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HOMOMORPHISMS BETWEEN CHEVALLEY GROUPS OF TYPES C_n AND G_2 OVER FINITE FIELDS

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Abstract

It is known that there exists an isogeny sort of Chevalley groups $G(\Sigma, F)$ associated to any indecomposable root system Σ and any field F. In this paper the author determines all nontrivial homomorphisms from $G(\Sigma, k)$ to $G(\Sigma, K)$ when the root system Σ is of type C_n or G_2, and the fields k and K are finite fields of characteristic p.

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§1. Introduction

Given data $\{\Sigma, F, L\}$, where Σ is an indecomposable root system, F is any field and L is a lattice between the weight lattice and the root lattice of Σ, we can associate a unique Chevalley group $G(\Sigma, F, L)$ over the field F to the data. The automorphism groups of the Chevalley groups have been determined by Steinberg[1]. We know that any automorphism of a Chevalley group over a perfect field can be expressed as a product of an inner, a diagonal, a graph and a field automorphism. The further development is to study homomorphisms between the Chevalley groups, which was initiated by Borel, Tits[2] and Weisfeiler[3]. They considered abstract homomorphisms between subgroups of algebraic groups provided that the images of the homomorphisms are Zariski dense subsets and that the fields over which algebraic groups are defined are infinite. Recently, Chen[4], Dicks and Hartley[5] have studied homomorphisms between the Chevalley groups of type A_1 over any fields and division rings without assumption of Zariski density and of infinitude of fields. In the present paper we are going to determine homomorphisms between the Chevalley groups of types C_n and G_2 over finite fields.

For any root $\alpha \in \Sigma$, let $U_\alpha(F)$ denote the root subgroup of $G(\Sigma, F, L)$ corresponding to α, which is isomorphic to the additive group of the field F. Fixing a simple system $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of Σ, we denote the diagonal subgroup of $G(\Sigma, F, L)$ by $H(F)$, which is an abelien group generated by $H_{\alpha_i}(F), \alpha_i \in \Pi$. Each $H_{\alpha_i}(F)$ is isomorphic to the multiplicative group of F^* and $\prod\limits_{i=1}^{n} H_{\alpha_i}(t_i) = 1$ if and only if $\prod\limits_{i=1}^{n} t_i^{\langle \lambda, \alpha_i \rangle} = 1$ for any $\lambda \in L$, where $t_i \in F^*$ and $\langle \lambda, \alpha_i \rangle$ is the Cartan inner product of λ and α_i.

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Lemma 1.1 *Let k and K be any fields, σ be a homomorphism from k to K. The map $\sigma : U_\alpha(t) \mapsto U_\alpha(t^\sigma)$ on the root subgroups can be extended to a homomorphism from $G(\Sigma, k, L_1)$ to $G(\Sigma, K, L_2)$ if and only if $L_1 \supseteq L_2$.*

Proof. If the map $\sigma : U_\alpha(t) \mapsto U_\alpha(t^\sigma)$ can be extended to a homomorphism from $G(\Sigma, k, L_1)$ to $G(\Sigma, K, L_2)$, then $\sigma(H_{\alpha_i}(t)) = H_{\alpha_i}(t^\sigma)$ for any $\alpha_i \in \Pi$. Thus, $\prod_{i=1}^{n} t_i^{\langle \lambda, \alpha_i \rangle} = 1$ for any $\lambda \in L_1$ implies that $\prod_{i=1}^{n} H_{\alpha_i}(t_i^\sigma) = 1$ and $\prod_{i=1}^{n} t_i^{\langle \mu, \alpha_i \rangle} = 1$ for any $\mu \in L_2$. So $L_1 \supseteq L_2$. Conversely, if $L_1 \supseteq L_2$, then $G(\Sigma, k, L_2)$ is a factor group of $G(\Sigma, k, L_1)$, and it is clear that there exists a homomorphism $\sigma : G(\Sigma, k, L_2) \longrightarrow G(\Sigma, K, L_2)$ such that $\sigma(U_\alpha(t)) = U_\alpha(t^\sigma)$ for any $\alpha \in \Sigma$.

Such a homomorphism σ, if any, is called a group homomorphism induced from a field homomorphim, and we will denote it by $x \mapsto x^\sigma$ for any $x \in G(\Sigma, k, L_1)$. We have the following conjecture.

Conjecture. *Let k and K be any fields of the same characteristic. If*

$$\phi : G(\Sigma, k, L_1) \longrightarrow G(\Sigma, K, L_2)$$

is a nontrivial homomorphism, then there exists a homomorphism σ induced by a field homomorphism and an automorphism γ of $G(\Sigma, K, L_2)$ such that $\phi(x) = \gamma(x^\sigma)$ for any $x \in G(\Sigma, k, L_1)$.

We will verify the conjecture for the Chevalley groups of types C_n and G_2 over finite fields of characteristic p, where $p \neq 2$ in the case of type C_n and $p \neq 2, 3$ in the case of type G_2.

§2. Normalizing Procedure

Let k and K be finite fields of characteristic p,

$$\phi : G(\Sigma, k, L_1) \longrightarrow G(\Sigma, K, L_2)$$

a nontrivial homomorphism. We can normalize ϕ in the following way.

(1) For any field F, let $U_+(F)$ (resp. $U_-(F)$) denote the maximal unipotent subgroup of $G(\Sigma, F, L)$ generated by all $U_\alpha(F)$ corresponding to positive roots α (resp. corresponding to negative roots). If F is a finite field of characteristic p, $U_+(F)$ is a p-Sylow subgroup of $G(\Sigma, F, L)$. Thus, by Sylow's theorem the homomorphism ϕ can be normalized by multiplication by an inner automorphism of $G(\Sigma, K, L_2)$ so that $\phi(U_+(k)) \subseteq U_+(K)$. From now on we always assume that the nontrivial homomorphism ϕ has the property of sending $U_+(k)$ to $U_+(K)$.

In order to further normalize the homomorphism ϕ we need the following lemmas.

Lemma 2.1. *For any field F, let*

$$U_+(F) = U_+^{(1)}(F) \supset U_+^{(2)}(F) \supset \cdots U_+^{(k)}(F) \supset \cdots$$

be the lower central series of $U_+(F)$. Assume that $\mathrm{char} F \neq 2$ *if all roots of Σ are not equal in length and Σ is not of type G_2, and* $\mathrm{char} F \neq 2, 3$ *if Σ is of type G_2. Then $U_+^{(k)}(F)$ is generated by all $U_\alpha(F)$ with $ht(\alpha) \geq k$.*

The same result also holds for $U_-(F)$.

Proof. Assume that the conclusion holds for $U_+^{(k)}(F)$. We make induction on the height of roots to assert that the conclusion also holds for $U_+^{(k+1)}(F)$. It is clear that $U_+^{(k+1)}(F) = (U_+(F), \quad U_+^{(k)}(F))$ is contained in the subgroup generated by all root subgroups $U_\alpha(F)$ with $ht(\alpha) \geq k+1$. So it is enough to prove that each root subgroup $U_\alpha(F)$ with $ht(\alpha) \geq k+1$ is contained in $U_+^{(k+1)}(F)$. For any positive root γ, if γ is not a simple root, there exist a simple root α and a positive root β such that $\gamma = \alpha + \beta$. By the Chevalley commutator formula

$$(U_\alpha(t), \quad U_\beta(u)) = \prod U_{i\alpha+j\beta}(c_{ij}t^i u^j)$$

and $c_{11} = \pm(r+1)$, when $\beta - r\alpha, \cdots, \beta - \alpha, \beta, \beta+\alpha, \cdots$ is an α-string through β. We know[6] that $r = 0$ if only one root length occurs in Σ, $r \leq 1$ if Σ is not of type G_2 and $r \leq 2$ if Σ is of type G_2. If $ht(\gamma) \geq k+1$, by induction hypothesis $U_\gamma(c_{11}tu) \in U_+^{(k+1)}(F)$ for any $t, u \in F$, and by the assumption on the characteristic of F, $c_{11} \neq 0$. Then it follows that $U_\gamma(F) \subseteq U_+^{(k+1)}(F)$.

Lemma 2.2. *For any fields F' and F, if there exists a nontrivial homomorphism*

$$\sigma : SL_2(F') \longrightarrow G(\Sigma, F, L)$$

such that

$$\sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in U_\alpha(F), \qquad \sigma \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in U_\beta(F)$$

for any $t \in F'$, then $\alpha + \beta = 0$.

Proof. Applying σ to the both sides of the identity

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}$$

yields $U_\alpha(a)U_\beta(b)U_\alpha(a) = U_\beta(b)U_\alpha(a)U_\beta(b)$ for some $a, b \in F$. Then we have $(U_\alpha(a), U_\beta(b)) = U_\beta(-b)U_\alpha(a)$. If $\alpha + \beta \neq 0$, the left side should be $\prod U_{i\alpha+j\beta}(c_{ij}a^i b^j)$ by the Chevalley commutator formula. It forces $a = b = 0$ since any expression of product of root subgroups on the set $S = \{i\alpha + j\beta\}$ of roots is unique in any fixed order. This implies that σ is trivial, a contradiction.

(2) Let θ be a unique highest root of the root system Σ. By Lemma 2.1, $\phi(U_\theta(k)) \subseteq U_\theta(K)$. The homomorphism ϕ can be further normalized by an inner automorphism σ_u induced by $u \in U_+(K)$ such that $\sigma_u \circ \phi(U_{-\theta}(k)) \subseteq U_{-\theta}(K)$.

Proof of (2). Denote by σ_g any inner automorphism induced by $g \in G(\Sigma, K, L_2)$. There exists $g \in G(\Sigma, K, L_2)$ such that $\sigma_g \circ \phi(U_-(k)) \subseteq U_-(K)$ and by Lemma 2.1, $\sigma_u \circ \phi(U_{-\theta}(k)) \subseteq U_{-\theta}(K)$. Using the Birkhoff decomposition of $G(\Sigma, K, L_2)$, we can write $g = vwu$, where $v \in U_-(K), w \in W$ and $u \in U_+(K)$. Then

$$\sigma_u \circ \phi(U_{-\theta}(k) \subseteq w^{-1}U_{-\theta}(K)w \subseteq U_{-w^{-1}(\theta)}(K).$$

Since there exists the homomorphism

$$\phi_\theta : SL_2(k) \longrightarrow \langle U_\theta(k), \quad U_{-\theta}(k)\rangle$$

such that

$$\phi_\theta \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = U_\theta(t), \qquad \phi_\theta \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = U_{-\theta}(t)$$

for any $t \in k$, applying Lemma 2.2 to the homomorphism $\sigma_u \circ \phi \circ \phi_\theta$ yields $-w^{-1}\theta = -\theta$.

We introduce the following notation. Let I be the subset of Π consisting of all simple roots α_i with $(\alpha_i, \theta) = 0$, and Σ_I be the subsystem of roots of Σ with simple system I. Denote by $P_I^+(F)$ (resp. $P_I^-(F)$) the parabolic subgroup associated to I containing $B_+(F) = H(F)U_+(F)$ (resp. $B_-(F) = H(F)U_-(F)$), which is the normalizer of $U_\theta(F)$ (resp. $U_{-\theta}(F)$) in $G(\Sigma, F, L)$. $L_I(F) = P_I^+(F) \cap P_I^-(F)$ is called a Levi factor of $P_I^+(F)$, and it is clear that the derived subgroup $L_I'(F)$, which is generated by $U_{\pm\alpha_i}(F), \alpha_i \in I$, is still a Chevalley group associated to the data $\{\Sigma_I, F, L\}$.

Lemma 2.3. *For any fixed $a \in F^*, g \in G(\Sigma, F, L)$, $gU_\theta(a)g^{-1} \in U_\theta(F)$ if and only if $g \in P_I^+(F)$.*

Similarly, $gU_{-\theta}(a)g^{-1} \in U_{-\theta}(F)$ if and only if $g \in P_I^-(F)$.

Proof. Write $g = uwv$, where $u, v \in U_+(F)$ and $w \in W$. For a fixed $a \in F^*, gU_\theta g^{-1} \in U_\theta(F)$ if and only if $w(\theta) = \theta$. Since θ is the highest root of Σ, θ lies in the fundamental domain of the action of W. Then $w(\theta) = \theta$ if and only if w can be expressed as a product of the simple reflections s_{α_i} with $\alpha_i \in I$, and it follows that $g \in P_I^+(F)$.

(3) By Lemma 2.3, the normalized homomorphism $\sigma_u \circ \phi$ in (2) satisfies

$$\sigma_u \circ \phi(P_I^\pm(k)) \subseteq P_I^\pm(K).$$

Then $\sigma_u \circ \phi$ sends $L_I'(k)$ to $L_I'(K)$.

In order to verify the conjecture for the Chevalley groups of types C_n and G_2, we hope that we can apply induction hypothesis to $L_I'(k)$. So we present our main theorem in a slightly strengthening form.

Theorem 2.1. *Let Σ be an indecomposable root system of type C_n or type G_2, k and K be any finite fields of characteristic p, and $p \neq 2$ in the case of type C_n and $p \neq 2, 3$ in the case of type G_2. For any nontrivial homomorphism $\phi : G(\Sigma, k, L_1) \longrightarrow G(\Sigma, K, L_2)$, if ϕ satisfies $\phi(U_+(k)) \subseteq U_+(K)$, then there exist a field homomorphism $\sigma : k \longrightarrow K$, an inner automorphism σ_u induced by $u \in U_+(K)$ and a diagonal automorphism χ such that $\chi \circ \sigma_u \circ \phi(x) = x^\sigma$ for any $x \in G(\Sigma, k, L_1)$.*

Proof. We make induction on the rank of Σ. Note that Theorem 2.1 is true for the Chevalley groups of type A_1[4,7]. By normalizing procedure (2) we can assume that ϕ satisfies $\phi(U_{\pm\theta}(k)) \subseteq U_{\pm\theta}(K)$. Then by procedure (3) the restriction ϕ_I of ϕ to $L_I'(k)$ is a nontrivial homomorphism of $L_I'(k)$ to $L_I'(K)$. We will distinguish the cases of type C_n and type G_2 to verify the Theorem.

§3. The Chevalley Groups of Type C_n

The notation used in the present section is the same as those in the previous ones.

Proof of Theorem 2.1 for type C_n. For the root system Σ of type C_n, if $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a simple system of roots, where α_n is a long root, then the unique highest root

$$\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$$

and $I = \Pi - \{\alpha_1\}$. So Σ_I is a root system of type C_{n-1} and the induction hypothesis can be applied to the homomorphism ϕ_I. There exist a field homomorphism $\sigma : k \longrightarrow K$, an inner automorphism σ_u induced by $u \in U_I(K)$, where $U_I(K)$ is the subgroup generated by $U_\alpha(K)$ with $\alpha \in \Sigma_I^+$, and a diagonal automorphism χ determined by $\chi \in \mathrm{Hom}(\Sigma_I, K)$ such that $\chi\circ\sigma_u\circ\phi(x)=x^\sigma$ for any $x\in L'_I(k)$. It is clear that χ can be extended to a diagonal automorphism of $G(\Sigma,K,L_2)$ by defining $\chi(\alpha_1)$ arbitrarily. Since $\chi\circ\sigma_u\circ\phi(U_{\pm\theta}(k))\subseteq U_{\pm\theta}(K)$, by the results[4,7] on homomorphisms between the Chevalley groups of type A_1, there exists a field homomorphism $\sigma':k\longrightarrow K$ and $a\in K^*$ such that

$$\chi\circ\sigma_u\circ\phi(U_{\pm\theta}(t))=U_{\pm\theta}(a^{\pm1}t^{\sigma'})$$

for any $t\in k$. For simplicity, write $\psi=\chi\circ\sigma_u\circ\phi$. We claim that $\psi(x)=x^\sigma$ for any $x\in G(\Sigma,k,L_1)$ by defining $\chi(\alpha_1)$ properly. The proof will be divided in several steps.

1. Let w_0 denote the longest element in W_I. For any positive root $\alpha\in\Sigma^+$, $w_0(\alpha)\in\Sigma^-$ if and only if $\alpha\in\Sigma_I^+$, and w_0 permutes the elements of $\Sigma^+-\Sigma_I^+$ since w_0 is the product of the simple reflections $s_{\alpha_i},\alpha_i\in I$. Let s_θ denote the reflection along the hyperplane orthogonal to θ. s_θ fixes Σ_I elementwise, and by direct computation, $s_\theta(\alpha)\in\Sigma^-$ for any $\alpha\in\Sigma^+-\Sigma_I^+$. Then $s_\theta w_0(\alpha)\in\Sigma^-$ for any $\alpha\in\Sigma^+$. For any positive root α, let $n_\alpha=U_\alpha(1)U_{-\alpha}(-1)U_\alpha(1)$, which represents the reflection s_α along the hyperplane orthogonal to α. There exist n_θ and n_0 representing s_θ and w_0, respectively, and we have $(n_\theta n_0)U_+(k)(n_\theta n_0)^{-1}=U_-(k)$. Since $\psi(n_0)=n_0$ and $\psi(n_\theta)=n_\theta$ modulo $H(K)$, applying ψ to the both sides of the identity yields

$$\psi(U_-(k))\subseteq(n_\theta n_0)U_+(K)(n_\theta n_0)^{-1}\subseteq U_-(K).$$

2. Since $\theta-\alpha_1$ is a unique second highest root in Σ, by Lemma 2.1

$$\psi(U_{\theta-\alpha_1}(k))\subseteq U_{\theta-\alpha_1}(K)U_\theta(K).$$

By the fundamental relation

$$H_{\alpha_2}(t)U_{\theta-\alpha_1}(u)H_{\alpha_2}(t)^{-1}=U_{\theta-\alpha_1}(t^{\langle\theta-\alpha_1,\alpha_2\rangle}u)=U_{\theta-\alpha_1}(t^{-\langle\alpha_1,\alpha_2\rangle}u)=U_{\theta-\alpha_1}(tu),$$

we obtain $U_{\theta-\alpha_1}(k)=(H_{\alpha_2}(t),\quad U_{\theta-\alpha_1}(k))$ by choosing $t\neq1$. Applylig ψ yields

$$\psi(U_{\theta-\alpha_1}(k))\subseteq(H_{\alpha_2}(t^\sigma),\quad U_{\theta-\alpha_1}(K)U_\theta(K))\subseteq(H_{\alpha_2}(t^\sigma),\quad U_{\theta-\alpha_1}(K))\subseteq U_{\theta-\alpha_1}(K).$$

Similarly, we have $\psi(U_{-(\theta-\alpha_1)}(k))\subseteq U_{-(\theta-\alpha_1)}(K)$. Furthermore, since

$$n_\theta U_{\theta-\alpha_1}(k)n_\theta^{-1}=U_{-\alpha_1}(k),$$

applying ψ yields $\psi(U_{-\alpha_1}(k))\subseteq U_{-\alpha_1}(K)$. Similarly, $\psi(U_{\alpha_1}(k))\subseteq U_{\alpha_1}(K)$.

3. Now we can define $\chi(\alpha_1)$ properly so that $\psi(U_{\alpha_1}(1))=U_{\alpha_1}(1)$. Since

$$U_{\alpha_1}(a)U_{-\alpha_1}(b)U_{\alpha_1}(a)=U_{-\alpha_1}(b)U_{\alpha_1}(a)U_{-\alpha_1}(b)$$

if and only if $ab=-1$, which can be verified immediately in SL_2, after taking $a=1,b=-1$ and applying ψ we achieve $\psi(U_{-\alpha_1}(-1))=U_{-\alpha_1}(-1)$ and $\psi(U_{-\alpha_1}(1))=U_{-\alpha_1}(1)$. For any $t\in k^*$, $H_{\alpha_2}(t)^{-1}U_{\alpha_1}(1)H_{\alpha_2}(t)=U_{\alpha_1}(t)$, hence $\psi(U_{\alpha_1}(t))=U_{\alpha_1}(t^\sigma)$. Similarly, $\psi(U_{-\alpha_1}(t))=U_{-\alpha_1}(t^\sigma)$. But $U_{\pm\alpha_i}(k),\alpha_i\in\Pi$ are a generating system of subgroups of $G(\Sigma,k,L_1)$, so $\psi(x)=x^\sigma$ for any $x\in G(\Sigma,k,L_1)$.

It is well known that a Chevalley group of type C_n over any field F is the symplectic group $Sp(2n,F)$ or projective symplectic group $PSp(2n,F)$. Let

$$J=\begin{pmatrix}0&I\\-I&0\end{pmatrix}.$$

As a subgroup of $GL(2n,F)$, the symplectic group $Sp(2n,F)$ consists of those matrices $A\in GL(2n,F)$ satisfying $AJA'=J$, and $PSp(2n,F)=Sp(2n,F)/\{\pm I\}$. It is easy to observe that a diagonal automorphism of $Sp(2n, F)$ or $PSp(2n, F)$ can be realized by a diagonal matrix D under conjugation, where the matrix D satisfies $DJD' = aJ$ for some $a \in F^*$. A matrix $A \in GL(2n, F)$ satisfying $AJA' = aJ$ for some $a \in F^*$ is called a generalized symplectic matrix. Now we can restate the results on homomorphism between the Chevalley groups of type C_n over finite fields in matrix form.

Theorem 3.1. *Let k and K be any finite fields of characteristic p, where $p \neq 2$.*

(i) *If $\phi : Sp(2n, k) \longrightarrow Sp(2n, K)$ is a nontrivial homomorphism, there exist a field homomorphism $\sigma : k \longrightarrow K$ and a generalized symplectic matrix $P \in GL(2n, K)$ such that $\phi(A) = PA^\sigma P^{-1}$ for any $A \in Sp(2n, k)$.*

(ii) *Similar results hold for any nontrivial homomorphism $\phi : PSp(2n, k) \longrightarrow PSp(2n, K)$ or $\phi : Sp(2n, k) \longrightarrow PSp(2n, K)$.*

(iii) *Any homomorphisms $\phi : PSp(2n, k) \longrightarrow Sp(2n, K)$ are all trivial.*

It is obvious that Theorem 3.1 is a generalization of the results on automorphisms of symplectic groups and projective symplectic groups over finite fields.

§4. The Chevalley Groups of Type G_2

Proof of Theorem 2.1 for type G_2. The proof is almost the same as one given by the previous section, only a little change needs to be made. For the root system Σ of type G_2, if $\Pi = \{\alpha_1, \alpha_2\}$ is a simple system, where α_2 is a long root, then the highest root $\theta = 3\alpha_1 + 2\alpha_2$ and $I = \{\alpha_1\}$. The induction hypothesis can be applied, so we can futher assume

$$\phi(U_{\pm\alpha_1}(t)) = U_{\pm\alpha_1}(t^\sigma)$$

for any $t \in k$, where $\sigma : k \longrightarrow K$ is a field homomorphism. Since $s_\theta s_{\alpha_1}(\alpha) \in \Sigma^-$ for any positive root $\alpha \in \Sigma^+$, we can achieve $\phi(U_\pm(k)) \subseteq U_\pm(K)$. The root system Σ has a unique second highest root $\theta - \alpha_2$. Since $(H_{\alpha_1}(t),\quad U_{\theta-\alpha_2}(u)) = U_{\theta-\alpha_2}((t^3 - 1)u)$, by choosing t such that $t^3 \neq 1$, $(H_{\alpha_1}(t), U_{\theta-\alpha_2}(k)) = U_{\theta-\alpha_2}(k)$. Then $\phi(U_{\pm(\theta-\alpha_2)}(k)) \subseteq U_{\pm(\theta-\alpha_2)}(K)$, and

$$\phi(U_{\pm\alpha_2}(k)) = \phi(n_\theta U_{\pm(\theta-\alpha_2)}(k) n_\theta^{-1}) \subseteq U_{\pm\alpha_2}(K).$$

Normalizing by a diagonal automorphism defined properly, we have $\phi(U_{\pm\alpha_2}(t)) = U_{\pm\alpha_2}(t^{\sigma'})$ for some field homomorphism $\sigma' : k \longrightarrow K$. Since $H_{\alpha_2}(t)^{-1} U_{\alpha_1}(1) H_{\alpha_2}(t) = U_{\alpha_1}(t)$, applying ϕ yields $\sigma = \sigma'$, as requested.

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