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ON A MULTILINEAR OSCILLATORY SINGULAR INTEGRAL OPERATOR (I)**

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Abstract

The authors consider the multilinear oscillatory singular integral operator defined by

$$T_{A_1,\,A_2,\,\cdots,\,A_k} f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} \prod_{j=1}^{k} R_{m_j}(A_j;\, x, y) \frac{\Omega(x-y)}{|x-y|^{n+M}} f(y)\, dy,$$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, Ω is homogeneous of degree zero, $R_{m_j}(A_j;\, x, y)$ denotes the m_j-th order Taylor series remainder of A_j at x expanded about y, $M = \sum\limits_{j=1}^{k} m_j$. It is shown that if Ω belongs to the space $L\log^+ L(S^{n-1})$ and has vanishing moment up to order M, then

$$\|T_{A_1,\,A_2,\,\cdots,\,A_k} f\|_q \le C \prod_{j=1}^{k} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{r_j} \Big) \|f\|_p,$$

provided that $1 < p,\, q < \infty$, $1 < r_j \le \infty$ $(j = 1, 2, ..., k)$ and $1/q = 1/p + \sum\limits_{j=1}^{k} 1/r_j$. The corresponding maximal operator is also considered.

Keywords Multilinear operator, Oscillatory singular integral, Maximal operator
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§1. Introduction

Let us consider the following oscillatory singular integral operator

$$Tf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y)\, dy,$$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, $K(x)$ is homogeneous of degree $-n$ and has mean value zero on each sphere centered at the origin. Ricci and Stein [1] proved that if $K \in C^1(\mathbb{R}^n \backslash \{0\})$, then T is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, with bounds depending only on n, p and the total degree of P, but not on its coefficients. Subsequently, Chanillo and Christ [2] proved that T is also of weak type $(1,1)$. Since oscillatory singular integral operators with polynomial phases are very useful in the study of Hilbert transforms along curves, singular integrals supported on lower-dimensional varieties and singular Radon

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transforms (for example, see [3, 4]), recently, there are many works about these operators (see [5-7]).

The purpose of this paper is to consider the multilinear operator related to the oscillatory singular integral defined by

$$T_{A_1, A_2, \cdots, A_k} f(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} \prod_{j=1}^{k} R_{m_j}(A_j; x, y) \frac{\Omega(x-y)}{|x-y|^{n+M}} f(y)\, dy,$$

and the corresponding maximal operator

$$T^*_{A_1, A_2, \cdots, A_k} f(x) = \sup_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} e^{iP(x,y)} \prod_{j=1}^{k} R_{m_j}(A_j; x, y) \frac{\Omega(x-y)}{|x-y|^{n+M}} f(y)\, dy \Big|,$$

where Ω is homogeneous of degree zero and integrable in the unit sphere, $P(x, y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, $R_{m_j}(A_j; x, y)$ denotes the m_j-th Taylor series remainder of A_j at x expanded about y, i.e.,

$$R_{m_j}(A_j; x, y) = A_j(x) - \sum_{|\alpha|<m_j} \frac{D^\alpha A_j(y)}{\alpha!}(x-y)^\alpha.$$

As well-known, operators of this type related to the singular integral operators have been studied by Cohen and Gosselin [8]. In this paper, we will show that the multilinear oscillatory singular integral operators enjoy some properties which are parallel to that of the multilinear singular integral operators. Our main results can be stated as follows.

Theorem 1.1. *Let $\Omega(x)$ be homogeneous of degree zero and belong to $L\log^+ L(S^{n-1})$, $\int_{S^{n-1}} \Omega(x') {x'}^\alpha dx' = 0$ for all $|\alpha| \le M$. Suppose that for $j = 1, 2, \cdots, k$, $A_j(x)$ has derivatives of order m_j in L^{r_j} with $1 < r_j \le \infty$. If $1 < p, q < \infty$ and $1/q = 1/p + \sum_{j=1}^{k} 1/r_j$, then*

$$\|T_{A_1, A_2, \cdots, A_k} f\|_q \le C \|\Omega\|_{L\log^+ L} \prod_{j=1}^{k} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{r_j} \Big) \|f\|_p,$$

and C depends only on n, p and the total degree of P, but not on its coefficients.

Theorem 1.2. *Let Ω and A_j (j=1, 2, $\cdots$, k) satisfy the same conditions as in Theorem 1.1. If $1 < p, q < \infty$ and $1/q = 1/p + \sum_{j=1}^{k} 1/r_j$, then*

$$\|T^*_{A_1, A_2, \cdots, A_k} f\|_q \le C \|\Omega\|_{L\log^+ L} \prod_{j=1}^{k} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{r_j} \Big) \|f\|_p,$$

and C depends only on n, p and the total degree of P, but not on its cofficients.

§2. Some Lemmas

Lemma 2.1.[8] *Let $\Omega(x)$ be homogeneous of degree zero,belong to the space $L\log^+ L(S^{n-1})$, and $\int_{S^{n-1}} {x'}^\alpha \Omega(x') dx' = 0$ for all α with $|\alpha| \le M$. Suppose that for $j = 1, 2, \cdots, k$, $A_j(x)$ has derivatives of order m_j in L^{r_j} with $1 < r_j \le \infty$. Define the operator*

$$\bar{T}_{A_1, A_2, \cdots, A_k} f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+M}} \prod_{j=1}^{k} R_{m_j}(A_j; x, y) f(y)\, dy.$$

If $1 < p,\, q < \infty$ *and* $1/q = 1/p + \sum_{j=1}^{\infty} 1/r_j$, *then*

$$\|\bar{T}_{A_1, A_2, \cdots, A_k} f\|_q \le C\|\Omega\|_{L\log^+ L} \prod_{j=1}^{k} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{r_j} \Big) \|f\|_p,$$

where C *depends only on* $p,\, r_1,\, r_2, \cdots, r_k,\, M$, *and* n. *The same conclusion is also true for the corresponding maximal operator*

$$\tilde{T}^{*}_{A_1, A_2, \cdots, A_k} f(x) = \sup_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^{n+M}} \prod_{j=1}^{k} R_{m_j}(A_j;\, x, y) f(y) dy \Big|.$$

Lemma 2.2.[8] *Let* Ω_0 *be homogeneous of degree zero and* $\Omega_0 \in L^1(S^{n-1})$. *Suppose that for* $j = 1, 2, \cdots, k$, $A_j(x)$ *has derivatives of order* m_j *in* L^{r_j} *with* $1 < r_j \le \infty$. *Define the operator*

$$M^{\Omega_0}_{A_1, A_2, \cdots, A_k} f(x) = \sup_{r>0} r^{-(n+M)} \int_{|x-y|<r} \Big| \prod_{j=1}^{k} R_{m_j}(A_j; x, y) \Omega_0(x-y) f(y) \Big| dy.$$

If $1 < p,\, q < \infty$ *and* $1/q = 1/p + \sum_{j=1}^{k} 1/r_j$, *then*

$$\|M^{\Omega_0}_{A_1, A_2, \cdots, A_k} f\|_q \le C\|\Omega_0\|_1 \prod_{j=1}^{k} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{r_j} \Big) \|f\|_p.$$

Lemma 2.3. *Let* $1 < p < \infty$, Ω_0 *be homogeneous of degree zero and* Ω_0 *belong to the space* $L^\infty(S^{n-1})$. *Let* $P(x, y)$ *be a non-trivial real-valued polynomial which has the form*

$$P(x, y) = \sum_{|\alpha|=k_0,\, |\beta|=l_0} a_{\alpha\beta} x^\alpha y^\beta + \sum_{|\alpha|<k_0,\, |\beta|=l_0} a_{\alpha,\beta} x^\alpha y^\beta + \sum_{|\alpha|\le k_0, |\beta|<l_0} a_{\alpha,\beta} x^\alpha y^\beta,$$

with k_0 *and* l_0 *two positive integers and* $\sum_{|\alpha|=k_0, |\beta|=l_0} |a_{\alpha,\beta}| = 1$. *For* l *a positive integer, define the operator*

$$T^{\Omega_0, l}_{A_1, A_2, \cdots, A_k} f(x) = \int_{2^{l-1}<|x-y|\le 2^l} e^{iP(x,y)} \prod_{j=1}^{k} R_{m_j}(A_j; x, y) \frac{\Omega_0(x-y)}{|x-y|^{n+M}} f(y)\, dy.$$

Suppose that for $j = 1, 2, \cdots, k$, $A_j(x)$ *has derivatives of order* m_j *in* L^∞. *Then there exists a positive constant* ε *which depends only on* k_0 *and* l_0 *such that*

$$\|T^{\Omega_0, l}_{A_1, A_2, \cdots, A_k} f\|_p \le C 2^{-\varepsilon l} \|\Omega_0\|_{L^\infty(S^{n-1})} \prod_{j=1}^{k} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_\infty \Big) \|f\|_p.$$

This Lemma can be proved by the same arguement as that used in [6, pp.209-212], together with some techniques of Cohen and Gosselin [8]. We omit the details for brevity.

Lemma 2.4. *Let* Ω_0 *be homogeneous of degree zero and integrable on the unit sphere,* $b(x, y) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, *and* $A_j(x)$ $(j = 1, 2, \cdots, k)$ *have derivatives of order* m_j *in* L^{r_j} *with* $1 < r_j \le \infty$. *Let* $1 < p,\, q < \infty$ *and* $1/q = 1/p + \sum_{j=1}^{k} 1/r_j$. *Suppose that the operator*

$$Tf(x) = \int_{\mathbb{R}^n} b(x, y) \frac{\Omega(x-y)}{|x-y|^{n+M}} \prod_{j=1}^{k} R_{m_j}(A_j;\, x, y) f(y) dy$$

satisfies

$$\|Tf\|_q \le A\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\|f\|_p.$$

Then the truncated operator

$$T_1f(x)=\int_{|x-y|<1} b(x,y)\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{j=1}^{k}R_{m_j}(A_j;\,x,y)f(y)dy$$

satisfies

$$\|T_1f\|_q \le C(A+\|b\|_\infty)\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\|f\|_p.$$

Proof. For each fixed $h\in\mathbb{R}^n$, we split f into three parts

$$f=f_1+f_2+f_3,$$

where

$$f_1(y)=f(y)\chi_{\{|y-h|<1/2\}}(y),\;\; f_2(y)=f(y)\chi_{\{1/2\le|y-h|<5/4\}}(y).$$

Let $\varphi_h\in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_h\subset\{|y-h|<4\}$, $\varphi_h(y)=1$ if $|y-h|<2$, $\|D^\nu\varphi_h\|_\infty\le C$ for all multi-index ν. Set

$$A_j^h(y)=R_{m_j}(A_j;\,y,h)\varphi_h(y).$$

It is easy to verify that if $|x-h|<1/4$, then

$$T_1f_1(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{j=1}^{k}R_{m_j}(A_j^h;\,x,y)b(x,y)f_1(y)dy,$$

which in turn implies

$$\int_{|x-h|<1/4}|T_1f_1(x)|^qdx\le A^q\Big(\prod_{j=1}^{k}\Big(\sum_{|\beta|=m_j}\|D^\beta A_j^h\|_{r_j}\Big)\|f_1\|_p\Big)^q.$$

For each fixed multi-index β, $|\beta|=m_j$, write

$$D^\beta A_j^h(y)=\sum_{\beta=\nu+\mu}C_{\mu,\nu}R_{m_j-|\mu|}(D^\mu A_j;\,y,h)D^\nu\varphi_h(y).$$

Denote $(y-h)'=(y-h)/|y-h|$. With the aid of the formula

$$R_m(A;\,x,y)=\sum_{|\alpha|=m}\frac{m}{\alpha!}\int_0^{|y-h|}t^{m-1}D^\alpha A(y-t(y-h)')[(y-h)']^\alpha dt,$$

we have

$$|R_{m_j-|\mu|}(D^\mu A_j^h;\,y,h)|\le C\sum_{|\beta|=m_j}\int_0^{|y-h|}|D^\beta A_j(y-t(y-h)')|dt.$$

This leads to

$$\sum_{|\beta|=m_j}\|D^\beta A_j^h\|_{r_j}\le C\sum_{|\beta|=m_j}\Big(\int_{|y-h|<8}|D^\beta A_j(y)|^{r_j}dy\Big)^{1/r_j}.$$

Therefore

$$\int_{|x-h|<1/4}|T_1f_1(x)|^qdx\le A^q\prod_{j=1}^{k}\Big(\sum_{|\beta|=m_j}\int_{|y-h|<8}|D^\beta A_j(y)|^{r_j}dy\Big)^{q/r_j}\|f_1\|_p^q.$$

If $|x-h|<1/4$ and $1/2\le|y-h|<5/4$, then $1/4<|x-y|<3/2$. So we see that for $|x-h|<1/4$,

$$
\begin{aligned}
|T_1f_2(x)| &\le \|b\|_\infty \int_{1/4<|x-y|<5/4}\Big|\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{j=1}^{k}R_{m_j}(A_j;\,x,y)f_2(y)\Big|dy\\
&\le C\|b\|_\infty M^{\Omega}_{A_1^h,\,A_2^h,\cdots,A_k^h}f_2(x).
\end{aligned}
$$

In view of Lemma 2.2, we can deduce that

$$
\int_{|x-h|<1/4}|T_1f_2(x)|^q dx\le C\|b\|_\infty^q\prod_{j=1}^{k}\Big(\sum_{|\beta|=m_j}\int_{|y-h|<8}|D^\beta A_j(y)|^{r_j}dy\Big)^{q/r_j}\|f_2\|_p^q.
$$

Obviously, $T_1f_3(x)=0$ if $|x-h|<1/4$. Hence

$$
\begin{aligned}
&\int_{|x-h|<1/4}|T_1f(x)|^q dx\\
&\le C(A^q+\|b\|_\infty^q)\prod_{j=1}^{k}\Big(\sum_{|\beta|=m_j}\int_{|y-h|<8}|D^\beta A_j(y)|^{r_j}dy\Big)^{q/r_j}\Big(\int_{|y-h|<2}|f(y)|^p dy\Big)^{q/p}.
\end{aligned}
$$

Integrating the last inequality with respect to h yields the desired estimate.

§3. Proofs of Theorems

Proof of Theorem 1.1. We shall carry out our arguement by a double induction on the degrees in x and y of the polynomial. By Lemma 2.1 we know that Theorem 1.1 holds if the polynomial $P(x,y)$ depends only on x or only on y. Let k_0 and l_0 be two positive integers and the polynomial P have degree k_0 in x and l_0 in y. We assume that Theorem 1.1 holds for all polynomials which are sums of monomials of degree less than k_0 in x times monomials of any degree in y, together with monomials which are of degree k_0 in x times monomials which are of degree less than l_0 in y. Write

$$
P(x,y)=\sum_{|\alpha|=k_0,|\beta|=l_0}a_{\alpha\beta}x^\alpha y^\beta+R_0(x,y),
$$

where $R_0(x,y)$ satisfies the above induction assumption. By dilation-invariance we may assume that

$$
\sum_{|\alpha|=k_0,|\beta|=l_0}|a_{\alpha\beta}|=1.
$$

Decompose the operator $T_{A_1,\,A_2,\cdots,\,A_k}$ as

$$
\begin{aligned}
T_{A_1,\,A_2,\cdots,\,A_k}f(x)&=\int_{|x-y|\le1}e^{iP(x,y)}\prod_{j=1}^{k}R_{m_j}(A_j;x,y)\frac{\Omega(x-y)}{|x-y|^{n+M}}f(y)\,dy\\
&\quad+\int_{|x-y|>1}e^{iP(x,y)}\prod_{j=1}^{k}R_{m_j}(A_j;x,y)\frac{\Omega(x-y)}{|x-y|^{n+M}}f(y)\,dy\\
&=T^{\Omega,0}_{A_1,\,A_2,\cdots,\,A_k}f(x)+T^{\Omega,\infty}_{A_1,\,A_2,\cdots,\,A_k}f(x).
\end{aligned}
$$

We first consider the operator $T^{\Omega,\,0}_{A_1,\,A_2,\cdots,\,A_k}$. For each fixed $h\in\mathbb{R}^n$, let $\varphi_h\in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_h\subset\{|y-h|<4\}$, $\varphi_h(y)=1$ if $|y-h|<2$, $\|D^\nu\varphi_h\|_\infty\le C$ for all multi-index ν.

Set

$$A_j^h(y)=R_{m_j}(A_j;\,y,h)\varphi_h(y).$$

Write

$$P(x,y)=\sum_{|\alpha|=k_0,\,|\beta|=l_0}a_{\alpha\beta}(x-h)^\alpha(y-h)^\beta+R(x,y,h),$$

with the induction assumption applying to the polynomial $R(x,y,h)$. We have

$$\begin{aligned}&T^{\Omega,0}_{A_1,\,A_2,\,\cdots\,,\,A_k}f(x)\\&=\int_{|x-y|<1}e^{i[R(x,y,h)+\sum_{|\alpha|=k_0,|\beta|=l_0}a_{\alpha\beta}y^{\alpha+\beta}]}\prod_{j=1}^kR_{m_j}(A_j;x,y)\frac{\Omega(x-y)}{|x-y|^{n+M}}f(y)\,dy\\&\quad+\int_{|x-y|<1}\Big\{e^{iP(x,y)}-e^{i[R(x,y,h)+\sum_{|\alpha|=k_0,|\beta|=l_0}y^{\alpha+\beta}]}\Big\}\prod_{j=1}^kR_{m_j}(A_j;x,y)\frac{\Omega(x-y)}{|x-y|^{n+M}}f(y)\,dy\\&=T^{\Omega,\,01}_{A_1,\,A_2,\,\cdots\,,\,A_k}f(x)+T^{\Omega,\,02}_{A_1,A_2,\cdots,A_k}f(x).\end{aligned}$$

The induction assumption via Lemma 2.4 states that

$$\|T^{\Omega,01}_{A_1,\,A_2,\,\cdots\,,\,A_k}f\|_p\le C\|\Omega\|_{L\log^+L}\prod_{j=1}^k\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\|f\|_p.$$

Note that if $|x-h|<1/4$ and $|x-y|<1$, then $|y-h|<5/4$ and

$$\Big|e^{iP(x,y)}-e^{i[R(x,y,h)+\sum_{|\alpha|=k_0,|\beta|=l_0}a_{\alpha\beta}(y-h)^{\alpha+\beta}]}\Big|\le C\sum_{|\alpha|=k_0,|\beta|=l_0}|a_{\alpha\beta}||x-y|\le C|x-y|.$$

Thus for $|x-h|<1/4$,

$$\begin{aligned}|T^{\Omega,\,02}_{A_1,\,A_2,\cdots,\,A_k}f(x)|&\le\int_{|x-y|\le1}\Big|\prod_{j=1}^kR_{m_j}(A_j;\,x,y)\Big|\frac{|\Omega(x-y)|}{|x-y|^{n+M-1}}|f(y)|\chi_{B(h,\,5/4)}(y)dy\\&\le CM^\Omega_{A_1^h,A_2^h,\cdots,A_k^h}(f\chi_{B(h,5/4)})(x),\end{aligned}$$

which together with Lemma 2.2 yields

$$\|\chi_{B(h,\,1/4)}T^{\Omega,\,02}_{A_1,\,A_2,\,\cdots\,,\,A_k}f\|_q\le\|\Omega\|_1\prod_{j=1}^k\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j^h\|_{r_j}\Big)\|\chi_{B(h,\,5/4)}f(y)\|_p.$$

As in the proof of Lemma 2.4, we can obtain from the last inequality that

$$\|T^{\Omega,\,02}_{A_1,A_2,\cdots,A_k}f\|_q\le C\|\Omega\|_{L^1(S^{n-1})}\prod_{j=1}^k\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\|f\|_p.$$

Combining the estimates for operators $T^{\Omega,\,01}_{A_1,\,A_2,\,\cdots\,,\,A_k}$ and $T^{\Omega,\,02}_{A_1,\,A_2,\,\cdots\,,\,A_k}$ gives the desired estimate for $T^{\Omega,\,0}_{A_1,\,A_2,\,\cdots\,,\,A_k}$.

Now we turn our attention to the operator $T^{\Omega,\,\infty}_{A_1,\,A_2,\,\cdots\,,\,A_k}$. Write

$$\begin{aligned}T^{\Omega,\infty}_{A_1,\,A_2,\,\cdots\,,\,A_k}f(x)&=\sum_{l=1}^\infty\int_{2^{l-1}<|x-y|\le2^l}e^{iP(x,y)}\prod_{j=1}^kR_{m_j}(A_j;x,y)\frac{\Omega(x-y)}{|x-y|^{n+M}}f(y)\,dy\\&=\sum_{l=1}^\infty\sum_{d=0}^\infty\int_{2^{l-1}<|x-y|\le2^l}e^{iP(x,y)}\prod_{j=1}^kR_{m_j}(A_j;x,y)\frac{\Omega_d(x-y)}{|x-y|^{n+M}}f(y)\,dy\end{aligned}$$

$$= \sum_{l=1}^{\infty} \sum_{d=0}^{\infty} T_{A_1, A_2, \cdots, A_k}^{\Omega_d, l} f(x),$$

where

$$\Omega_d(x') = \Omega(x')\chi_{E_d(x')},$$
$$E_0 = \{x' \in S^{n-1};\ |\Omega(x')| < 1\},$$
$$E_d = \{x' \in S^{n-1};\ 2^{d-1} \le |\Omega(x')| < 2^d\}, \quad d \in \mathbb{N}.$$

We claim that there exists a positive constant δ which is independent of d such that for all $l \ge 1$,

$$\|T_{A_1, A_2, \cdots, A_k}^{\Omega_d, l} f\|_q \le C2^{-l\delta} \|\Omega_d\|_\infty \prod_{j=1}^{k} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{r_j} \Big) \|f\|_p. \tag{3.1}$$

In fact, for fixed p, q, r_j $(j = 1, 2, \cdots, k)$ with

$$1 > 1/p = 1/q + \sum_{j=1}^{k} 1/r_j,$$

let $1/q_k = 1/p + 1/r_k$. Choose $\tilde{r}_k$ such that

$$1 < \tilde{r}_k < r_k, \quad \text{and} \quad 1/p + 1/\tilde{r}_k = 1/\tilde{q}_k < 1.$$

Lemma 2.2 now says that

$$\|T_{A_1, A_2, \cdots, A_k}^{\Omega_d, l} f\|_{\tilde{q}_k} \le C \|\Omega_d\|_1 \prod_{j=1}^{k-1} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_\infty \Big) \Big(\sum_{|\alpha|=m_k} \|D^\alpha A_k\|_{\tilde{r}_k} \Big) \|f\|_p. \tag{3.2}$$

On the other hand, it follows from Lemma 2.3 that

$$\|T_{A_1, A_2, \cdots, A_k}^{\Omega_d, l} f\|_p \le C \|\Omega_d\|_\infty 2^{-\varepsilon l} \prod_{j=1}^{k-1} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_\infty \Big) \Big(\sum_{|\alpha|=m_k} \|D^\alpha A_k\|_\infty \Big) \|f\|_p, \tag{3.3}$$

with $\varepsilon > 0$. We view the operator $T_{A_1, A_2, \cdots, A_k}^{\Omega_d, l}$ as a linear operater of A_k. Let $0 < t < 1$ such that $1/r_k = t/\tilde{r}_k$. Then

$$1/q_k = (1-t)/p + t/\tilde{q}_k.$$

Interpolating between the inequalities (3.2) and (3.3) shows that for some positive constant ε_1,

$$\|T_{A_1, A_2, \cdots, A_k}^{\Omega_d, l} f\|_q \le C \|\Omega_d\|_\infty 2^{-\varepsilon_1 l} \|f\|_p \prod_{j=1}^{k-1} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_\infty \Big) \Big(\sum_{|\alpha|=m_k} \|D^\alpha A_k\|_{r_k} \Big). \tag{3.4}$$

Repeating the interpolating proceedure as above k times then establishes our claim.

We can now conclude the proof of Theorem 1.1. Let N be a positive integer such that $N > 2\delta^{-1}$. Write

$$\begin{aligned} \|T_{A_1, A_2, \cdots, A_k}^{\Omega, \infty} f\|_q &\le \sum_{l=1}^{\infty} \|T_{A_1, A_2, \cdots, A_k}^{\Omega_0, l} f\|_q + \sum_{d=1}^{\infty} \sum_{1 \le l \le Nd} \|T_{A_1, A_2, \cdots, A_k}^{\Omega_d, l} f\|_q \\ &\quad + \sum_{d=1}^{\infty} \sum_{l > Nd} \|T_{A_1, A_2, \cdots, A_k}^{\Omega_d, l} f\|_q \\ &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned}$$

By the estimate (3.1),

$$\begin{aligned}
\mathrm{I}_1 &\le C\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\sum_{i=1}^{\infty}2^{-l\delta}\|f\|_p \\
&\le C\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\|f\|_p.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{I}_3 &\le C\sum_{d=1}^{\infty}\sum_{l>Nd}2^{-l\delta}2^d\|f\|_p\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big) \\
&\le C\sum_{d=1}^{\infty}2^{-d(N\delta-1)}\|f\|_p\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big) \\
&\le C\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\|f\|_p.
\end{aligned}$$

On the other hand, it follows from Lemma 2.2 that

$$\begin{aligned}
\mathrm{I}_2 &\le C\sum_{d=1}^{\infty}\sum_{1\le l\le Nd}\|\Omega_d\|_1\|f\|_p\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big) \\
&\le C\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\sum_{d=1}^{\infty}d2^d|E_d|\|f\|_p \\
&\le C\prod_{j=1}^{k}\Big(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{r_j}\Big)\|\Omega\|_{L\log^+ L}\|f\|_p.
\end{aligned}$$

This finishes the proof of Theorem 1.1.

Proof of Theorem 1.2. We shall carry out the argument by a double induction on the degrees in x and y of the polynomial $P(x,y)$ as in the proof of Theorem 1.1. By Lemma 2.1 we know that Theorem 1.2 holds if the polynomial $P(x,y)$ depends only on x or only on y. Let k_0 and l_0 be two positive integers and the polynomial P have degree k_0 in x and l_0 in y. We assume that Theorem 1.2 holds for all polynomials which are sums of monomials of degree less than k_0 in x times monomials of any degree in y, together with monomials which are of degree k_0 in x times monomials which are of degree less than l_0 in y. For general polynomial $P(x,\,y)$ with degree k_0 in x and l_0 in y, write

$$P(x,\,y)=\sum_{|\alpha|=k_0,|\beta|=l_0}a_{\alpha\beta}x^\alpha y^\beta+R(x,\,y),$$

and $R(x,\,y)$ satisfies our induction assumption. We may assume that

$$\sum_{|\alpha|=k_0,|\beta|=l_0}|a_{\alpha\beta}|=1.$$

Decompose the operator $T^*_{A_1, A_2, \cdots, A_k}$ as

$$\begin{aligned}
&T^*_{A_1, A_2, \cdots, A_k} f(x) \\
&\le \sup_{0<\varepsilon<1} \Big| \int_{|x-y|>\varepsilon} e^{iP(x,y)} \frac{\prod_{j=1}^k R_{m_j}(A_j; x, y)\Omega(x-y)}{|x-y|^{n+M}} f(y)dy \Big| \\
&\quad + \sup_{\varepsilon\ge 1} \Big| \int_{|x-y|>\varepsilon} e^{iP(x,y)} \frac{\prod_{j=1}^k R_{m_j}(A; x, y)\Omega(x-y)}{|x-y|^{n+M}} F(y)\,dy \Big| \\
&\le \sup_{0<\varepsilon<1} \Big| \int_{\varepsilon<|x-y|<1} e^{iP(x,y)} \frac{\prod_{j=1}^k R_{m_j}(A_j; x, y)\Omega(x-y)}{|x-y|^{n+M}} f(y)dy \Big| \\
&\quad + \Big| \int_{|x-y|\ge 1} e^{iP(x,y)} \frac{\prod_{j=1}^k R_{m_j}(A_j; x, y)\Omega(x-y)}{|x-y|^{n+M}} f(y)dy \Big| \\
&\quad + \sup_{\varepsilon\ge 1} \Big| \int_{|x-y|>\varepsilon} e^{iP(x,y)} \frac{\prod_{j=1}^k R_{m_j}(A_j; x, y)\Omega(x-y)}{|x-y|^{n+M}} f(y)dy \Big| \\
&= T^{*,0}_{A_1, A_2, \cdots, A_k} f(x) + \Big| \int_{|x-y|\ge 1} e^{iP(x,y)} \frac{\prod_{j=1}^k R_{m_j}(A_j; x, y)\Omega(x-y)}{|x-y|^{n+M}} f(y)dy \Big| \\
&\quad + T^{*,\infty}_{A_1, A_2, \cdots, A_k} f(x).
\end{aligned}$$

By the familiar arguement involving Lemma 2.1 and the induction assumption, it is easy to verify that

$$\|T^{*,0}_{A_1,A_2,\cdots,A_k} f\|_q \le C\|\Omega\|_{L\log^+ L} \prod_{j=1}^k \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{r_j} \Big) \|f\|_p.$$

Observe that

$$\begin{aligned}
&T^{*,\infty}_{A_1, A_2, \cdots, A_k} f(x) \\
&\le \sup_{J\in\mathbb{N}} \int_{2^{J-1}\le|y|<2^J} \frac{|\prod_{j=1}^k R_{m_j}(A_j; x, x-y)\Omega(y)|}{|y|^{n+M}} |f(x-y)|\,dy \\
&\quad + \sup_{J\in\mathbb{N}} \sum_{l=J+1} \Big| \int_{2^{l-1}\le|x-y|<2^l} e^{iP(x,y)} \frac{\prod_{j=1}^k R_{m_j}(A_j; x, y)\Omega(x-y)}{|x-y|^{n+M}} f(y)dy \Big| \\
&\le CM^\Omega_{A_1,A_2,\cdots,A_k} f(x) + \sum_{l=1}^\infty |T^{\Omega,l}_{A_1,A_2,\cdots,A_k} f(x)|.
\end{aligned}$$

The same method as in the proof of Theorem 1.1 leads to

$$\|T^{*,\infty}_{A_1, A_2, \cdots, A_k} f\|_q \le C\|\Omega\|_{L\log^+ L} \prod_{j=1}^k \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{r_j} \Big) \|f\|_p,$$

C depending only on the total degree of P, but not on its coefficients. So we completes the proof of Theorem 1.2.

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