Chin. Ann. of Math.
18B: 2(1997),173-180.

CONFORMAL DEFORMATION OF COMPLETE SURFACE WITH NEGATIVE CURVATURE**

ZHOU DETANG*

Abstract

The author considers the problem of deforming the metric on a complete negatively curved surface conformal to another metric whose Gauss curvature is nonpositive.

Keywords Riemannian manifold, Conformal deformation, Gaussian curvature
1991 MR Subject Classification 53C20, 35J60
Chinese Library Classification O186.12, O175.29

§1. Introduction

Let (M,g) be a Riemannian manifold with or without boundary ($n=\dim M\geq 2$), and $\tilde{K}$ a continuous function on M. In this paper we consider the problem of deforming the given metric g conformal to another metric

$$\tilde{g}=\begin{cases} e^{2u}g, & n=2,\\ u^{\frac{4}{n-2}}g, & n\geq 3,\end{cases}$$

with the prescribed scalar curvature $\tilde{K}$. It is well known that this problem is equivalent to solving the following elliptic differential equations:

$$-\Delta_g u+S_g=\tilde{K}e^u,\qquad n=2,\tag{1.1}$$

$$\begin{cases} -4\dfrac{n-1}{n-2}\Delta_g u+S_g u=\tilde{K}u^{\frac{n+2}{n-2}},\\ u>0, \qquad\qquad n\geq 3,\end{cases}\tag{1.2}$$

where Δ_g is the Laplacian with respect to g, namely, $\Delta_g=\operatorname{trace}\nabla_g^2$, and S_g is the scalar curvature of g. This problem has been extensively investigated, mainly in the case that (M,g) is a compact manifold. As for the case that (M,g) is the Euclidean space (R^n,g_0), since Ni's paper [11] was published, many authors have refined and generalized his results. For noncompact Riemannian manifolds this problem was posed in [7] and [15].

We write (1.1) in the following form:

$$\Delta u+K(x)e^{2u}=f(x),\qquad x\in M,\tag{1.3}$$

where M is a noncompact complete Riemannian surface. In the case $M=R^2$, many existence and nonexistence results for equation (1.3) were proved (see [2, 6, 8, 10, 11, 13,

Manuscript received October 31, 1994. Revised August 30, 1995.
*Department of Mathematics, Shandong University, Jinan, Shandong 250100, China.
**Project supported by the Natural Science Foundation of Shandong Province and NNSFC.

17] and references therein). In their works $K(x) = O(|x|^l)$ as $|x| \longrightarrow +\infty$ is essential. In [2] the authors considered the case $M = D^2(-1)$, the Poincáre disk. For the manifolds with negative curvature it is natural to ask: if $K(x)$ does not satisfy the decay condition, for which kind of functions f's can equation (1.3) still have solutions? In this paper we obtain some results in this direction (see Theorem A).

The main results and some discussions are given in §2, and the proofs of the results are given in §3 and §4.

§2. Main Results and Some Discussions

One of the typical results is

Theorem A. *Let (M, g) be a connected smooth complete noncompact Riemannian surface with Gaussian curvature $f(x) < 0$. Then, for any negative locally Hölder continuous function $K(x)$ on M satisfying $f(x) \le b^2 K(x)$, there exists a conformal complete metric $\tilde{g}$ whose Gaussian curvature is $K(x)$. Furthermore such metric is unique provided $K(x) \le -a^2$ and $f(x) \ge -C(1+\rho^2(x))$ for some positive constants a and C, where $\rho(x) = \mathrm{dist}(x, o)$ and o is a fixed point.*

An easy consequence of this theorem is

Corrollary. If the Gaussian curvature $f(x)$ of a complete surface (M, g) satisfies

$$f(x) \le -a^2$$

for some constant $a > 0$, then there exists a complete conformal metric whose curvature is -1.

Theorem A can also be viewed as a generalization of the well-known Ahlfors-Schwarz Lemma[1]. In that case $K(x) \equiv -1$ and $f(x) \le -b^2 < 0$ for some constant b. Later Yau[16] generalized the lemma to Kähler manifolds and Troyanov[14] generalized this lemma to generalized Riemannian surfaces. Pan[12] considered the equation (1.3) on R^2 by using the Baras-Piere's technique. Unfortunately their method does not apply to the curved manifolds. Before proving our theorem we would like to give some discussions of the main theorem. Noting that we have not assumed that the manifolds is simply connected, our results can be applied to cylinders and some other manifolds.

For Poincare disk $H^2 = (R^2, ds^2)$ with $ds^2 = dr^2 + \sinh^2 r d\theta^2$, we can consider the conformal metric $ds^{*2} = u^2(r) ds^2$ for some positive continuous function $u(r)$. In order to guarantee completeness of the new metric we assume that

$$\int_0^{+\infty} u(r)\, dr = +\infty. \tag{2.1}$$

It is straightforward to calculate that the Gaussian curvature of $M = (R^2\ , ds^{*2})$ is

$$f(r) = -\frac{1}{u^2(r)}\left[1 + \frac{uu'' - u'^2}{u^2} + \frac{u'}{u}\coth r\right]. \tag{2.2}$$

So, if we can choose $u(r)$ such that (2.1) is satisfied and $f(r) < 0$, then by Theorem A any negative continuous function $K(r)$ satisfying $K(x) \ge bf(x)$ outside a compact subset for some positive constant b can be realized as a Gaussian curvature of a complete conformal metric $\tilde{g}$. In particular we have

Theorem B. *If $K(x)$ is a negative continuous function on H^2 satisfying $K(x) \geq -Cr^2$ at infinity for some positive constant C, then there exists a complete metric conformal to ds^2 with $K(x)$ as the Gaussian curvature.*

Proof. Choose $u(r) = q(p + r^2)^{-\frac{1}{2}}$ for $r \geq 0$ where p and q are positive constants. Then

$$u'(r) = -q(p + r^2)^{-\frac{3}{2}} r,$$
$$u''(r) = -q(p + r^2)^{-\frac{3}{2}} + 3q(p + r^2)^{-\frac{5}{2}} r^2,$$

and from (1.5)

$$\begin{aligned} f(r) =& - q^{-2}(p + r^2) \left[1 + \frac{-(p + r^2)^{-2} + 2(p + r^2)^{-3} r^2}{(p + r^2)^{-1}} - (p + r^2)^{-1} r \coth r \right] \\ =& - q^{-2}(p + r^2) \left[1 + \frac{-p + r^2}{(p + r^2)^2} - \frac{r}{p + r^2} \coth r \right]. \end{aligned}$$

We can choose p sufficiently large so that $f(r) < 0$ for all r. It is easy to see that $K(x) \geq f(r(x))$ holds outside some compact subset when q is sufficiently small and then Theorem A applies. This proves our conclusion.

Remark 2.1. In fact, we can obtain sharper results by choosing $v(r)$. We can also obtain similar results for models which are not simply connected.

The proof of Theorem A is a combination of the existence theorem and uniqueness theorem. We shall prove the existence theorem in section 3 and prove the uniqueness theorem in section 4, which have independent interests in partial differential equations.

After we finished the first draft of this paper we learned that [9] also obtained some results to the Theorem B by using a different method. We find out that our theorems allow many special cases which cannot be concluded from [9] and our proofs are simpler.

§3. Existence Theorem

In this section we will treat equation (1.3) in the viewpoint of partial differential equations. In [10] and [12], (1.3) was discussed when $M = R^2$ by different methods. Here we will use another technique which is essentially sub-upper solution method. In contrast to the previous papers this method is simpler.

Theorem 3.1. *Let (M, g) be a smooth noncompact complete Riemannian surface. Assume that $K(x)$ is negative and locally Hölder continuous on M. Then for any locally Hölder continuous function f which satisfies*

$$\liminf_{d(x,x_0) \to +\infty} \frac{f(x)}{K(x)} > 0, \tag{3.1}$$

where x_0 is a fixed point, equation (1.3) *possesses a continuous solution. Furthermore if $f(x) \leq K(x)$ for any $x \in M$ then the solution is nonnegative on M.*

Proof. By the condition there exists a compact set $M_0 \subset M$ and a constant k_0 such that $f(x) \leq k_0 K(x)$ for any $x \in M \backslash M_0$. Let $\phi \in C_0^\infty(M)$ be the function such that $\Delta_g u = 1$ in

M_0 and otherwise arbitrary and $\varphi = \alpha + \beta\phi$, where α , β are constants to be determined later. Define a new conformal metric $g_1 = e^{2\varphi} g$. Its Gauss curvature

$$\begin{aligned} f_1(x) &= e^{-2\varphi}(f - \Delta_g \varphi) \\ &= e^{-2\alpha - 2\beta\phi}(f - \beta), \quad \forall x \in M_0. \end{aligned}$$

So we can choose α and β such that $f_1(x) \le \tilde{K}(x)$ and it suffices to prove our theorem in case of $f(x) \le K(x)$ for all $x \in M$. In what follows we only need to prove the theorem for $f(x) \le K(x) < 0$.

For $R > 0$, set $B(R) = \{x \in M : d(o, x) \le R\}$. By the assumption, there are constants $M(R) > m(R) > 0$ such that

$$-M(R) \le K(x) \le -m(R), \qquad x \in B_R.$$

We shall divide the following proof into two steps.

Step 1. Consider the elliptic problem in the ball B_R.

$$\begin{cases} \Delta_g u + K(x) e^{2u} = f(x), & \text{for} \quad x \in B_R, \\ u|_{\partial B_R} = 0. \end{cases} \tag{3.2}$$

It is easy to see that $u_- = 0$ is the subsolutions of (3.2). Since $f(x)$ is continuous and negative, there exist constants $N(R) \ge n(R) > 0$ such that

$$-N(R) \le f(x) \le -n(R).$$

So we can choose the $u^+ = C$ such that

$$e^{2C} n(R) \ge N(R),$$

thus u^+ is an upper solution of (3.2). By the standard iteration technique (see for example [13]) we know that there exists a nonnegative continuous solution of (3.2) u on B_R. We can see that such solution is unique. Otherwise, if there are two solutions u, v satisfying (3.2), then

$$\begin{cases} \Delta(u - v) = -K(x)(e^{2u} - e^{2v}), \\ u - v\,|_{\partial B_R} = 0\ . \end{cases}$$

Denote $u-\ v$ by w. Suppose that $\inf w < 0$ is achieved at some point $x_0 \in M$. Then

$$0 \le \Delta w(x_0) = -K(x_0) e^{2v(x_0)}(e^{2w(x_0)} - 1) < 0,$$

this contradiction shows that $\inf w \ge 0$. Similarly $\sup w \le 0$. This shows that $w \equiv 0$, i.e., $u \equiv v$.

Denote $B_n = \{x \in M,\ d(x, 0) < n\}$. We conclude that, for any $n \ge 1$, problem (3.2) has a unique solution $u_n \in H_0^1(B_n) \cap C(B_n)$ and $u_n \ge 0$ for all $x \in B_n$. Extend $u_n(x)$ to all M by setting $u_n(x) = 0$ $x \in M \backslash \bar{B}_n$. Then u_n is continuous.

Step 2. Uniform upper bound for $\{u_n\}$ on compact domain. Fix $R > 0$. We are going to give a uniform bound for $\{u_n\}$ on the ball B_R. Let $\xi \in C_0^\infty(B_{2R})$ such that

$$\xi(x) = \begin{cases} 1, & x \in B_R, \\ 0, & x \in M \backslash B_R, \end{cases}$$

and $|\nabla\xi| \le C_1$, $|\Delta\xi| \le C_1$. From (3.2), when $n > 2R$,

$$\int_{B_{2R}} u_n \Delta(\xi^4)\, dv + \int_{B_{2R}} K(x) e^{2u_n} \xi^4\, dv = \int_{B_{2R}} f\xi^4\, dv.$$

Therefore

$$\begin{aligned}\int_{B_{2R}} |K(x)| e^{2u_n} \xi^4\, dv &\le \int_{B_{2R}} |f|\, dv + C_2 \int_{B_{2R}} u_n \xi^2\, dv \\ &\le \int_{B_{2R}} |f|\, dv + C \left(\int_{B_{2R}} u^2 \xi^4\, dv \right)^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

Since

$$\int_{B_{2R}} |K(x)| e^{2u_n} \xi^4\, dv \ge 2m(2R) \int_{B_{2R}} u_n^2 \xi^4\, dv, \tag{3.4}$$

where $m(2R) = \min\{|K(x)| : d(x, 0) \le 2R\}$, from (3.3), (3.4)

$$\int_{B_{2R}} u_n^2 \xi^4\, dv \le C_4 \left(1 + \int_{B_{2R}} |f|\, dv \right),$$

where the constant C_4 which depends on R, $K(x)$ is independent of u_n. Then

$$\int_{B_R} u_n^2\, dv \le C_5 \left(1 + \int_{B_{2R}} |f|\, dv \right),$$

$$\int_{B_{2R}} |K(x)| e^{2u_n} \xi^4\, dv \le C_5 \left(1 + \int_{B_{2R}} |f|\, dv \right), \tag{3.5}$$

where the constant C_5 which depends on R, $K(x)$ is independent of u_n. From (3.2)

$$\int_{B_R} |\Delta u_n|\, dv \le C_6 \left(1 + \int_{B_{2R}} |f|\, dv \right).$$

Since

$$\Delta u_n = -K(x) e^{2u_n} + f \ge C_7 u_n - C_8 \quad \text{for all} \quad x \in B_{2R},$$

by Theorem 8.17 in [5] we know

$$\sup_{x \in B_R} u_n \le C_9 R^{-\frac{n}{p}} \|u_n\|_{L^p(B_{2R})},$$

where C_8, C_9 and $p > 1$ are constants independent of n, and by (3.5) we know that there exists a constant $C_0(R)$ independent of n such that

$$\sup_{x \in B_R} u_n \le C_0(R). \tag{3.6}$$

The estimate (3.6) implies that $\{u_n\}$ is a bounded subset of $W^{2,2}_{\text{loc}}(M)$. By the diagonal method we can select a subsequence, denoted again by $\{u_n\}$, such that $u_n \to u_0$ in $W^{2,2}_{\text{loc}}$ and $u_n(x) \to u(x)$ for a.e. $x \in M$. From Lemma 2.4.1 in [3] we know

$$K(x) e^{2u_n(x)} \to K(x) e^{2u_0} \quad \text{in} \quad L^2_{\text{loc}}.$$

Letting $n \to +\infty$ we see that u_0 is a solution of (1.3). From elliptic regularity theory $u_0 \in W^{2,2}_{\text{loc}}(M)$. From Sobolev's imbedding theorem u_0 is continuous in M. It is clear that $u_0(x)$ is also nonnegative in M.

Since $K(x)$ and $f(x)$ are locally Hölder continuous functions, $u_0(x)$ is a classical solution of (1.3).

Remark 3.1. It is not hard to show that the solution obtained under the condition (3.1) is bounded below by a constant.

§4. Uniqueness Theorem

In this section we will prove the uniqueness result by using the generalized maximum principle of Chen and Xin[4].

Theorem 4.1. *Let (M, g) be a Riemannian surface with Gaussian curvature bounded below by $-C(1+\rho^2(x))$ for some constant C, where $\rho(x) = \operatorname{dist}(o, x)$ for some fixed point $o \in M$. Let $K(x)$ be a continuous function satisfying $K(x) \le -a^2$. If $f(x) \le K(x)$ for all $x \in M$, then any C^2 solution of (1.3) is nonnegative.*

Proof. If u is a continuous solution of (1.3), consider the set

$$N = \{x \in M : u(x) < 0\}.$$

Our aim is to show that $N = \emptyset$. Notice that

$$\Delta u = f(x) - K(x)e^{2u} \le K(x)(1 - e^{2u}) \le 0.$$

Case 1. u achieves its minimum and $\partial N \ne \emptyset$. Since u is a superharmonic function on N and $u(x) \ge 0$ on ∂N, we have $u(x) \ge 0$ on N. This shows that $N = \emptyset$.

Case 2. u achieves its minimum at $x_0 \in N$ and $\partial N = \emptyset$. Since N is an open set in M, we have either $N = \emptyset$ or $N = M$. If $N = M$, u is a superharmonic function on M and $u(x) < 0$ on M and

$$0 \le \Delta u(x_0) \le K(x_0)(1 - e^{2u(x_0)}) < 0,$$

then $N = \emptyset$.

Case 3. u does not reach its minimum. Suppose that $N \ne$ emptyset. Then we can find a point $x_0 \in N$ such that $u(x_0) = 2\delta_0 < 0$. We can define a new function as follows:

$$w(x) = \frac{1}{1 + e^{-u(x)}}. \tag{4.1}$$

So $w(x)$ is bounded from below. By the generalized maximum principle in [4] we can find a sequence $\{x_n\}$ such that

$$w(x_n) \to \inf_{x \in M} w(x),$$

$$|\nabla w(x_n)| < \frac{1}{n}, \quad \Delta w(x_n) \ge -\frac{1}{n}.$$

Then a straightforward calculation gives

$$\nabla w(x_n) = \frac{e^{-u(x_n)}}{\left(1 + e^{-u(x_n)}\right)^2} \nabla u(x_n),$$

$$\Delta w(x_n) = \frac{e^{-u(x_n)}}{\left(1 + e^{-u(x_n)}\right)^2} \Delta u(x_n) + \frac{e^{-u(x_n)} - 1}{e^{-u(x_n)}\left(1 + e^{-u(x_n)}\right)} |\nabla w(x_n)|^2.$$

When n is sufficiently large, $u(x_n) \leq \delta_0$. Then

$$\begin{aligned} -\frac{1}{n} &\leq -\frac{e^{-u(x_n)}}{\left[1+e^{-u(x_n)}\right]^2} a^2 \left[1-e^{2u(x_n)}\right] + \frac{1}{n^2} \frac{e^{u(x_n)}}{1+e^{u(x_n)}} \\ &\leq -a^2 \frac{(1-e^{u(x_n)})}{1+e^{-u(x_n)}} + \frac{1}{n^2} \frac{e^{\delta_0}}{1+e^{\delta_0}} \\ &= -a^2 \frac{(1-e^{u(x_n)})e^{u(x_n)}}{1+e^{u(x_n)}} + \frac{1}{n^2} \frac{e^{\delta_0}}{1+e^{\delta_0}} \\ &\leq -a^2 \frac{(1-e^{\delta_0})e^{\delta_0}}{2} + \frac{1}{n^2} \frac{e^{\delta_0}}{1+e^{\delta_0}}. \end{aligned} \tag{4.2}$$

Let $n \to +\infty$.We have arrived at a contradiction which shows that $N = \emptyset$. This completes our proof.

Remark 4.1. Since our proof of Theorem 3.1 relies heavily on Chen-Xin's generalized maximum principle which has been shown to be optimal in some sense (see [4]), it would be interesting to know whether the conclusion of the Theorem 3.1 holds in the other cases.

Theorem 4.2. *Let* (M, g) *be a noncompact complete Riemannian surface with Gaussian curvature* $f(x)$ *bounded below by* $-C(1+\rho^2(x))$ *for some positive constant* C*, where* $\rho(x) = \mathrm{dist}(o, x)$ *for some fixed point* $o \in M$*. If* $K(x) \leq -a^2$ *for any* $x \in M$ *and some constant* a*, then equation* (1.3) *possesses at most one solution.*

Proof. If u, v are two solutions of (1.3), then u, v are nonnegative by Theorem 4.1. Writing $w = u - v$ we can conclude that

$$\begin{aligned} \Delta w &= -K(x)e^{2v}(e^{2w} - 1) \\ &= K(x)e^{2v}(1 - e^{2w}). \end{aligned}$$

Similar to the proof of Theorem 3.1 we can prove that $w \geq 0$. Since the same technique applies to $-w = v - u$, i.e. we also have $w \leq 0$ for any $x \in M$, we have shown that $w \equiv 0$ and $u \equiv v$. The conclusion of the theorem follows.

Acknowledgment. The main part of the present paper was done while the author was at Fudan University. The author would like to thank Professors Hu Hesheng and Xin Yuanlong for their encouragements and kindly help. He also wishes to thank Professor P. Li for his interest.

References

[1] Ahlfors, L. V., An extension of Schwarz's lemma, *Trans. Amer. Math. Soc.,* **43** (1938), 359-364.
[2] Aviles, P. & McOwen, R., Conformal deformation of complete manifolds with negative curvature, *J. Diff. Geometry,* **21** (1985), 269-281.
[3] Chang, K. C., Critical point theory and its applications (in Chinese), Shanghai Science Press, Shanghai, 1986.
[4] Chen, Q. & Xin, Y. L., A generalized maximum principle and its applications in geometry, *Amer. J. Math.,* **114** (1992), 355-366.
[5] Gilbarg, D. & Trudinger, N. S., Elliptic partial differential equations of second order, second edition, Springer, 1983.
[6] Hong, C. W., Equation $\Delta u + K(x)e^{2u} = f(x)$ on R^2 via stereographic projection, *J. Math. Anal. Appl.,* **130** (1988), 484-492.
[7] Kazdan, J. L., Prescribing the curvature of a Riemannian manifold, *Regional Conference Series in Math.,* **57** (1985), USA 1985.

[8] Kenig, C. E. & Ni, W. M., On the elliptic equation $Lu - k + K(x)e^{2u} = 0$, Univ. of Minnesota Math. Report, 83-147.

[9] Kalka, M. & Yang, D., On conformal deformation of nonpositive curvature on noncompact manifolds, *Duke Math. J.,* **72** (1993), 405-430.

[10] McOwen, R., On the equation $\Delta u + Ke^{2u} = f$ and prescribed negative curvature in R^2, *J. Math. Anal. Appl.,* **103** (1984), 365-370.

[11] Ni, W. M., On the elliptic equation $\Delta u + K(x)e^{2u} = 0$ with prescribed Gaussian curvature, *Invent. Math.,* **66** (1982), 343-352.

[12] Pan, X., Solutions of elliptic equations $\Delta u + K(x)e^{2u} = f(x)$, *J. Partial Diff. Equations,* **4** (1991), 36-44.

[13] Sattinger, D. H., Conformal metrics in R^2 with prescribed curvature, *Indiana Univ. Math. J.,* **22** (1974), 1-4.

[14] Troyanov, M., The Schwarz lemma for nonpositively curved Riemannian surfaces, *Manuscript Math.,* **72** (1991), 251-156.

[15] Yau, S. T., Seminar on differential geometry: Problem Section, *Ann. of Math. Studies,* **102** (1982).

[16] Yau, S. T., A general Schwarz Lemma for Kähler manifolds, *Amer. J. Math.,* **100** (1978), 197-203.

[17] Zhou Detang, On the solutions of second order elliptic equations, *Chinese Science Bulletin,* **36** (1991), 617-620.

[18] Zhou Detang, On scalar curvatures on noncompact Riemann manifolds, *Chin. Ann. of Math.,* **16B:**1 (1995), 87-92.