k=1}^{l-2} l^{-2} \|f\|^2_{0,\Omega_k} \leq C \|f\|^2_0,$$

which gives the estimate of second order derivatives in (3.12). It is easy to see that $\|u\|^2_{1,\Omega_l}$ is also bounded by the right hand side of (3.6). The proof for (3.13) is analogous.

§4. General Equations

The proof of Theorem 1.1 proceeds in four steps.

Step 1. We consider the disc $\Omega = S(o, 1)$ and assume that o is the unique singular point. Let $u \in H^1(\Omega)$ be a solution to the equation (1.1). Let $v = u \circ T_{k-1}$ for $k \geq 1$. Then v satisfies

$$\frac{\partial}{\partial x_j} \Big(a_{ij} \frac{\partial v}{\partial x_i} \Big) = \xi^{2(k-1)} \Big(f - b_i \frac{\partial u}{\partial x_i} - cu \Big) \circ T_{k-1}$$

on $\Omega \setminus \overline{\xi^3\Omega}$. Applying the interior estimate we have

$$|v|_{2,\Omega_1\cap S_m} \le C\left\{|v|_{1,\Omega\setminus\overline{\xi^3\Omega}} + \|\xi^{2(k-1)}\Big(f - b_i\frac{\partial u}{\partial x_i} - cu\Big)\circ T_{k-1}\|_{0,\Omega\setminus\overline{\xi^3\Omega}}\right\}$$
$$\le C\left\{\|u\|_{1,\xi^{k-1}\Omega\setminus\overline{\xi^{k+2}\Omega}} + \xi^{k-1}\|f\|_{0,\xi^{k-1}\Omega\setminus\overline{\xi^{k+2}\Omega}}\right\},$$

which gives

$$\|rD^2u\|^2_{0,\xi\Omega\cap S_m} \le C\{\|u\|^2_{1,\Omega} + \|rf\|^2_{0,\Omega}\}. \tag{4.1}$$

Step 2. We rewrite the equation (1.1) as

$$a_0(u,v) = \int_\Omega a_{ij}(0)\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}\,dx$$
$$= \int_\Omega (a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}\,dx + \int_\Omega \Big(f - b_i\frac{\partial u}{\partial x_i} - cu\Big)\bar v\,dx, \quad \forall v \in H^1_0(\Omega).$$

Integrating by parts we get

$$\int_\Omega (a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}\,dx$$
$$= \int_\Omega \frac{\partial}{\partial x_j}\Big((a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}\Big)\bar v\,dx + \sum_m \int_{\partial S_m} n_j\frac{\partial u}{\partial x_i}(a_{ij}(0) - a_{ij})\bar v\,ds,$$

where $n = (n_1, n_2)$ is the unit exterior normal vector on ∂S_m.

We consider one sector S_m. For simplicity we drop the subscript m. We define a function $\chi \in C_0^\infty(\Omega \setminus \overline{\xi^3\Omega})$ such that $\chi \ge 0$ and $\chi > 0$ on Ω_1. Then we define $\chi_k = \chi \circ T_{1-k}$. Replace χ_k by $\chi_k/\sum_k \chi_k$, still denoted by χ_k. By the inverse trace theorem[9], there exist $\tilde z_k \in H^2(\Omega \setminus \overline{\xi^3\Omega})$ such that we have

$$\tilde z_k = 0, \qquad \frac{\partial \tilde z_k}{\partial n} = \xi^{k-1}\Big\{\chi_k\frac{(a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}n_j}{n^TAn}\Big\}\circ T_{k-1}$$

on $\partial S \cap \{\Omega \setminus \overline{\xi^3\Omega}\}$, and

$$\|\tilde z_k\|_2 \le C\Big\|\xi^{k-1}\Big\{\chi_k\frac{(a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}n_j}{n^TAn}\Big\}\circ T_{k-1}\Big\|_1,$$

where the matrix $A = (a_{ij}(0))$. Let $z_k = \tilde z_k \circ T_{1-k}$. Then

$$\frac{\partial z_k}{\partial n} = \chi_k\frac{(a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}n_j}{n^TAn}, \tag{4.2}$$

and

$$\|z_k\|_2 \le C\Big\{\Big|\frac{(a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}n_j}{n^TAn}\Big|_{1,\xi^{k-1}\Omega\setminus\overline{\xi^{k+2}\Omega}}$$
$$+ \xi^{-(k-1)}\Big\|\frac{(a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}n_j}{n^TAn}\Big\|_{0,\xi^{k-1}\Omega\setminus\overline{\xi^{k+2}\Omega}}\Big\}. \tag{4.3}$$

Let $z = \sum_{k=1}^\infty z_k$. Then by (4.2) we have

$$\frac{\partial z}{\partial n} = \frac{(a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}n_j}{n^TAn}$$

on the boundary. But z vanishes on the boundary, hence

$$a_{ij}(0)\frac{\partial z}{\partial x_i}n_j = n^TA\nabla z = n^TA\frac{\partial z}{\partial n}n = (a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}n_j. \tag{4.4}$$

Since $|a_{ij}(0) - a_{ij}| \leq Cr$, using the weighted estimate (4.1) and (4.3) we get

$$\|z\|^2_{2,\xi\Omega\cap S_m} \leq C\{\|u\|^2_{1,\Omega} + \|rf\|^2_{0,\Omega}\}. \tag{4.5}$$

The equation becomes

$$a_0(u,v) = -\int_{\xi\Omega} \frac{\partial}{\partial x_j}\Big((a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}\Big)\bar{v}\,dx + \sum_m \int_{\partial S_m} a_{ij}(0)\frac{\partial z}{\partial x_i} n_j \bar{v}\,ds$$
$$+ \int_{\xi\Omega}\Big(f - b_i\frac{\partial u}{\partial x_i} - cu\Big)\bar{v}\,dx, \quad \forall v \in H^1_0(\xi\Omega). \tag{4.6}$$

Step 3. Integrating by parts we deduce from (4.6) that

$$a_0(u-z,v) = \int_{\xi\Omega}\Big\{-\frac{\partial}{\partial x_j}\Big((a_{ij}(0) - a_{ij})\frac{\partial u}{\partial x_i}\Big) + \frac{\partial}{\partial x_j}\Big(a_{ij}(0)\frac{\partial z}{\partial x_i}\Big)$$
$$+ f - b_i\frac{\partial u}{\partial x_i} - cu\Big\}\bar{v}\,dx. \tag{4.7}$$

By Lemma 3.4 and (4.1), (4.5) there is a particular solution w to the equation (4.7) such that

$$\|w\|_{1,\xi^2\Omega} + \sum_m \Big\|\frac{D^2 w}{(|\log r| + 1)^M}\Big\|_{0,\xi^2\Omega\cap S_m}$$
$$\leq C\Big\{\|u\|_{1,\xi\Omega} + \sum_m \big(\|rD^2 u\|_{0,\xi\Omega\cap S_m} + \|z\|_{2,\xi\Omega\cap S_m}\big) + \|f\|_{0,\xi\Omega}\Big\}.$$
$$\leq C(\|u\|_{1,\Omega} + \|f\|_{0,\Omega}). \tag{4.8}$$

There is no singular points in Ω_1, so the above estimate also holds in Ω_1. The function $u - z - w$ satisfies the equation (2.1) in $\xi\Omega$, and from (4.8) we have

$$\|u - z - w\|_{1,\xi\Omega} \leq C(\|u\|_{1,\Omega} + \|f\|_{0,\Omega}).$$

Then by Lemma 2.2 we have $u - z - w = u_1 + u_2$, and

$$\|u_2\|_{2,\xi\Omega\cap S_m} \leq C(\|u\|_{1,\Omega} + \|f\|_{0,\Omega}), \quad \|u_1\|_{1,\xi\Omega} \leq C(\|u\|_{1,\Omega} + \|f\|_{0,\Omega}).$$

Step 4. Let us regard $z + w + u_2$ as the function w in Theorem 1.1, u_1 the function v in Theorem 1.1. Then (1.5) is verified for one singular point. The proof for (1.6) is analogous.

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