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SPECTRAL SEQUENCES OF COHOMOLOGIES OF SINGULARITIES OF C^∞ MAPPINGS**

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Abstract

Two spectral sequences are defined respectively for cohomologies $H^p(\Omega^{\cdot}_{f,k-\cdot,0})$ and $H^p(\Lambda^{\cdot}_{f,k-\cdot,0})$ of singularities of C^∞ mappings. They are finitely dimensional new invariances under right equivalences and contact transformations respectively. Formulae to be computed by linear algebra are proved.

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Suppose that U is an open set in R^N with coordinates $(x_1, \cdots, x_N)$. $0 \in U$, O_U (or $O_U(x)$) is the sheaf of C^∞ function germs on U. $U^{(k)}$ is the k-th infinitesimal neighbourhood of U with structure sheaf $O_{U,k} = \sum\limits_{|\alpha| \le k} O_U(y-x)^\alpha$, where $\alpha = (\alpha_1, \cdots, \alpha_N)$, α_i are nonegative integers. $\Omega^1_{U,l} = \sum\limits_{i=1}^N O_{U,l} dy_i$ is the differential form module with respect to $dy_1, \cdots, dy_N$. $\Omega^p_{U,l} = \bigwedge^p \Omega^1_{U,l}$. D is the partial differential with respect to $y_1, \cdots, y_N$. D induces the exterior differential $D: \Omega^p_{U,l} \longrightarrow \Omega^{p+1}_{U,l-1}$ (see [1, 2, 3]).

$f: (U,0) \longrightarrow (R,0)$ is a C^∞ mapping.

$$Q_f = Q_f(x) = \frac{O_{U,0}}{(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N}) O_{U,0}(x)},$$

where $O_{U,0}$ is the stalk of O_U at 0. $F = \sum\limits_{|\alpha| \le k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} (y-x)^\alpha$ is the Taylor expansion of f in $O_{U,k}$. In this paper we suppose $\dim_R Q_f \le \infty$.

In [1, 2, 3] we defined the following modules and complexes

$$\Omega^0_{f,k} = O_{U,k}, \quad \Omega^p_{f,k-p} = \frac{\Omega^p_{U,k-p}}{DF \wedge \Omega^{p-1}_{U,k-p}}, \quad p \ge 1.$$

$$\Lambda^0_{f,k} = \frac{O_{U,k}}{FO_{U,0}}, \quad \Lambda^p_{f,k-p} = \frac{\Omega^p_{U,k-p}}{F\Omega^p_{U,k-p} + DF \wedge \Omega^{p-1}_{U,k-p}}, \quad p \ge 1.$$

$$\Omega^0_{V,k} = \frac{O_{U,k}}{fO_{U,0} + FO_{U,0}}, \quad \Omega^p_{V,k-p} = \frac{\Omega^p_{U,k-p}}{f\Omega^p_{U,k-p} + F\Omega^p_{U,k-p} + DF \wedge \Omega^{p-1}_{U,k-p}}, \quad p \ge 1,$$

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and complexes $\{\Omega^p_{f,k-p}, D\}_{p\geq 0}$, $\{\Lambda^p_{f,k-p}, D\}_{p\geq 0}$, and $\{\Omega^p_{V,k-p}, D\}_{p\geq 0}$. We consider their stalks at 0, e.g., $\Omega^p_{f,k-p,0}$, $\Lambda^p_{f,k-p,0}$, etc.

In [3] we used a spectral sequence to prove formulae of cohomological groups $H^p(\Omega^{\cdot}_{V,k-\cdot,0})$ for real analytic and complex holomorhpic cases. By these formulae $H^p(\Omega^{\cdot}_{V,k-\cdot,0})$ can be computed by linear algebra. In this paper we introduce spectral sequences of $H^p(\Omega^{\cdot}_{f,k-\cdot,0})$ and $H^p(\Lambda^{\cdot}_{f,k-\cdot,0})$. These spectral sequences are new invariances under right equivalences and contact transformations for $H^p(\Omega^{\cdot}_{f,k-\cdot,0})$ and $H^p(\Lambda^{\cdot}_{f,k-\cdot,0})$, respectively. They provide filtrations for $H^p(\Omega^{\cdot}_{f,k-\cdot,0})$ and $H^p(\Lambda^{\cdot}_{f,k-\cdot,0})$.

Let $\Pi_l : O_{U,l} \to O_{U,l-1}$ be the natural projection. It induces natural projections Π_l : $\Omega^p_{U,l} \to \Omega^p_{U,l-1}$, $\Pi_l : \Omega^p_{f,l} \to \Omega^p_{f,l-1}$ and $\Pi_l : \Lambda_{f,l} \to \Lambda_{f,l-1}$. Let

$$W^{N-1}_{f,l} = \frac{\Omega^{N-1}_{f,l,0}}{D\Omega^{N-2}_{f,l+1,0}}, \quad l \geq 0.$$

$\dim_R W^{N-1}_{f,l} = \binom{l+N-1}{N+1}\dim_R Q_f$ (see [1]).

In [1] we proved

Theorem A.

$$H^0(\Omega^{\cdot}_{f,k-\cdot,0}) = \bigoplus_{i=0}^{k} O_{U,0}F^i, \quad H^p(\Omega^{\cdot}_{f,k-\cdot,0}) = 0, \quad =\neq 0, N-1,$$

$$H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) \overset{\delta_{k-N+1}}{\cong} W^{N-1}_{f,k-N}.$$

If $[a] \in H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0})$, $Da = DF \wedge b$, *then* $\delta_{k-N+1}[a] = [b]$.

$$\dim_R H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) = \begin{pmatrix} k \\ N+1 \end{pmatrix} \dim_R Q_f.$$

Define a filtration on $W^{N-1}_{f,k-N+1}$:

$$F_0 W^{N-1}_{f,k-N+1} = W^{N-1}_{f,k-N+1},$$
$$F_r W^{N-1}_{f,k-N+1} = \delta^{-1}_{k-N+1} \cdots \delta^{-1}_{k-N+2-r} W^{N-1}_{f,k-N+1-r}, \quad r \geq 1.$$

$$F_0 \supset F_1 \supset \cdots \supset F_r \supset \cdots \supset F_{k-N+1} = 0. \quad F_1 W^{N-1}_{f,k-N+1} = H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}).$$

The main results of this paper are:

Theorem 1. $f : (R^N, 0) \longrightarrow (R, 0)$ *is a* C^∞ *mapping,* $\dim_R Q_f < \infty$. *Associated with* f*, there is a spectral sequence* $\{'E^{p,q}_r\}_{r\geq 2}$ *which is invariant under right equivalences and satisfies the following properties:*

(1) $'E^{p,0}_2 = H^p(\Omega^{\cdot}_{f,k-\cdot,0}), \quad p \geq 0,$

$$'E^{p,0}_r = H^p(\Omega^{\cdot}_{f,k-\cdot,0}), \quad 0 \leq p \leq N-2, \quad r \geq 3;$$

(2) $H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) =' E^{N-1,0}_2 \supset \cdots \supset' E^{N-1,0}_r \supset' E^{N-1,0}_{r+1} \supset \cdots \supset' E^{N-1,0}_{k-N+2} = 0,$

$$'E^{N-1,0}_r \cong F_{r-1} W^{N-1}_{f,k-N+1}, \quad r \geq 2;$$

(3) $'E^{p,N-p}_r \cong \Omega^N_{f,k-p,0}, \quad r \leq p+1-N,$

$$'E^{p,N-p}_r = 0, \quad r \geq p+2-N;$$

(4) $'E^{p,q}_r = 0, \quad$ *otherwise.*

Theorem 2. *$f : (R^N, 0) \longrightarrow (R, 0)$ is a C^∞ mapping, $\dim_R Q_f < \infty$. Associated with f, there is a spectral sequence $\{'E_r^{p,q}\}_{r\geq 2}$ which is invariant under contact transformations and satisfies the following properties:*

(1) $'E_2^{p,0} = H^p(\Lambda^{\cdot}_{f,k-\cdot,0}), \quad p \geq 0,$

$$'E_r^{p,0} = H^p(\Lambda^{\cdot}_{f,k-\cdot,0}), \quad p \neq N-2, N-1, \quad r \geq 3;$$

(2) $'E_2^{N-2,0} = 0, \quad r \geq 3;$

(3) $H^{N-1}(\Lambda^{\cdot}_{f,k-\cdot,0}) = {}' E_2^{N-1,0} \supset \cdots \supset' E_r^{N-1,0} \supset' E_{r+1}^{N-1,0} \supset \cdots \supset' E_{k-N+2}^{N-1,0} = 0,$

$$'E_r^{N-1,0} \cong \frac{F_{r-1}W^{N-1}_{f,k-N+1}}{FW^{N-1}_{f,k-N+1} \bigcap F_{r-1}W^{N-1}_{f,k-N+1}}, \quad r \geq 2;$$

(4) $'E_r^{p,q} = 0, \quad p+q \neq N-1, N, \quad q \leq -1, \quad r \geq 2;$

(5) $\dim'_R E_r^{p,q} < \infty, \quad r \geq 2,$ *except* $'E_r^{0,0}$.

The following results proved in [1] are useful in this paper.

Lemma A. *If $a \in \Omega^p_{U,l,0}, 0 \leq p \leq N-1, \quad DF \wedge a = 0$, there exists $b \in \Omega^{p-1}_{U,l,0}, \quad a = DF \wedge b$.*

Lemma B. *The sequence*

$$0 \to O_{U,0} \to O_{U,k,0} \xrightarrow{D} \Omega^1_{U,k-1,0} \to \cdots \xrightarrow{D} \Omega^p_{U,k-p,0} \xrightarrow{D} \Omega^{p+1}_{U,k-p-1,0} \to \cdots \xrightarrow{D} \Omega^N_{U,k-N,0} \to 0$$

is exact.

In [2] we proved

Theorem B.

$$H^0(\Lambda^{\cdot}_{f,k-\cdot,0}) = \frac{O_{U,0}}{f^{k+1}O_{U,0}}, \quad H^p(\Lambda^{\cdot}_{f,k-\cdot,0}) = 0, \quad p \neq 0, N-2, N-1,$$

$$H^{N-2}(\Lambda^{\cdot}_{f,k-\cdot,0}) = \ker(W^{N-1}_{f,k-N+1} \xrightarrow{F} W^{N-1}_{f,k-N+1}),$$

$$H^{N-1}(\Lambda^{\cdot}_{f,k-\cdot,0}) = \frac{H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0})}{FW^{N-1}_{f,k-N+1} \cap H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0})}.$$

From now on we denote the elements of $\Omega^l_{U,k-p,0}$ by $a_p(l), b_p(l)$, etc. Sometimes we omit p or l, e.g. by $a(l), b(l), a, b$, etc. If $a \in \Omega^{N-1}_{U,l,0}$, $[a]$ is the coset of a in W^{N-1}_{k-N+1}.

Now we prove Theorem 1.

Let DB be an indeterminate element. Define its degree: $\deg DB = 2$. Define $\deg a = 0, \forall a \in O_{U,l,0}$ and $\deg dy_i = 1, i = 1, \cdots, N$.

Convention: $(DB)^0 = 1$, and $(DB)^n = 0$, if $n < 0$.

$\Omega_k = \bigoplus_{p=0}^{N} \Omega^p_{U,k-p,0}$ is the exterior algebra over $O_{U,k,0}$. Let $\widetilde{\Omega}^{p,q}_{k-p} = \Omega^{p-q}_{U,k-p,0}(DB)^q$. $\widetilde{\Omega}^{p,q}_{k-p} = 0$, if $p < 0$ or $q < 0$. Let $\widetilde{\Omega}_k(DB) = \sum_{p,q} \widetilde{\Omega}^{p,q}_{k-p}$. It is a commutative algebra over $O_{U,k,0}$. Here "commutative" means: If a, b are homogeneous elements of $\widetilde{\Omega}_k(DB)$, $ab = (-1)^{\deg a \deg b} ba$.

The partial exterior differential $D : \Omega_k \to \Omega_k$ induces the differential $D : \widetilde{\Omega}_k(DB) \to \widetilde{\Omega}_k(DB)$ as follows. Let $D(DB) = 0$. If $a \in \Omega^p_{U,l,0}, \quad D(a(DB)^q) = Da(DB)^q$. $\deg D = 1$ and $DD = 0$. $D : \widetilde{\Omega}^{p.q}_{k-p} \to \widetilde{\Omega}^{p+1.q}_{k-p-1}$. For fixed q, $\{\widetilde{\Omega}^{p.q}_{k-p}, D\}_{p\geq 0}$ is a complex. By Lemma A

$$H(\widetilde{\Omega}_k(DB), D) = \bigoplus_{q=0}^{k} O_{U,0}(DB)^q,$$

$$H^0(\widetilde{\Omega}^{\cdot,q}_{k-\cdot}, D) = O_{U,0}(DB)^q, \quad H^p(\widetilde{\Omega}^{\cdot,q}_{k-\cdot}, D) = 0, \quad p \geq 1.$$

Define the second differential $\partial : \widetilde{\Omega}_k(DB) \to \widetilde{\Omega}_k(DB)$ as follows. $\deg\partial = -1$, $\partial\Omega^p_{U,l,0} = 0, p = 0, \cdots, N$, $\partial(DB) = -DF$. For homogeneous elements $a, b \in \widetilde{\Omega}_k(DB)$, $\partial(ab) = (\partial a)b + (-1)^{\deg a} a\partial b$. It is clear that $\partial\partial = 0$ and $\partial D + D\partial = 0$. ∂ induces $\partial : \widetilde{\Omega}^{p,q}_{k-p} \to \widetilde{\Omega}^{p,q-1}_{k-p}$.

$$\partial(a(DB)^q) = (-1)^{p-q+1} qa \wedge DF(DB)^{q-1}, \quad a \in \Omega^{p-q}_{U,k-p,0}.$$

Let $K^{p,q} = \widetilde{\Omega}^{p,-q}_{k-p}$. $'d = D, "d = \partial$. It is a double complex. $K^n = \sum\limits_{p+q=n} K^{p,q}$. Clearly $K^{p,q} = 0$, if $p+q \leq -1$, or $p+q \geq N+1$ or $p \geq k+1$ or $p \leq -1$ or $q \geq 1$. $K^{p,0} = \Omega^p_{U,k-p,0}$.

$$"E^{q,p}_1 =' H^p(K^{\cdot,q}, 'd),$$

$$\text{hence} \quad "E^{q,p}_1 = O_{U,0}(DB)^q, \quad p+q = 0,$$

$$"E^{q,p}_1 = 0, \quad p+q \neq 0,$$

$$H^0(K^{\cdot}) = \bigoplus_{q=0}^{k} O_{U,0}(DB)^q,$$

$$H^n(K^{\cdot}) = 0, \qquad n \neq 0,$$

$$'E^{p,q}_1 = "H^q(K^{p,\cdot}, "d),$$

$$\text{hence} \quad 'E^{p,q}_1 = \Omega^p_{f,k-p,0}, \quad q = 0, p \geq 0,$$

$$'E^{p,q}_1 = \Omega^N_{f,k-p,0}(DB)^{-q}, \quad p+q = N, q \leq 0,$$

$$'E^{p,q}_1 = 0, \quad \text{otherwise}.$$

$$'E^{p,q}_2 =' H^p("H^q(K^{\cdot,\cdot}, "d), 'd),$$

$$\text{hence} \quad 'E^{p,q}_2 = H^p(\Omega^{\cdot}_{f,k-\cdot,0}, \quad q = 0, p \geq 0,$$

$$'E^{p,q}_2 = \Omega^N_{f,k-p,0}(DB)^{-q}, \quad p+q = N, q \leq -1,$$

$$'E^{p,q}_2 = 0, \qquad \text{otherwise}.$$

So (1) and (4) hold.

Now we prove (2) and (3). Clearly

$$'E^{N-1,0}_2 \supset \cdots \supset' E^{N-1,0}_r \supset \cdots \supset' E^{N-1,0}_{k-N+2} = 0.$$

$$'E^{p,q}_r = \frac{'Z^{p,q}_r}{'Z^{p+1,q-1}_{r-1} +' B^{p,q}_{r-1}},$$

$'Z^{p,q}_r \supset' F_{p+r}K^{p+q}, 'Z^{p+1,q-1}_{r-1} \supset' F_{p+r}K^{p+q}$.

$$'E^{p,q}_r = \frac{\frac{'Z^{p,q}_r}{'F_{p+r}K^{p+q}}}{\frac{'Z^{p+1,q-1}_{r-1} +' B^{p,q}_{r-1}}{'F_{p+r}K^{p+q}}}.$$

Let

$$\begin{aligned} Z^{p,q}_r &=' Z^{p,q}_r \bmod 'F_{p+r}K^{p+q}, \\ &= \Big\{ \sum_{i+j=p+q, p\leq i\leq p+r-1} u^{i,j} | u^{i,j} \in K^{i,j}, \partial u^{p,q} = 0, \\ &\quad Du^{p,q} + \partial u^{p+1,q-1} = 0, \cdots, Du^{p+r-2,q-r+2} + \partial u^{p+r-1,q-r+1} = 0 \Big\}, \end{aligned} \tag{1}$$

$$\begin{aligned} Z_{r-1}^{p+1,q-1} &=' Z_{r-1}^{p+1,q-1} \mathrm{mod}\, 'F_{p+r}K^{p+q}, \\ &= \Big\{ \sum_{i+j=p+q, p+1\le i\le p+r-1} v^{i,j} | v^{i,j} \in K^{i,j}, \partial v^{p+1,q-1} = 0, \\ &\quad \cdots, Dv^{p+r-2,q-r+2} + \partial v^{p+r-1,q-r+1} = 0 \Big\}, \end{aligned} \tag{2}$$

$$\begin{aligned} B_{r-1}^{p,q} &=' B_{r-1}^{p,q} \mathrm{mod}' F_{p+r}K^{p+q}, \\ &= \{(Dw^{p-1,q} + \partial v^{p,q-1}) + \cdots + (Dw^{p+r-2,q-r+1} + \partial v^{p+r-1,q-r})| \\ &\quad w^{i,j} \in K^{i,j}, \exists w^{p-r+1,q+r-2} \in K^{p-r+1,q+r-2}, \cdots, w^{p-2,q+1} \in K^{p-2,q+1}, \\ &\quad \text{s.t.} \partial w^{p-r+1,q+r-2} = 0, Dv^{p-r+1,q+r-2} + \partial v^{p-r+2,q+r-3} = 0, \\ &\quad \cdots, Dv^{p-2,q+1} + \partial v^{p-1,q} = 0\} \end{aligned} \tag{3}$$

$$'E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}}.$$

$$\begin{aligned} Z_r^{N-1,0} = \Big\{ &\sum_{i=N-1}^{N+r-2} a_i(N-1)(DB)^{i-N+1} | a_i(N-1) \in \Omega_{U,k-i,0}^{N-1}, \\ &Da_{N-1}(N-1) + (-1)^N a_N(N-1) \wedge DF = 0, \cdots, \\ &(Da_i(N-1) + (-1)^N (i+2-N) a_{i+1}(N-1) \wedge DF)(DB)^{i+1-N} = 0, \cdots, \\ &(Da_{N+r-3}(N-1) + (-1)^N a_{N+r-2}(N-1) \wedge DF)(DB)^{r-2} = 0 \Big\}. \end{aligned}$$

Let $[a_i(N-1)]$ be the coset of $a_i(N-1)$ in $W_{f,k-i}^{N-1}$. We have

$$\delta_{k-N+1}[a_{N-1}(N-1)] = [a_N(N-1)], \cdots, \delta_{k-i}[a_i(N-1)] = (i+2-N)[a_{i+1}(N-1)],$$

$$\cdots \delta_{k-N-r+3}[a_{N+r-3}(N-1)] = (r-1)[a_{N+r-2}(N-1)].$$

$$[a_{N-1}(N-1)] \in \delta_{k-N+1}^{-1} \cdots \delta_{k-N-r+3}^{-1} W_{f,k-N-r+2}^{N-1} = F_{r-1} W_{k-N+1}^{N-1}.$$

Let $\pi_r^{N-1,0} : Z_r^{N-1,0} \longrightarrow F_{r-1} W_{k-N+1}^{N-1}$,

$$\pi_r^{N-1,0}\Big(\sum_{i=N-1}^{N+r-2} a_i(N-1)(DB)^{i-N+1} \Big) = [a_{N-1}(N-1)].$$

It is clear that $\pi_r^{N-1,0}$ is surjective.

$$\begin{aligned} Z_{r-1}^{N,-1} = \Big\{ &\sum_{i=N}^{N+r-2} b_i(N-1)(DB)^{i-N+1} | b_i(N-1) \in \Omega_{U,k-i,0}^{N-1}, \\ &(-1)^N b_N(N-1) \wedge DF = 0, \\ &(Db_N(N-1) + (-1)^N 2 b_{N+1}(N-1) \wedge DF)(DB) = 0, \\ &\cdots, (Db_{N+r-3}(N-1) + (-1)^N (r-1) b_{N+r-2}(N-1) \wedge DF)(DB)^{r-2} = 0\}. \\ &b_N(N-1) = (-1)^N 2 e_N(N-2) \wedge DF, \\ &(De_N(N-2) - b_{N+1}(N-1)) \wedge DF = 0, \\ &b_{N+1}(N-1) = De_N(N-2) + (-1)^{N-1} 3 e_{N+1}(N-2) \wedge DF, \\ &\cdots\cdots \\ &b_{N+r-2}(N-1) = De_{N+r-1}(N-2) + (-1)^{N-1} r e_{N+r-2}(N-2) \wedge DF. \end{aligned}$$

$$\begin{aligned}B_{r-1}^{N-1,0} = \{&(Dc_{N-2}(N-2) + (-1)^{N-1}c_{N-1}(N-2)\wedge DF) + \cdots \\ &+ (Dc_i(N-2) + (-1)^{N-1}c_{i+3-N}(N-2)\wedge DF)(DB)^{i+2-N} + \cdots \\ &+ (Dc_{N-3+r}(N-2) + (-1)^{N-1}c_{N-2+r}(N-2)\wedge DF)(DB)^{r-1}\}.\end{aligned}$$

Hence $Z_{r-1}^{N,-1} \subset B_{r-1}^{N-1,0}$. Clearly $\pi_r^{N-1,0}(B_{r-1}^{N-1,0}) = 0$.

Suppose $u = \sum\limits_{i=N-1}^{N+r-2} a_i(N-1)(DB)^{i-N+1} \in Z_r^{N-1,0}, \pi_r^{N-1,0}(u) = 0$. Because δ's are isomorphisms, $[a_{N+r-2}(N-1)] = 0$ in $W_{f,k-N-r+2}^{N-1}$.

$$\begin{aligned}&a_{N+r-2}(N-1) = Dc_{N-3+r}(N-2) + (-1)^{N-1}rc_{N-2+r}(N-2)\wedge DF, \\ &D(a_{N+r-3}(N-1) + (-1)^N(r-1)c_{N-3+r}(N-2)\wedge DF) = 0, \\ &a_{N-3+r}(N-1) = Dc_{N+r-4}(N-2) + (-1)^{N-1}(r-1)c_{N-3+r}(N-2)\wedge DF, \\ &\quad\cdots\cdots \\ &a_{N-1}(N-1) = Dc_{N-2}(N-2) + (-1)^{N-1}c_{N-1}(N-2)\wedge DF.\end{aligned}$$

So $u \in B_r^{N-1,0}$. $\pi_r^{N-1,0}$ induces isomorphism $'E_r^{N-1,0} \cong F_{r-1}W_{f,k-N+2-r}^{N-1}$.

Now we prove (3).

$$\begin{aligned}Z_r^{p,N-p} &= \{u^{p,N-p} + u^{p+1,N-p-1} + \cdots + u^{p+r-1,N-p-r+1}\} \\ &= \{a_p(N)(DB)^{p-N} + \cdots + a_{p+r-1}(N)(DB)^{p+r-N-1}\}, \\ Z_{r-1}^{p+1,N-p-1} &= \{v^{p+1,N-p-1} + \cdots + v^{p+r-1,N-p-r+1}\} \\ &= \{b_{p+1}(N)(DB)^{p+1-N} + \cdots + b_{p+r-1}(N)(DB)^{p+r-N-1}\}, \\ B_{r-1}^{p,N-p} &= \{(Dc_{p-1}(N-1) + (-1)^{N-1}(p+1-N)c_p(N-1)\wedge DF)(DB)^{p-N} \\ &\quad + (Dc_p(N-1) + (-1)^{N-1}(p+2-N)c_{p+1}(N-1)\wedge DF)(DB)^{p+1-N} \\ &\quad + \cdots + (Dc_{p+r-2}(N-1) \\ &\quad + (-1)^{N-1}(p+r-N)c_{p+r-1}(N-1)\wedge DF)(DB)^{p+r-N-1}| \\ &\quad \cdot(-1)^{N-1}(p+2-N-r)c_{p-r+1}(N-1)\wedge DF(DB)^{p+1-N-r} = 0, \\ &\quad (Dc_{p-r+1}(N-1) + (-1)^{N-1}(p+3-N-r)c_{p-r+2}(N-1)\wedge DF) \\ &\quad \cdot(DB)^{p+2-N-r} = 0, \\ &\quad \cdots\cdots \\ &\quad (Dc_{p-2}(N-1) + (-1)^{N-1}(p-N)c_{p-1}(N-1)\wedge DF)(DB)^{p-1-N} = 0\}.\end{aligned}$$

If $r \geq p+2-N$, $(DB)^{p+1-N-r} = 0$. For any $c_{p-1}(N-1)$ there are $c_{p-2}(N-1), \cdots, c_{N-1}(N-1)$ such that $Dc_{N-1}(N-1) + (-1)^{N-1}c_N(N_1)\wedge DF = 0, \cdots, Dc_{p-2}(N-1) + (-1)^{N-1}(p-N)c_{p-1}(N_1)\wedge DF = 0$. So $'E_r^{p,N-p} = 0, r \geq p+2-N$.

Suppose $r \leq p+1-N$,

$$\begin{aligned}&c_{p-r+1}(N-1) = (-1)^{N-2}(p+3-N-r)e_{p-r+1}(N-2)\wedge DF, \\ &c_{p-r+2}(N-1) = De_{p-r+1}(N-2) + (-1)^{N-2}(p+4-N-r)e_{p-r+2}(N-2)\wedge DF, \\ &\qquad\cdots\cdots \\ &c_{p-1}(N-1) = De_{p-2}(N-2) + (-1)^{N-2}(p+1-N)e_{p-1}(N-2)\wedge DF.\end{aligned}$$

Hence

$$Dc_{p-1}(N-1)+(-1)^{N-1}(p+1-N)c_p(N-1)\wedge DF$$
$$=(-1)^{N-2}(p+1-N)(De_{p-1}(N-2)-c_p(N-1))\wedge DF.$$

Because $c_p(N-1)$ is arbitrary, ${}'E_r^{p,N-p}\cong\Omega^N_{f,k-p,0}$, $r\le p+1-N$.

Suppose that $U'\subset R^N$ is an open set with coordinates $(x'_1,\cdots,x'_N)$, $0\in U'$, $\phi:(U',0)\to(U,0)$ is a local diffeomorphism. $f'(x')=f(\phi(x'))=\phi^*(f)$. For f' there are

$$\widetilde{\Omega'}^{p,q}_{k-p}=\Omega^{p-q}_{U',k-p,0}(DB')^q \quad\text{and}\quad \widetilde{\Omega}'_k(DB')=\sum_{p,q}\widetilde{\Omega'}^{p,q}_{k-p}.$$

Let $\phi^*:\widetilde{\Omega}^{p,q}_{k-p}\to\widetilde{\Omega'}^{p,q}_{k-p}$. $\phi^*(a(DB)^q)=\phi(a)(DB')^q$. It is clear that $\phi^*:\widetilde{\Omega}^{p,q}_{k-p}\to\widetilde{\Omega'}^{p,q}_{k-p}$ is an isomorphism. Hence ${}'E_r^{p,q}$ are invariant under right equivalent. The proof of Theorem 1 has been completed.

Now we prove Theorem 2.

Let B and DB be indeterminate elements. Define their degrees: $\deg B=1$, $\deg DB=2$. Define $\deg a=0,\forall a\in O_{U,l,0}$ and $\deg dy_i=1, i=1,\cdots,N$. Hence $BB=0$.

Convention: $(B)^0=1=(DB)^0$ and $B^n=0=(DB)^n, n<0$.

Let

$$\Lambda^{p,q}_{k-p}=\Omega^{p-q+1}_{U,k-p,0}B(DB)^{q-1}+\Omega^{p-q}_{U,k-p,0}B(DB)^q.$$

$\Lambda^{p,q}_{k-p}=0$, if $p\le-1$ or $q\le-1$ or $p-q\le-2$ or $p-q\ge N+1$ or $p\ge k+1$. Let $\Lambda_k(B,DB)=\sum\limits_{p,q}\Lambda^{p,q}_{k-p}$. It is a commutative algebra over $O_{U,k,0}$. If a,b are homogeneous elements of $\Lambda_k(B,DB)$, $ab=(-1)^{\deg a\deg b}ba$.

The partial exterior diferential $D:\Omega_k\to\Omega_k$ induces the differential $D:\Lambda_k(B,DB)\to\Lambda_k(B,DB)$ as follows. $D(B)=DB, D(DB)=0$. If $aB(DB)^{q-1}+b(DB)^q\in\Lambda^{P,q}_{k-p}$,

$$D(aB(DB)^{q-1}+b(DB)^q)=DaB(DB)^{q-1}+((-1)^{p-q+1}\Pi_{k-p}a+Db)(DB)^q.$$

$\deg D=1, DD=0$. For fixed q, $D:\Lambda^{p,q}_{k-p}\to\Lambda^{p+1,q}_{k-p-1}$. $\{\Lambda^{p,q}_{k-p},D\}_{p\ge0}$ is a complex.

Proposition 1. *For the differential algebra* $\Lambda_k(B,DB)$

$$H(\Lambda_k(B,DB))=O_{U,0}\oplus O_{U,0}B(DB)^k,$$
$$H^0(\Lambda^{\cdot,0}_{k-\cdot},D)=O_{U,0},\quad H^k(\Lambda^{\cdot,k+1}_{k-\cdot},D)=O_{U,0}B(DB)^k$$
$$H^p(\Lambda^{\cdot,q}_{k-\cdot},D)=0,\quad \textit{otherwise}\ .$$

Proof. Define a filtration

$$F_n\Lambda_k(B,DB)=\sum_{\substack{s,t\\ m\ge n}}\Omega^s_{U,k-s-m,0}B^t(DB)^m,$$

$$E_0^n=\frac{F_n}{F_{n+1}}=\sum_{s,t}\Omega^s_{U,k-s-n,0}B^t(DB)^n,\quad E_1^n=\sum_{\substack{t=0,1\\0\le n\le k}}O_{U,0}B^t(DB)^n,$$

$$E_2^0=O_{U,0},\quad E_2^n=0,\quad 1\le n\le k-1,\quad E_2^k=O_{U,0}B(DB)^k.$$

$E_\infty^n=E_2^n$. The proof of the proposition has been completed.

Define the second differential $\partial:\Lambda_k(B,DB)\to\Lambda_k(B,DB)$. $\deg\partial=-1$, $\partial\Omega^p_{U,l,0}=0, p=0,\cdots,N$. $\partial B=F$, $\partial(DB)=-DF$. If $a,b\in\Lambda_k(B,DB)$ are homogeneous elements,

$\partial(ab)=(\partial a)b+(-1)^{dega}a\partial b$. It is clear that $\partial\partial=0$ and $\partial D+D\partial=0$. $\partial:\Lambda^{p,q}_{k-p}\to\Lambda^{p,q-1}_{k-p}$. If $aB(DB)^{q-1}+b(DB)^q\in\Lambda^{p,q}_{k-p}$,

$$\begin{aligned}&\partial(aB(DB)^{q-1}+b(DB)^q)\\&=(-1)^{p-q+2}(q-1)a\wedge DFB(DB)^{q-2}+(-1)^{p-q+1}(aF+qb\wedge DF)(DB)^{q-1}.\end{aligned}$$

Let $K^{p,q}=\Lambda^{p,-q}_{k-p}$. $'d=D,''d=\partial$. It is a double complex. $K^{p,q}=0$, if $p\le -1$ or $q\ge 1$ or $p+q\le -2$ or $p+q\ge N+1$ or $p\ge k+1$. $K^{p,0}=\Omega^p_{U,k-p,0}$. $K^n=\sum\limits_{p+q=n}K^{p,q}$.

$$\begin{aligned}&''E_1^{q,p}={}'H^p(K^{\cdot,q},{}'d),\quad ''E_1^{0,0}=O_{U,0},\quad ''E_1^{-k-1,k}=O_{U,0}B(DB)^k,\\&''E_1^{q,p}=0,\ \text{otherwise}.\\&H^{-1}(K^{\cdot})=O_{U,0}B(DB)^k,\quad H^0(K^{\cdot})=O_{U,0},\\&H^n(K^{\cdot})=0,\ \text{otherwise}.\end{aligned}$$

Proposition 2.

$$\begin{aligned}&'E_1^{p,0}=\Lambda^p_{f,k-p,0},\\&'E_1^{p,q}=\frac{\{a(N)\in\Omega^N_{U,k-p,0}|a(N)F=qb(N-1)\wedge DF\}(DB)^{-q}}{(-1)^N(-q)\Omega^{N-1}_{U,k-p,0}\wedge DF},\\&p+q=N-1,\quad q\le -1,\\&'E_1^{p,N-p}=\Lambda^N_{f,k-p,0}(DB)^{p-N},\\&'E_1^{p,q}=0,\quad \textit{otherwise},\\&\dim'_R E_1^{p,q}\le\infty,\quad \textit{except }{}'E_1^{p,0},\ 0\le p\le N-1.\end{aligned}$$

Proof. If $p+q=-1$, $u\in K^{p,q}$, $u=a(0)B(DB)^{-q-1}$,

$$\partial u=(q+1)a(0)DFB(DB)^{-q-2}+a(0)F(DB)^{-q-1}=0,$$

then $a(0)=0$. $'E_1^{p,q}=0,p+q=-1$.

If $0\le p+q\le N-2,q\le -1$,

$$\begin{aligned}u^{p,q}&=a(p+q+1)B(DB)^{-q}+b(p+q)(DB)^{-q}\in K^{p,q},\\\partial u^{p,q}&=(-1)^{p+q+2}(-q-1)a(p+q+1)\wedge DFB(DB)^{-q-2}\\&\quad+(-1)^{p+q+1}(a(p+q+1)F-qb(p+q)\wedge DF)(DB)^{-q-1}=0.\\a(p+q+1)&=(-1)^{p+q+1}(-q)a(p+q)\wedge DF,\\b(p+q)&=a(p+q)F-(q-1)b(p+q-1)\wedge DF.\end{aligned}$$

Hence $u^{p,q}=\partial(a(p+q)B(DB)^{-q}+b(p+q-1)(DB)^{-q+1})$ and $'E_1^{p,q}=0,0\le p+q\le N-2,q\le -1$.

Suppose $p+q=N$, $K^{p,q}=\Omega^N_{U,k-p,0}(DB)^{-q}$,

$$\begin{aligned}K^{p,q-1}&=\Omega^N_{U,k-p,0}B(DB)^{-q}+\Omega^{N-1}_{U,k-p,0}(DB)^{-q+1},\\\partial K^{p,q-1}&=(\Omega^N_{U,k-p,0}F+(-1)^{N-1}(-q+1)\Omega^{N-1}_{U,k-p,0}\wedge DF)(DB)^{-q}.\end{aligned}$$

Hence $'E_1^{p,q}=\Lambda^N_{f,k-p.0},\quad p+q=N$.

Suppose $p+q=N-1,q\le -1$, $u=a(N)B(DB)^{-q-1}+b(N-1)(DB)^{-q}\in K^{p,q}$.

$$\partial u=0\Leftrightarrow a(N)F=qb(N-1)\wedge DF,\quad 'E_1^{p,q}=\frac{\ker(K^{p,q}\xrightarrow{\partial}K^{p,q+1})}{\partial K^{p,q-1}}.$$

Let $\phi : \ker(K^{p,q} \xrightarrow{\partial} K^{p,q+1}) \longrightarrow \Omega^N_{U,k-p,0}, \quad \phi(u) = a(N),$

$$\mathrm{Im}\phi = \{a(N) \in \Omega^N_{U,k-p,0} | a(N)F = qb(N-1) \wedge DF\}.$$

$v = a(N-1)B(DB)^{-q} + b(N-2)(DB)^{-q+1} \in K^{p,q-1},$

$$\begin{aligned} \partial v &= (-1)^{N+1} qa(N-1) \wedge DFB(DB)^{-q-1} \\ &\quad + (-1)^{N-1}(a(N-1)F - (q-1)b(N-2) \wedge DF)(DB)^{-q}, \end{aligned}$$

$$\phi(\partial K^{p,q-1}) = (-1)^{N+1} q \Omega^{N-1}_{U,k-p,0} \wedge DF.$$

Suppose

$$u = a(N)B(DB)^{-q-1} + b(N-1)(DB)^{-q} \in \ker(K^{p,q} \xrightarrow{\partial} K^{p,q+1}),$$

$$\phi(u) = a(N) = (-1)^{N+1} qa(N-1) \wedge DF \in \phi(\partial K^{p,q-1}).$$

$$b(N-1) = (-1)^{N-1}(a(N-1)F - (q-1)b(N-2) \wedge DF).$$

Hence $u \in \partial K^{p,q-1}$.

$$'E_1^{p,q} = \frac{\{a(N) \in \Omega^N_{U,k-p,0} | a(N)F = qb(N-1) \wedge DF\}(DB)^{-q}}{(-1)^N(-q)\Omega^{N-1}_{U,k-p,0} \wedge DF}.$$

Now we continue to prove Theorem 2. We only need to compute $'E_r^{p,0}, p = N-2, N-1$. Similar to Theorem 1

$$'E_r^{p,q} = \frac{Z_r^{p,q}}{B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1}}, \quad Z_r^{p,q} =' Z_r^{p,q} \mathrm{mod}' F_{p+r} K^{p+q},$$

$$Z_{r-1}^{p+1,q-1} =' Z_{r-1}^{p+1,q-1} \mathrm{mod}' F_{p+r} K^{p+q}, \quad B_{r-1}^{p,q} =' B_{r-1}^{p,q} \mathrm{mod}' F_{p+r} K^{p+q}.$$

If $u^{i,j} = a_i(i+j+1)B(DB)^{-j-1} + b_i(i+j)(DB)^{-j} \in K^{i,j}$,

$$\begin{aligned} u^{i+1,j-1} &= a_{i+1}(i+j+1)B(DB)^{-j} + b_{i+1}(i+j)(DB)^{-j+1} \in K^{i+1,j-1}, \\ Du^{i,j} + \partial u^{i+1,j-1} &= (Da_i(i+j+1) + (-1)^{i+j+1} j a_{i+1}(i+j+1) \wedge DF)B(DB)^{-j-1} \\ &\quad + (Db_i(i+j) + (-1)^{i+j+1}(\Pi_{k-i} a_i(i+j+1) + a_{i+1}(i+j+1)F \\ &\quad - (j-1)b_{i+1}(i+j) \wedge DF))(DB)^{-j}. \end{aligned}$$

First we compute $'E_3^{N-2,0}$. If $u = u^{N-2,0} + u^{N-1,-1} + u^{N,-2} \in Z_3^{N-2,0}$,

$$Du^{N-2,0} + \partial u^{N-1,-1} = Db_{N-2}(N-2) + (-1)^{N-1}(a_{N-1}(N-1)F + b_{N-1}(N-2) \wedge DF) = 0. \tag{4}$$

$$Du^{N-1,-1} + \partial u^{N,-2} = 0 \Longleftrightarrow Da_{N-1}(N-1) + (-1)^{N-1} a_N(N-1) \wedge DF = 0. \tag{5}$$

$$Db_{N-1}(N-2) + (-1)^{N-1}(\Pi_{k-N+1} a_{N-1}(N-1) + a_N(N-1)F + 2b_N(N-2) \wedge DF) = 0. \tag{6}$$

By (1)

$$\begin{aligned} 0 &= D(a_{N-1}(N-1)F + b_{N-1}(N-2) \wedge DF) \\ &= FDa_{N-1}(N-1) + (-1)^{N-1}\Pi_{k-N+1} a_{N-1}(N-1) \wedge DF \\ &\quad + Db_{N-1}(N-2) \wedge DF \\ &= ((-1)^N Fa_N(N-1) + (-1)^{N-1}\Pi_{K-N+1} a_{N-1}(N-1) + Db_{N-1}(N-2)) \wedge DF. \end{aligned}$$

$$Db_{N-1}(N-2) + (-1)^{N-1}(\Pi_{k-N+1} a_{N-1}(N-1) - a_N(N-1)F + 2b'_N(N-2) \wedge DF) = 0. \tag{7}$$

$$(3) - (4) \Rightarrow Fa_N(N-1) + (b_N(N-2) - b'_N(N-2)) \wedge DF = 0,$$

$$Fa_N(N-1)\wedge DF=0,\quad a_N(N-1)=(-1)^{N-1}2c_N(N-2)\wedge DF. \tag{8}$$

Substituing (8) into (5), we have $Da_{N-1}(N-1)=0$,

$$a_{N-1}(N-1)=Dc_{N-2}(N-2). \tag{9}$$

Substituting into (4), we have

$$\begin{aligned}&Db_{N-2}(N-2)+(-1)^{N-1}(FDc_{N-2}(N-2)+b_{N-1}(N-2)\wedge DF)=0,\\&D(b_{N-2}(N-2)+(-1)^{N-1}Fc_{N-2}(N-2))+\\&(-1)^{N-1}((-1)^{N-1}\Pi_{k-N+2}c_{N-2}(N-2)+b_{N-1}(N-2))\wedge DF=0.\end{aligned} \tag{10}$$

By $H^{N-2}(\Omega^{\cdot}_{f,k-\cdot,0})=0$,

$$b_{N-2}(N-2)=(-1)^{N-2}Fc_{N-2}(N-2)+De_{N-3}(N-3)+(-1)^{N-2}e_{N-2}(N-3)\wedge DF.$$

Substituing into (10), we have

$$\begin{aligned}&(-1)^{N-2}De_{N-2}(N-3)\wedge DF\\&+(-1)^{N-1}((-1)^{N-1}\Pi_{k-N+2}c_{N-2}(N-2)+b_{N-1}(N-2))\wedge DF=0,\end{aligned}$$

$$b_{N-1}(N-2)=De_{N-2}(N-3)+(-1)^{N-2}(\Pi_{k-N+2}c_{N-2}(N-2)+2e_{N-1}(N-3)\wedge DF).$$

Substituting into (6), we have

$$\begin{aligned}&2(-1)^{N-2}De_{N-1}(N-3)\wedge DF\\&+(-1)^{N-1}((-1)^{N-1}2Fc_N(N-2)\wedge DF+2b_N(N-2)\wedge DF)=0,\end{aligned}$$

$$b_N(N-2)=De_{N-1}(N-3)+(-1)^{N-2}(Fc_N(N-2)+3e_N(N-3)\wedge DF).$$

Let $v^{N-3,0}=e_{N-3}(N-3)$, $\quad v^{N-2,-1}=c_{N-2}(N-2)B+e_{N-2}(N-3)DB$, $v^{N-1,-2}=e_{N-1}(N-3)$, $\quad v^{N,-3}=c_N(N-2)B(DB)^2+e_N(N-3)DB$.

$$u=(Dv^{N-3,0}+\partial v^{N-2,-1})+(Dv^{N-2,-1}+\partial v^{N-1,-2})+(Dv^{N-1,-2}+\partial v^{N,-3}).$$

Hence $u\in B_2^{N_2,0}$, ${}'E_3^{N-2,0}=0$ and ${}'E_r^{N-2,0}=0$, $\quad r\geq 3$.

Now we compute ${}'E_r^{N-1,0}=\frac{Z_r^{N-1,0}}{B_{r-1}^{N-1,0}+E_{r-1}^{N,-1}}$, $\quad r\geq 2$. If $u=u^{N-1,0}+u^{N,-1}+\cdots+u^{N-2+r,1-r}\in Z_r^{N-1,0}$,

$$\begin{aligned}0&=Du^{N-1,0}+\partial u^{N,-1}=Db_{N-1}(N-1)+(-1)^N(a_N(N)F+b_N(N-1)\wedge DF),\\0&=Du^{i,N-1-i}+\partial u^{i+1,N-2-i}\\&=(Db_i(N-1)+(-1)^N(\Pi_{k-i}a_i(N)+a_{i+1}(N)F\\&\quad+(i-N+2)b_{i+1}(N-1)\wedge DF))(DB)^{i-N+1},\quad N\leq i\leq N-3+r.\end{aligned}$$

Let $a^r_{N-2+r}(N-1)=0$, $Da^r_{N-3+r}(N-1)=a_{N-2+r}(N)$,

$$Da^r_{i-1}(N-1)+(-1)^N(i-N+1)a^r_i(N-1)\wedge DF=a_i(N),\quad N-3+r\geq i\geq N.$$

Substituing into above equations, we have

$$\begin{aligned}&D(b_{N-1}(N-1)+(-1)^NFa^r_{N-1}(N-1))\\&+(-1)^{N-1}((-1)^{N-1}\Pi_{k-N+1}a^r_{N-1}(N-1)+(-1)^Na^r_N(N-1)F+b_N(N-1))\wedge DF=0,\\&D(b_i(N-1)+(-1)^N\Pi_{k-i+1}a^r_{i-1}(N-1)+(-1)^NFa^r_i(N-1))\\&+(-1)^N(i-N+2)((-1)^N\Pi_{k-i}a^r_i(N-1)+(-1)^Na^r_{i+1}(N-1)F\\&+b_{i+1}(N-1))\wedge DF=0,\quad N\leq i\leq N-3+r.\end{aligned} \tag{11}$$

Define $\pi_r^{N-1,0}: Z_r^{N-1,0} \longrightarrow W_{f,k-N+2-r}^{N-1}$,

$$\pi_r^{N-1,0}(u^{N-1,0}+\cdots+u^{N-2+r,1-r})[(-1)^N\Pi_{k-N+3-r}a_{N-3+r}^r(N-1)+b_{N-2+r}(N-1)].$$

$\pi_r^{N-1,0}$ is independt of the choice of $a_{N-3+r}^r(N-1),\cdots,a_N^r(N-1)$, for $a_{N-2+r}^r(N-1)=0$ and $Da_{N-3+r}^r(N-1)=a_{N-2+r}(N)$. Clearly $\pi_r^{N-1,0}$ is surjective.

Suppose

$$\begin{aligned}&\pi_r^{N-1,0}(u^{N-1,0}+\cdots+u^{N-2+r,1-r})\\&=[(-1)^N\Pi_{k-N+3-r}a_{N-3+r}^r(N-1)+b_{N-2+r}(N-1)]=0.\end{aligned}$$

Then

$$\begin{aligned}b_{N-2+r}(N-1)&=Db_{N-3+r}^r(N-2)\\&\quad+(-1)^{N-1}(\Pi_{k-N+3-r}a_{N-3+r}^r(N-1)+rb_{N-2+r}^r(N-2)\wedge DF),\\b_i(N-1)&=Db_{i-2}^r(N-2)(-1)^{N-1}(\Pi_{k-i+1}a_{i-1}^r(N-1)\\&\quad+a_i^r(N-1)F+(i-N+2)b_i^r(N-2)\wedge DF),\quad N-3+r\geq i\geq N,\\b_{N-1}(N-1)&=Db_{N-2}^r(N-2)+(-1)^{N-1}(a_{N-1}^r(N-1)F+b_{N-1}^r(N-2)\wedge DF).\end{aligned}$$

Let $v^{N-2,0}=b_{N-2}^r(N-2)$, $v^{i,N-2-i}=a_i^r(N-1)B(DB)^{i-N+1}+b_i^r(N-2)(DB)^{i-N+2}$, $N-1\leq i\leq N-2+r$.

$$\begin{aligned}u&=u^{N-1,0}+\cdots+u^{N-2+r,1-r}\\&=(Dv^{N-2,0}+\partial v^{N-1,-1})+\cdots+(Dv^{i-1,N-i}+\partial v^{i,N-1-i})\\&\quad+\cdots(Dv^{N-3+r,1-r}+\partial v^{N-2+r,-r}).\end{aligned}$$

So $u\in B_{r-1}^{N-1,0}$. Let

$$\begin{aligned}C_{r-1}^{N-1,0}=\{&(Dv^{N-2,0}+\partial v^{N-1,-1})+\cdots+(Dv^{N-3+r,1-r}+\partial v^{N-2+r,-r})|\\&v^{i,N-2-i}\in K^{i,N-2-i},\quad i=N-2,\cdots,N-2+r,\\&v^{N-2+r,-r}=b'_{N-2+r}(N-2)(DB)^R\}\subset B_{r-1}^{N-1,0}.\end{aligned}$$

Then $\ker\pi_r^{N-1,0}\subset C_{r-1}^{N-1,0}$. Clearly $\pi_r^{N-1,0}(C_{r-1}^{N-1,0})=0$. Hence

$$\pi_r^{N-1,0}:\ \frac{Z_r^{N-1,0}}{C_r^{N-1,0}}\cong W_{f,k-N+1}^{N-1}.$$

Suppose $v\in B_{r-1}^{N-1,0}$,

$$\begin{aligned}&v=(Dv^{N-2,0}+\partial v^{N-1,-1})+(Dv^{N-1,-1}+\partial v^{N,-2})\\&\quad+\cdots+(Dv^{N-3+r,1-r}+\partial v^{N-2+r,-r}),\\&\pi_r^{N-1,0}((Dv^{N-2,0}+\partial v^{N-1,-1})+Dv^{N-1,-1})=0,\\&\partial v^{N,-2}+\cdots+(Dv^{N-3+r,1-r}+\partial v^{N-2+r,-r})\in Z_{r-1}^{N,-1}.\end{aligned}$$

We need only to compute $\pi_r^{N-1,0}(Z_{r-1}^{N-1,0})$.

$$Z_{r-1}^{N,-1}=\{u^{N-1,0}+\cdots+u^{N-2+r,1-r}\in Z_r^{N-1,0}|u^{N-1,0}=b_{N-1}(N-1)=0\}.$$

Hence the equations (11) for $u \in Z_{r-1}^{N,-1}$ become

$$\begin{aligned}
&\delta_{k-N+1}((-1)^N[Fa_{N-1}^r(N-1)])\\
&=[(-1)^N\Pi_{k-N+1}a_{N-1}^r(N-1)+(-1)^NFa_N^r(N-1)+b_N(N-1)],\\
&\delta_{k-i}([b_i(N-1)+(-1)^N\Pi_{k-i+1}a_{i-1}^r(N-1)+(-1)^NFa_i^r(N-1)])\\
&=(i-N+2)[b_{i+1}(N-1)+(-1)^N\Pi_{k-i}a_i^r(N-1)+(-1)^NFa_{i+}^r(N-1)],\\
&i=N,\cdots,N-3+r.
\end{aligned}$$

But $\delta_{k-N+3-r}\cdots\delta_{k-N+1}: F_{r-1}W_{f,k-N+1}^{N-1}\cong W_{f,k-N+2-r}^{N-1}$. Therefore

$$\begin{aligned}
{}'E_r^{N-1,0}\overset{\pi_r^{N-1,0}}{\cong}&\frac{W_{f,k-N+2-r}^{N-1}}{\delta_{k-N+3-r}\cdots\delta_{k-N+1}(FW_{f,k-N+1}^{N-1}\cap F_{r-1}W_{f,k-N+1}^{N-1})}\\
\cong&\frac{F_{r-1}W_{f,k-N+1}^{N-1}}{FW_{f,k-N+1}^{N-1}\cap F_{r-1}W_{f,k-N+1}^{N-1}}.
\end{aligned}$$

Because $K^{p,q}=0, q>0$, we have ${}'E_{r+1}^{p,0}\subset{}' E_r^{p,0}$.

Now we prove that ${}'E_r^{p,q}$ are invariant under contact transformations. Suppose that $U'\subset R^N$ is an open set with coordinates $(x_1',\cdots,x_N'), 0\in U'$. $f':(U',0)\longrightarrow(R,0)$ is a C^∞ mapping. Suppose that there is a contact transformation H such that $Hf=f'$. It is equivalant to a local diffeomorphism $\phi:(U,0)\to(U',0)$ and $\theta(x)\in O_{U,0}$, $\theta(0)\neq 0$, such that $f'(\phi(x))=\theta(x)f(x)$ (see [4]). ϕ induces $\phi^*:\Omega_{U',k-p,0}^p\cong\Omega_{U,k-p,0}^p,\quad p\geq 0$.

Suppose that

$$\Lambda_{k-p}^{'p,q}=\Omega_{U',k-p,0}^{p-q+1}B'(DB')^{q-1}+\Omega_{U',k-p,0}^{p-q}(DB')^q,\quad \Lambda_k(B',DB')=\sum_{p,q}\Lambda_{k-p}^{'p,q}$$

are the modules for f'. Define $\phi^*(B')=\theta(y)B$,

$$\phi^*(DB')=D(\theta(y)B)=d\theta(y)B+\theta(y)DB.$$

Clearly $\partial\phi^*(B')=\phi^*\partial B'$, and $\partial\phi^*(DB')=\phi^*\partial DB'$. It induces $\phi^*:\Lambda_{k-p}^{'p,q}\to\Lambda_{k-p}^{p,q}$ and $\phi^*:\Lambda_k'(B',DB')\to\Lambda_k(B,DB)$. Because θ is invertible in $O_{U,0}$. $B=\frac{1}{\theta(y)}\phi^*(B')$,

$$DB=d\Big(\frac{1}{\theta(y)}\Big)\phi^*(B')+\frac{1}{\theta(y)}\phi^*(DB').$$

$\phi^*:\Lambda_{k-p}^{'p,q}\cong\Lambda_{k-p}^{p,q}$ and $\phi^*:\Lambda_k'(B',DB')\cong\Lambda_k(B,DB)$. Hence ${}'E_r^{p,q}$ are invariant under contact transformations.

The proof of Thoerem 2 has been completed.

Corollary. *Under the hypothesis of Theorem* 2,

$$\begin{aligned}
&\frac{{}'E_r^{N-1,0}}{{}'E_{r+1}^{N-1,0}}\\
&\cong\frac{\Omega_{f,k-N+1-r,0}^N}{D\delta_{k-N+3-r}\cdots\delta_{k-N+1}(FW_{f,k-N+1}^{N-1}\cap F_{r-1}W_{f,k-N+1}^{N-1})+F\Omega_{f,k-N+1-r,0}^N}.
\end{aligned}$$

Proof. Suppose

$$u=u^{N-1,0}+\cdots+u^{N-2+r,1-r}+u^{N-1+r,-r}\in Z_{r+1}^{N-1,0}\subset Z_r^{N-1,0}.$$

By the definition

$$\begin{aligned}
\pi_{r+1}^{N-1,0}(u) &= [(-1)^N \Pi_{k-N+2-r} a_{N-2+r}^{r+1}(N-1) + b_{N-1+r}(N-1)],\\
\pi_{r}^{N-1,0}(u) &= [(-1)^N \Pi_{k-N+3-r} a_{N-3+r}^{r}(N-1) + b_{N-2+r}(N-1)],\\
a_{N-1+r}^{r+1}(N-1) &= 0, Da_{N-2+r}^{r+1}(N-1) = a_{N-1+r}(N), a_{N-2+r}^{r}(N-1) = 0.\\
&Da_{N-3+r}^{r+1}(N-1) + (-1)^N (r-1) a_{N-2+r}^{r+1}(N-1) \wedge DF\\
&= a_{N-2+r}(N) = Da_{N-3+r}^{r}(N-1).
\end{aligned}$$

Let

$$\begin{aligned}
Dc_{N-3+r}^{r+1}(N-1) &= (-1)^{N-1} a_{N-2+r}^{r+1}(N-1) \wedge DF,\\
\delta_{k-N+3-r}[c_{N-3+r}^{r+1}(N-1)] &= [a_{N-2+r}^{r+1}(N-1)],\\
Da_{N-3+r}^{r}(N-1) &= D(a_{N-3+r}^{r+1}(N-1) - (r-1)c_{N-3+r}^{r+1}(N-1)).
\end{aligned}$$

We can choose $a_{N-3+r}^{r}(N-1) = a_{N-3+r}^{r+1}(N-1) - (r-1)c_{N-3+r}^{r+1}(N-1)$,

$$\begin{aligned}
\pi_{r}^{N-1,0}(u) &= [(-1)^N \Pi_{k-N+3-r} a_{N-3+r}^{r+1}(N-1) + b_{N-2+r}(N-1)]\\
&\quad (-1)^{N+1}(r-1)[\Pi_{k-N+3-r} c_{N-3+r}^{r+1}(N-1)]\\
&= [(-1)^N \Pi_{k-N+3-r} a_{N-3+r}^{r+1}(N-1) + b_{N-2+r}(N-1)]\\
&\quad (-1)^{N+1}(r-1)\Pi_{k-N+3-r}\delta_{k-N+3-r}^{-1}[a_{N-2+r}^{r+1}(N-1)].
\end{aligned}$$

But

$$\begin{aligned}
&\delta_{k-N+2-r}[b_{N-2+r}(N-1) + (-1)^N \Pi_{k-N+3-r} a_{N-3+r}^{r+1}(N-1)(-1)^N a_{N-2+r}^{r+1}(N-1)F]\\
&= r[(-1)^N \Pi_{k-N+2-r} a_{N-2+r}^{r+1}(N-1) + b_{N-1+r}(N-1)],\\
&\delta_{k-N+2-r}^{-1}\pi_{r+1}^{N-1,0}(u)\\
&= \delta_{k-N+2-r}^{-1}[(-1)^N \Pi_{k-N+2-r} a_{N-2+r}^{r+1}(N-1) + b_{N-1+r}(N-1)]\\
&= \frac{1}{r}[b_{N-2+r}(N-1) + (-1)^N \Pi_{k-N+3-r} a_{N-3+r}^{r+1}(N-1)](-1)^N \frac{1}{r}[a_{N-2+r}^{r+1}(N-1)F]\\
&= \frac{1}{r}\pi_{r}^{N-1,0}(u)\\
&\quad + (-1)^N(\frac{r-1}{r}\Pi_{k-N+3-r}\delta_{k-N+3-r}^{-1} + \frac{1}{r}F)[a_{N-2+r}^{r+1}(N-1)F]\pi_{r}^{N-1,0}(u)\\
&= r\delta_{k-N+2-r}^{-1}\pi_{r+1}^{N-1,0}(u)(-1)^{N-1}((r-1)\delta_{k-N+2-r}^{-1}\Pi_{k-N+2-r} + F)[a_{N-2+r}^{r+1}(N-1)].
\end{aligned}$$

Hence $\pi_{r}^{N-1,0}(Z_{r+1}^{N-1,0}) \subset F_1 W_{f,k-N+2-r}^{N-1} + FW_{f,k-N+2-r}^{N-1}$.

Now we prove $\pi_{r}^{N-1,0}(Z_{r+1}^{N-1,0}) = F_1 W_{f,k-N+2-r}^{N-1} + FW_{f,k-N+2-r}^{N-1}$.
Suppose $[w] \in F_1 W_{f,k-N+2-r}^{N-1}$, $[a_{N-2+r}^{r+1}(N-1)] \in W_{f,k-N+2-r}^{N-1}$. There are $u = u^{N-1,0} + \cdots + u^{N-2+r,1-r} \in Z_{r}^{N-1,0}$ such that $\pi_{r}^{N-1,0}(u) = [w] + (-1)^{N-1}F[a_{N-2+r}^{r+1}(N-1)]$. For $u^{N-1,0}+\cdots+u^{N-2+r,1-r}$ there are $a_N(N), \cdots, a_{N-2+r}(N), b_{N-1}(N-1), \cdots, b_{N-2+r}(N-1)$ and $a_N^r(N-1), \cdots, a_{N-3+r}^r(N)$ satisfying corresponding equations in above context. Let

$$\begin{aligned}
&a_{N-1+r}(N) = Da_{N-2+r}^{r+1}(N-1),\\
&Da_{N-3+r}^{r+1}(N-1) + (-1)^N(r-1)a_{N-2+r}^{r+1}(N-1) \wedge DF = a_{N-2+r}(N)\\
&= Da_{N-3+r}^{r}(N-1),\\
&\cdots\cdots
\end{aligned}$$

$$Da^{r+1}_{N-1}(N-1)+(-1)^N a^{r+1}_N(N-1)\wedge DF=a_N(N)$$
$$=Da^r_{N-1}(N-1)+(-1)^N a^r_N(N-1)\wedge DF.$$

Let $Dc^{r+1}_{N-3+r}(N-1)=DF\wedge a^{r+1}_{N-2+r}(N-1)$. We may suppose

$$\begin{aligned} a^r_{N-3+r}(N-1)&=a^{r+1}_{N-3+r}(N-1)-(r-1)c^{r+1}_{N-3+r}(N-1),\\ \pi^{N-1,0}_r(u)&=[(-1)^N\Pi_{k-N+3-r}a^r_{N-3+r}(N-1)+b_{N-2+r}(N-1)]\\ &=[(-1)^N\Pi_{k-N+3-r}(a^{r+1}_{N-3+r}(N-1)-(r-1)c^{r+1}_{N-3+r}(N-1)\\ &\quad+b_{N-2+r}(N-1)]\\ &=[w]+(-1)^{N-1}F[a^{r+1}_{N-2+r}(N-1)]. \end{aligned}$$

Hence

$$\begin{aligned} &[b_{N-2+r}(N-1)+(-1)^N\Pi_{k-N+3-r}a^{r+1}_{N-3+r}(N-1)+(-1)^N Fa^{r+1}_{N-2+r}(N-1)\\ &+(-1)^{N+1}(r-1)\Pi_{k-N+3-r}c^{r+1}_{N-3+r}(N-1)]\in F_1W^{N-1}_{f,k-N+2-r}. \end{aligned}$$

Choose $b_{N-1+r}(N-1)$ suitably. Then

$$\begin{aligned} &D([b_{N-2+r}(N-1)+(-1)^N\Pi_{k-N+3-r}a^{r+1}_{N-3+r}(N-1)+(-1)^N Fa^{r+1}_{N-2+r}(N-1))\\ &=(-1)^{N-1}r((-1)^N\Pi_{k-N+2-r}a^{r+1}_{N-2+r}(N-1)+b_{N-1+r}(N-1))\wedge DF. \end{aligned}$$

Let $u^{N-1+r,-r}=a_{N-1+r}(N)B(DB)^{r-1}+b_{N-1+r}(N-1)(DB)^r$,

$$\begin{aligned} u'&=u+u^{N-1+r,-r}\in Z^{N-1,0}_{r+1}.\\ \pi^{N-1,0}_{r+1}(u')&=[(-1)^N\Pi_{k-N+2-r}a^{r+1}_{N-2+r}(N-1)+b_{N-1+r}(N-1)],\\ \pi^{N-1,0}_r(u')&=[(-1)^N\Pi_{k-N+3-r}a^{r+1}_{N-3+r}(N-1)+b_{N-2+r}(N-1)]\\ &=[w]+(-1)^{N-1}F[a^{r+1}_{N-2+r}(N-1)]. \end{aligned}$$

Hence $\pi^{N-1,0}_r(Z^{N-1,0}_{r+1})=F_1W^{N-1}_{f,k-N+2-r}+FW^{N-1}_{f,k-N+2-r}$.

$$\frac{'E^{N-1,0}_r}{'E^{N-1,0}_{r+1}}\overset{\pi^{N-1,0}_r}{\cong}$$
$$\frac{W^{N-1}_{f,k-N+2-r}}{\delta_{k-N+3-r}\cdots\delta_{k-N+1}(FW^{N-1}_{f,k-N+1}\cap F_{r-1}W^{N-1}_{f,k-N+1})+FW^{N-1}_{f,k-N+2-r}+F_1W^{N-1}_{f,k-N+2-r}}$$
$$\cdot\frac{\Omega^N_{f,k-N+1-r,0}}{D\delta_{k-N+3-r}\cdots\delta_{k-N+1}(FW^{N-1}_{f,k-N+1}\cap F_{r-1}W^{N-1}_{f,k-N+1})+F\Omega^N_{f,k-N+1-r,0}}.$$

All the results and proofs are true for real analytic and complex holomorphic cases.

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