

A GLOBAL ALGORITHM IN THE NUMERICAL RESOLUTION OF THE VISCOUS/INVISCID COUPLED PROBLEM

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Abstract

This paper deals with the spectral approximation of an incompressible viscous/inviscid coupled model. An efficient Uzawa algorithm based on a new variational formulation is proposed. The generalization to the coupling between the Navier-Stokes equations and the Euler equations is discussed.

Keywords Coupled equations, Navier-Stokes equations, Euler equations,
Spectral discretization, Global Uzawa algorithm

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§1. Introduction

The strategy of coupling different mathematical models, as a particular implementation of domain decomposition ideas, allows faster solution of partial differential equations in many cases.

Indeed, in the simulation of the fluid flow past an obstacle for instance, often a complex and expensive model is only needed in a small fraction of domain, outside this region, one can use a simpler and cheaper model where the diffusion effects are negligible. Three major points in the coupled models consist in:

- (i) finding correct conditions on the interfaces separating the viscous and inviscid subdomains;
- (ii) proposing an efficient numerical discretization;
- (iii) choosing an appropriate numerical algorithm.

The first point is essential. The correct interface conditions could guarantee the well-posedness of the coupled problem. We use an artificial regularization technique to find the interface conditions. A spectral method is proposed to approximate the coupled problem based on a global variational formulation. In the existing literature, the numerical algorithm to solve the resultant discrete equations was iteration-by-subdomain resolution (known as the Schwarz alternating algorithm). An effective iterative procedure requires exact convergence analysis and certain numbers of repeat resolutions to reach the convergence, which is often theoretically non trivial and numerically costly. Instead, the coupled technique introduced

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here allows us to solve globally the coupled problem. This global resolution method does not require the convergence analysis of the interface iterative procedure and, on the other hand, avoids repeat computations. It offers potential advantages in regard to the overall computational cost in many cases.

§2. Viscous/Inviscid Coupled Model

We assume that Ω is a bounded, connected, open subset of R^2 , with a Lipschitz boundary $\partial\Omega$; Ω^- and Ω^+ are two open subsets of Ω , with $\Omega^- \cap \Omega^+ = \emptyset$, $\bar{\Omega}^- \cup \bar{\Omega}^+ = \bar{\Omega}$. Let $\Gamma^k = \partial\Omega \cap \partial\Omega^k$, $k = -, +$; $\Gamma = \partial\Omega^- \cap \partial\Omega^+$. \vec{n} is the normal on $\partial\Omega$ to Ω , and \vec{n}^-, \vec{n}^+ the normals on Γ to Ω^-, Ω^+ respectively. For any real number s , we consider the classical Hilbert Sobolev space $H^s(\Omega)$, provided with the usual norm $\|\cdot\|_{s,\Omega}$, and also, when s is an integer, with the semi-norm $|\cdot|_{s,\Omega}$. When s is an integer, we denote by $H^{s-\frac{1}{2}}(\partial\Omega)$ the trace space of $H^s(\Omega)$. The dual space of $H^{\frac{1}{2}}(\partial\Omega)$ is denoted by $H^{\frac{1}{2}}(\partial\Omega)'$. For any integer $s \geq 1$, $H_0^s(\Omega)$ stands for the closure in $H^s(\Omega)$ of the space of infinitely differentiable functions with compact support in Ω . $L_0^2(\Omega) = \{v; v \in L^2(\Omega), \int_{\Omega} v dx = 0\}$.

Throughout this paper, with any function φ defined in Ω , we identify by φ^k the restriction in Ω^k of φ , $k = -, +$. Reciprocally, for the functions φ^k defined in Ω^k , we denote by φ the pair (φ^-, φ^+) . In all that follows, C, C_1, C_2, \dots are generic positive constants independent of the discretization parameters.

Consider first the viscous/inviscid coupled problem: for \vec{f} given in $L^2(\Omega)^2$ and α a positive constant, find two function pairs (\vec{u}^-, p^-) , (\vec{u}^+, p^+) defined in Ω^- and Ω^+ respectively, such that

$$\begin{cases} \alpha \vec{u}^- - \nu \Delta \vec{u}^- + \nabla p^- = \vec{f}^-, & \nabla \cdot \vec{u}^- = 0 \text{ in } \Omega^-, \\ \alpha \vec{u}^+ + \nabla p^+ = \vec{f}^+, & \nabla \cdot \vec{u}^+ = 0 \text{ in } \Omega^+, \end{cases} \quad (2.1)$$

with the boundary conditions $\vec{u}^-|_{\Gamma^-} = 0$, $\vec{u}^+ \cdot \vec{n}|_{\Gamma^+} = 0$. Obviously, appropriate conditions on the interface Γ are required. In order to find them, we introduce the space

$$\mathbf{W} = \{\vec{v}; \vec{v} \in H^1(\Omega)^2, \nabla \cdot \vec{v} = 0, \vec{v}|_{\Gamma^- \cup \Gamma^+} = 0\}$$

and define

$$\mu_{\varepsilon}(x) = \begin{cases} \nu & \text{if } x \in \Omega^-, \\ \varepsilon & \text{if } x \in \Omega^+, \end{cases}$$

where $\varepsilon > 0$ tending to zero.

Now consider global viscous variational problem: Find $\vec{u}_{\varepsilon} \in \mathbf{W}$ such that

$$\alpha(\vec{u}_{\varepsilon}, \vec{v}) + (\mu_{\varepsilon} \nabla \vec{u}_{\varepsilon}, \nabla \vec{v}) = (\vec{f}, \vec{v}), \quad \forall \vec{v} \in \mathbf{W}, \quad (2.2)$$

where (\cdot, \cdot) is the scalar product of $L^2(\Omega)^2$. The following theorem is well known^[6]:

Theorem 2.1. *For all $\varepsilon > 0$, problem (2.2) admits one unique solution.*

It is noted that all solutions of (2.2) satisfy the relationships on the interface Γ :

$$\begin{cases} \nu \frac{\partial \vec{u}_{\varepsilon}^-}{\partial \vec{n}} - p_{\varepsilon}^- \cdot \vec{n}^- = -\varepsilon \frac{\partial \vec{u}_{\varepsilon}^+}{\partial \vec{n}^+} + p_{\varepsilon}^+ \cdot \vec{n}^+ & \text{in the sense of } H^{\frac{1}{2}}(\Gamma)', \\ \vec{u}_{\varepsilon}^- = \vec{u}_{\varepsilon}^+ & \text{on } \Gamma. \end{cases} \quad (2.3)$$

In order to pass to limit in (2.3), we need the following standard estimates.

Lemma 2.1. *There exists a constant $C > 0$ such that, for all $\varepsilon > 0$,*

$$\|\vec{u}_{\varepsilon}\|_{0,\Omega}^2 \leq C, \quad \nu \|\vec{u}_{\varepsilon}^-\|_{1,\Omega^-}^2 \leq C, \quad \varepsilon \|\vec{u}_{\varepsilon}^+\|_{1,\Omega^+}^2 \leq C.$$

Using weak convergence theory and noting that $p^+ \in H^1(\Omega^+)$, we see that Lemma 2.1 gives the following interfaces conditions:

$$\begin{cases} \nu \frac{\partial \vec{u}^-}{\partial \vec{n}^-} - p^- \cdot \vec{n}^- = p^+ \cdot \vec{n}^+ & \text{on } \Gamma, \\ \vec{u}^- \cdot \vec{n}^- = -\vec{u}^+ \cdot \vec{n}^+ & \text{on } \Gamma. \end{cases} \quad (2.4)$$

Consider now the equations (2.1) and (2.4). We use variational method to prove its well-posedness. Define two real Hilbert spaces:

$$\begin{aligned} X &= \{\vec{v}; \vec{v}|_{\Omega^-} \in H^1(\Omega^-)^2, \vec{v}|_{\Omega^+} \in L^2(\Omega^+)^2, \vec{v}|_{\Gamma^-} = 0\}, \\ M &= \{q; q|_{\Omega^-} \in L^2(\Omega^-), q|_{\Omega^+} \in H^1(\Omega^+)\} \cap L_0^2(\Omega) \end{aligned}$$

with respectively the norms

$$\|\vec{v}\|_X = \|\vec{v}^-\|_{1,\Omega^-} + \|\vec{v}^+\|_{0,\Omega^+}, \quad \text{and} \quad \|q\|_M = \|q^-\|_{0,\Omega^-} + |q^+|_{1,\Omega^+}.$$

We write (2.1) and (2.4) under the variational formulation: Find $\vec{u} \times p \in X \times M$, such that

$$\begin{aligned} \alpha(\vec{u}, \vec{v}) + \nu(\nabla \vec{u}^-, \nabla \vec{v}^-)_- - (p^-, \nabla \cdot \vec{v}^-)_- + (\nabla p^+, \vec{v}^+)_+ - (p^+ \cdot \vec{n}^+, \vec{v}^-)_\Gamma &= (\vec{f}, \vec{v}), \quad \forall \vec{v} \in X, \\ (\nabla \cdot \vec{u}^-, q^-)_- - (\vec{u}^+, \nabla q^+)_+ - (\vec{u}^- \cdot \vec{n}^-, q^+)_\Gamma &= 0, \quad \forall q \in M, \end{aligned} \quad (2.5)$$

where $(\cdot, \cdot)_k, (\cdot, \cdot)_\Gamma$ are defined by

$$(\varphi, \psi)_k = \int_{\Omega^k} \varphi \psi, \quad (\varphi, \psi)_\Gamma = \int_\Gamma \varphi \psi, \quad k = -, +.$$

Theorem 2.2. *For all α and ν positive, problem (2.5) has one unique solution in $X \times M$.*

Proof. Define two bilinear forms a and b as follow:

$$\begin{aligned} a(\vec{u}, \vec{v}) &= \alpha(\vec{u}, \vec{v}) + \nu(\nabla \vec{u}^-, \nabla \vec{v}^-)_- \quad \forall \vec{u}, \vec{v} \in X, \\ b(\vec{v}, q) &= -(q^-, \nabla \cdot \vec{v}^-)_- + (\nabla q^+, \vec{v}^+)_+ + (q^+, \vec{v}^- \cdot \vec{n}^-)_\Gamma, \quad \forall \vec{v} \in X, q \in M. \end{aligned}$$

Problem (2.5) can be rewritten in the saddle-point form: Find $\vec{u} \times p \in X \times M$, such that

$$\begin{cases} a(\vec{u}, \vec{v}) + b(\vec{v}, p) = (\vec{f}, \vec{v}), & \forall \vec{v} \in X, \\ b(\vec{u}, q) = 0, & \forall q \in M. \end{cases} \quad (2.6)$$

Applying the saddle-point theory^[6], we prove the theorem by checking the following properties:

(i) First, the continuity and coercivity of the mapping $(\vec{u}, \vec{v}) \mapsto a(\vec{u}, \vec{v})$ in $X \times X$ are trivial.

(ii) The form b is continuous. In fact

$$\begin{aligned} b(\vec{v}, q) &\leq \|q^-\|_{0,\Omega^-} |\vec{v}^-|_{1,\Omega^-} + |q^+|_{1,\Omega^+} \|\vec{v}^+\|_{0,\Omega^+} + \|q^+\|_{0,\Gamma} \|\vec{v}^- \cdot \vec{n}^-\|_{0,\Gamma} \\ &\leq \gamma \|q\|_M \|\vec{v}\|_X, \end{aligned}$$

where γ is a positive constant depending on the continuous trace mapping constant from $H^1(\Omega^-)$ or $H^1(\Omega^+)$ to $H^{1/2}(\Gamma)$.

(iii) The “inf-sup” condition of b in $X \times M$. The “inf-sup” condition of the form

$$b_-(\vec{v}^-, q^-) \stackrel{\text{def.}}{=} (q^-, \nabla \cdot \vec{v}^-)_-$$

in the space pair $H^1(\Omega^-)^2 \times L^2(\Omega^-)$ is given in [2], the one of the form

$$b_+(\vec{v}^+, q^+) \stackrel{\text{def.}}{=} (\nabla q^+, \vec{v}^+)_+ \text{ in } L^2(\Omega^+)^2 \times H^1(\Omega^+)$$

is given in [2]. But for the form b , due to the presence of the interface term $(q^+, \vec{v}^- \cdot \vec{n}^-)_\Gamma$, its proof of compatibility condition in $X \times M$ requires much techniques, which is given below:

Let $q \in M$, decompose q^- by

$$q^- = q_0^- + \tilde{q}^- \quad (2.7)$$

such that $q_0^- \in L_0^2(\Omega^-)$, and \tilde{q}^- is constant in Ω^- . It is known that for $q_0^- \in L_0^2(\Omega^-)$, there exists a $\vec{v}_0^- \in H_0^1(\Omega^-)^2$ such that

$$\nabla \cdot \vec{v}_0^- = -q_0^- \quad \text{and} \quad \|\vec{v}_0^-\|_{1,\Omega^-} \leq \frac{1}{\beta^-} \|q_0^-\|_{0,\Omega^-}, \quad (2.8)$$

where β^- is a positive constant. Now we fix a function $\vec{v}' \in X$ which satisfies $\int_{\Gamma} \vec{v}' \cdot \vec{n}^- = 1$, and let \vec{w}_0 be the solution of the following problem

$$\int_{\Omega^-} (\nabla \cdot \vec{w})q = \int_{\Omega^-} (\nabla \cdot \vec{v}')q \quad \forall q \in L_0^2(\Omega^-). \quad (2.9)$$

If define $\tilde{\vec{v}}^- = \vec{v}' - \vec{w}_0$, then the function $\tilde{\vec{v}}^-$ satisfies

$$\int_{\Omega^-} (\nabla \cdot \tilde{\vec{v}}^-)q = 0, \quad \forall q \in L_0^2(\Omega^-) \quad \text{and} \quad \int_{\Gamma} \tilde{\vec{v}}^- \cdot \vec{n}^- = 1. \quad (2.10)$$

Let $\vec{v}^- = \vec{v}_0^- - \tilde{q}^- \tilde{\vec{v}}^-$. We obtain by using (2.7), (2.8) and (2.10),

$$-\int_{\Omega^-} q^- \nabla \cdot \vec{v}^- = -\int_{\Omega^-} (q_0^- + \tilde{q}^-) \nabla \cdot (\vec{v}_0^- - \tilde{q}^- \tilde{\vec{v}}^-) = \|q_0^-\|_{0,\Omega^-}^2 + \tilde{q}^{-2}. \quad (2.11)$$

In the subdomain Ω^+ , the same decomposition as (2.7) gives

$$q^+ = q_0^+ + \tilde{q}^+ \quad (2.12)$$

with $q_0^+ \in H^1(\Omega^+) \cap L_0^2(\Omega^+)$, and \tilde{q}^+ is constant in Ω^+ . Let $\vec{v}_0^+ = \nabla q_0^+$. Then

$$\frac{\int_{\Omega^+} (\nabla q_0^+) \cdot \vec{v}_0^+}{\|\vec{v}_0^+\|_{0,\Omega^+}} = \|\nabla q_0^+\|_{0,\Omega^+} = |q_0^+|_{1,\Omega^+} = |q^+|_{1,\Omega^+}. \quad (2.13)$$

Let $\vec{z} \in L_0^2(\Omega^+)^2$ such that

$$\int_{\Omega^+} \vec{z} \nabla q = \int_{\Gamma} q (\tilde{\vec{v}}^- \cdot \vec{n}^-), \quad \forall q \in H^1(\Omega^+) \cap L_0^2(\Omega^+). \quad (2.14)$$

Taking $\vec{v}^+ = \vec{v}_0^+ + \tilde{q}^- \vec{z}$ and noting that $\tilde{q}^- |\Omega^-| + \tilde{q}^+ |\Omega^+| = 0$, we have

$$\int_{\Omega^+} (\nabla q^+) \vec{v}^+ + \int_{\Gamma} q^+ (\vec{v}^- \cdot \vec{n}^-) = |q^+|_{1,\Omega^+}^2 + \frac{|\Omega^+|}{|\Omega^-|} \tilde{q}^{+2}, \quad (2.15)$$

where $|\Omega^k|$ is the measure of Ω^k . To estimate \vec{v}^- and \vec{v}^+ , we use (2.8) and obtain

$$\begin{aligned} \|\vec{v}^-\|_{1,\Omega^-} &= \|\vec{v}_0^- - \tilde{q}^- \tilde{\vec{v}}^-\|_{1,\Omega^-} \\ &\leq \frac{1}{\beta^-} \|q_0^-\|_{0,\Omega^-} + C \tilde{q}^- \|\tilde{\vec{v}}^-\|_{1,\Omega^-} \quad (\text{by (2.9) and the definition of } \tilde{\vec{v}}^-) \\ &\leq \frac{C_1}{\beta^-} \|q^-\|_{0,\Omega^-}, \end{aligned}$$

where C, C_1 depend on \vec{w}_0 and \vec{v}' . Using (2.14) we have

$$\|\vec{v}^+\|_{0,\Omega^+} \leq |q_0^+|_{1,\Omega^+} + \tilde{q}^- \|\tilde{\vec{v}}^-\|_{1,\Omega^-} \leq C_2 \|q\|_M.$$

Taking $\vec{v} = (\vec{v}^-, \vec{v}^+)$, we have then $\vec{v} \in X$; furthermore using (2.11) and (2.15), we get

$$\begin{aligned} \frac{b(\vec{v}, q)}{\|\vec{v}\|_X} &= \frac{-\int_{\Omega^-} q^-(\nabla \cdot \vec{v}^-) + \int_{\Omega^+} (\nabla q^+) \cdot \vec{v}^+ + (q^+, \vec{v}^- \cdot \vec{n}^-)_\Gamma}{\|\vec{v}^-\|_{1,\Omega^-} + \|\vec{v}^+\|_{0,\Omega^+}} \\ &\geq \frac{\|q_0^-\|_{0,\Omega^-}^2 + \tilde{q}^{-2} + \frac{|\Omega^+|}{|\Omega^-|} \tilde{q}^{+2} + |q^+|_{1,\Omega^+}^2}{(C_1/\beta^-)\|q^-\|_{0,\Omega^-} + C_2\|q\|_M} \\ &\geq \beta\|q\|_M \end{aligned}$$

with $\beta = \frac{C\beta^-}{1 + \beta^-}$.

§3. Spectral Discretizations and Error Estimations

We approximate the coupled variational problem (2.5) by a spectral method. For the sake of simplicity, consider the domain $\Omega = (-2, 2) \times (-1, 1)$, which is partitioned by $\Omega^- = (-2, 0) \times (-1, 1)$ and $\Omega^+ = (0, 2) \times (-1, 1)$. We notice \mathbb{P}_N , the space of all polynomials of degree $\leq N$, $(\xi_{i,k}^1, \xi_{j,k}^2)$ and w_{ij}^k ($i, j = 0, \dots, N$) denote respectively Gauss-Lobatto points and weights of degree N corresponding to the subdomain Ω^k ($k = -, +$). We introduce the discrete bilinear form

$$\begin{aligned} (\vec{u}, \vec{v})_{k,N} &= \sum_{i,j=0}^N (\vec{u} \cdot \vec{v})(\xi_{i,k}^1, \xi_{j,k}^2) w_{ij}^k, \quad k = -, +, \\ (\vec{u}, \vec{v})_N &= \sum_{k=-,+} (\vec{u}, \vec{v})_{k,N}. \end{aligned}$$

Let $\Xi^k = \{(\xi_{i,k}^1, \xi_{j,k}^2); i, j = 0, \dots, N\}$, $k = -, +$. The following well-known identity and inequality^[2] will be used:

$$\sum_{i,j=0}^N \varphi(\xi_{i,k}^1, \xi_{j,k}^2) w_{ij}^k = \int_{\Omega^k} \varphi dx, \quad \forall \varphi \in \mathbb{P}_{2N-1}(\Omega^k), k = -, +, \quad (3.1)$$

$$\int_{\Omega^k} \varphi^2 dx \leq (\varphi, \varphi)_{k,N} \leq 9 \int_{\Omega^k} \varphi^2 dx, \quad \forall \varphi \in \mathbb{P}_N(\Omega^k), k = -, +. \quad (3.2)$$

We state the following result which can be found in [2].

Lemma 3.1. *There exist projection operators Π_N^k from $L^2(\Omega^k)$ in $\mathbb{P}_N(\Omega^k)$, $k = -, +$, $\Pi_N^{-,1}$ from $\{\vec{v}; \vec{v} \in H^1(\Omega^-), \vec{v}|_\Gamma = 0\}$ in $\{\vec{v}_N; \vec{v}_N \in \mathbb{P}_N(\Omega^-), \vec{v}_N|_\Gamma = 0\}$, $\Pi_N^{+,1}$ from $H^1(\Omega^+)$ in $\mathbb{P}_N(\Omega^+)$ such that*

$$\begin{aligned} \|\varphi - \Pi_N^k \varphi\|_{0,\Omega^k} &\leq CN^{-m} \|\varphi\|_{m,\Omega^k}, \quad \forall \varphi \in H^m(\Omega^k) \quad k = -, +, \quad m \geq 0, \\ \|\varphi - \Pi_N^{-,1} \varphi\|_{1,\Omega^-} &\leq CN^{1-m} \|\varphi\|_{m,\Omega^-}, \quad \forall \varphi \in H^m(\Omega^-) \quad m \geq 1, \\ \|\varphi - \Pi_N^{+,1} \varphi\|_{1,\Omega^+} &\leq CN^{1-m} \|\varphi\|_{m,\Omega^+}, \quad \forall \varphi \in H^m(\Omega^+) \quad m \geq 1. \end{aligned}$$

A classical method of solving coupled problem consists of exhibiting its solution as a limit of solutions of two subproblems within Ω^- and Ω^+ . The effectiveness of this strategy, on one hand, depends on the convergence result of the iterative procedure; on the other hand, requires a certain number of repeat resolution to reach the convergence. But here, we choose

the strategy called “global Uzawa resolution”. We are going to see that this method is very effective to the type of the coupled problem considered here.

We introduce two discrete spaces

$$X_N = X \cap (\mathbb{P}_N(\Omega^-) \times \mathbb{P}_N(\Omega^+)) , \quad M_N = M \cap (\mathbb{P}_{N-2}(\Omega^-) \times \mathbb{P}_N(\Omega^+)) ,$$

and consider the coupled discrete problem: Find $\vec{u}_N \times p_N \in X_N \times M_N$, such that

$$\begin{cases} a_N(\vec{u}_N, \vec{v}_N) + b_N(\vec{v}_N, p_N) = (\vec{f}, \vec{v}_N)_N, & \forall \vec{v}_N \in X_N, \\ b_N(\vec{u}_N, q_N) = 0, & \forall q_N \in M_N, \end{cases} \quad (3.2)$$

where a_N, b_N are two bilinear forms, defined by

$$a_N(\vec{u}_N, \vec{v}_N) = \alpha(\vec{u}_N, \vec{v}_N)_N + \nu(\nabla \vec{u}_N^-, \nabla \vec{v}_N^-)_{-,N}, \quad \forall \vec{u}_N, \vec{v}_N \in X_N, \quad (3.4)$$

$$\begin{aligned} b_N(\vec{v}_N, q_N) &= -(q_N^-, \nabla \cdot \vec{v}_N^-)_{-,N} + (\nabla q_N^+, \vec{v}_N^+)_{+,N} + (q_N^+, \vec{v}_N^- \cdot \vec{n}^-)_{\Gamma,N}, \\ &\quad \forall \vec{v}_N \in X_N, \quad q_N \in M_N, \end{aligned} \quad (3.5)$$

where $(\varphi, \psi)_{\Gamma,N} = \sum_{j=0}^N (\varphi\psi)(\xi_{N,-}^1, \xi_{j,-}^2) w_{Nj}^-$ (or equivalently, $= \sum_{j=0}^N (\varphi\psi)(\xi_{0,+}^1, \xi_{j,+}^2) w_{0j}^+$).

Theorem 3.1. *The discrete problem (3.3) is well posed in the space $X_N \times M_N$.*

Proof. The proof is done, as in Theorem 2.2, by verifying the four properties: continuity and ellipticity of the form a_N ; continuity and compatibility of the form b_N . The three firsts can be proven in a classical way, by using the identity and inequality (3.1) and (3.2). The verification of the “inf-sup” condition of the form b_N is given in Lemma 3.2 below.

Lemma 3.1. *There exists a constant $\beta_N > 0$, possibly depending on N , such that*

$$\inf_{q_N \in M_N} \sup_{\vec{v}_N \in X_N} \frac{b_N(\vec{v}_N, q_N)}{\|\vec{v}_N\|_X \|q_N\|_M} \geq \beta_N. \quad (3.6)$$

Proof. The proof follows the same lines as in the proof of Theorem 2.2. We need only replace the spaces X by X_N , M by M_N , \dots . We ignore the details of the proof, but give the estimation of the “inf-sup” constant β_N ,

$$\beta_N \geq \frac{C\beta_N^-}{1 + \beta_N^-}, \quad (3.7)$$

where β_N^- is local “inf-sup” constant for the viscous part.

Remark 3.1. It has been theoretically proven that the local “inf-sup” constant β_N^- satisfies $\beta_N^- \geq CN^{-1/2}$ and numerical evidences show^[4] a comportment as $O(N^{-1/4})$. Therefore theoretically

$$\beta_N \simeq CN^{-1/2}. \quad (3.8)$$

Define the space \mathbf{V}_N by

$$\mathbf{V}_N = \{\vec{v}_N; \vec{v}_N \in X_N, b_N(\vec{v}_N, q_N) = 0, \forall q_N \in M_N\}. \quad (3.9)$$

The error estimations are given in the following theorem.

Theorem 3.2. *Assume that the solutions of the problem (2.5) satisfy $\vec{u} = (\vec{u}^-, \vec{u}^+) \in H^l(\Omega^-)^2 \times H^{m-1}(\Omega^+)^2$, $p = (p^-, p^+) \in H^{l-1}(\Omega^-) \times H^m(\Omega^+)$, where l and m are real numbers, $l \geq 2$, $m \geq 2$; furthermore, assume $\vec{f} \in H^\sigma(\Omega)^2$, where σ is a real number ≥ 2 .*

Then the approximate solutions of (3.3) $\vec{u}_N = (\vec{u}_N^-, \vec{u}_N^+)$, $p_N = (p_N^-, p_N^+)$ verify

$$\begin{aligned} & \|\vec{u} - \vec{u}_N\|_X + \beta_N \|p - p_N\|_M \\ & \leq C \left[N^{1-l} \left(\frac{1}{\beta_N} \|\vec{u}^-\|_{l, \Omega^-} + \|p^-\|_{l-1, \Omega^-} \right) \right. \\ & \quad \left. + N^{1-m} \left(\frac{1}{\beta_N} \|\vec{u}^+\|_{m-1, \Omega^+} + \|p^+\|_{m, \Omega^+} \right) + N^{-\sigma} \|\vec{f}\|_{\sigma, \Omega} \right]. \end{aligned} \quad (3.10)$$

Proof. It is standard. We have first the following classical result^[2]:

$$\begin{aligned} & \|\vec{u} - \vec{u}_N\|_X + \beta_N \|p - p_N\|_M \\ & \leq C \left[\inf_{\vec{v}_N \in \mathbf{V}_N} \left(\|\vec{u} - \vec{v}_N\|_X + \sup_{\vec{w}_N \in \mathbf{V}_N} \frac{(a - a_N)(\vec{v}_N, \vec{w}_N)}{\|\vec{w}_N\|_X} \right) \right. \\ & \quad \left. + \inf_{q_N \in M_N} \left(\|p - q_N\|_M + \sup_{\vec{w}_N \in \mathbf{V}_N} \frac{(b - b_N)(\vec{w}_N, q_N)}{\|\vec{w}_N\|_X} \right) + \sup_{\vec{w}_N \in \mathbf{V}_N} \frac{(\vec{f}, \vec{w}_N) - (\vec{f}, \vec{w}_N)_N}{\|\vec{w}_N\|_X} \right]. \end{aligned} \quad (3.11)$$

According to the definitions of a, b and a_N, b_N , and by using (3.1), it is easy to see that the second and the fourth terms in the right-hand side of (3.11) vanish for all $\vec{v}_N \in X_{N-1}$, and respectively, for all $q_N \in M_{N-1}$. The last term is bounded^[2],

$$\sup_{\vec{w}_N \in \mathbf{V}_N} \frac{(\vec{f}, \vec{w}_N) - (\vec{f}, \vec{w}_N)_N}{\|\vec{w}_N\|_X} \leq CN^{-\sigma} \|\vec{f}\|_{\sigma, \Omega}. \quad (3.12)$$

Therefore, from (3.11), we get

$$\begin{aligned} & \|\vec{u} - \vec{u}_N\|_X + \beta_N \|p - p_N\|_M \\ & \leq C \left[\inf_{\vec{v}_N \in \mathbf{V}_N \cap X_{N-1}} \|\vec{u} - \vec{v}_N\|_X + \inf_{q_N \in M_{N-1}} \|p - q_N\|_M + N^{-\sigma} \|\vec{f}\|_{\sigma, \Omega} \right]. \end{aligned} \quad (3.13)$$

Noting that

$$\begin{aligned} \inf_{\vec{v}_N \in \mathbf{V}_N} \|\vec{u} - \vec{v}_N\|_X & \leq \frac{C}{\beta_N} \inf_{\vec{w}_N \in X_N} \left(\|\vec{u} - \vec{w}_N\|_X + \sup_{q_N \in M_N} \frac{(b - b_N)(\vec{w}_N, q_N)}{\|q_N\|_M} \right) \\ & \leq \frac{C}{\beta_N} \inf_{\vec{w}_N \in X_{N-1}} \|\vec{u} - \vec{w}_N\|_X \quad (\text{by (3.1)}), \end{aligned} \quad (3.14)$$

we obtain from (3.13)

$$\begin{aligned} & \|\vec{u} - \vec{u}_N\|_X + \beta_N \|p - p_N\|_M \\ & \leq C \left[\frac{1}{\beta_N} \inf_{\vec{v}_N \in X_{N-1}} \|\vec{u} - \vec{v}_N\|_X + \inf_{q_N \in M_{N-1}} \|p - q_N\|_M + N^{-\sigma} \|\vec{f}\|_{\sigma, \Omega} \right] \\ & \leq C \left[\frac{1}{\beta_N} \|\vec{u}^- - \Pi_{N-1}^{-,1} \vec{u}^-\|_{1, \Omega^-} + \frac{1}{\beta_N} \|\vec{u}^+ - \Pi_{N-1}^+ \vec{u}^+\|_{0, \Omega^+} \right. \\ & \quad \left. + \|p^- - \Pi_{N-2}^- p^-\|_{0, \Omega^-} + \|p^+ - \Pi_{N-1}^{+,1} p^+\|_{1, \Omega^+} + N^{-\sigma} \|\vec{f}\|_{\sigma, \Omega} \right]. \end{aligned} \quad (3.15)$$

Finally (3.10) follows by using Lemme 3.1.

§4. Description of the Algorithm

Rewriting the problem (3.3) by expressing $\vec{u}_N, \vec{v}_N, p_N$ and q_N in Lagrangian interpolants, and choosing each test function \vec{v}_N, q_N to be nonzero at only one global collocation point,

we arrive at the following matrix statement:

$$\begin{pmatrix} \mathbb{L}^- & 0 \\ 0 & \mathbb{L}^+ \end{pmatrix} \begin{pmatrix} U^- \\ U^+ \end{pmatrix} + \begin{pmatrix} -(D^-)^T & \mathbb{I}_\Gamma \\ 0 & D^+ \end{pmatrix} \begin{pmatrix} P^- \\ P^+ \end{pmatrix} = \begin{pmatrix} B^- & F^- \\ B^+ & F^+ \end{pmatrix}, \quad (4.1)$$

$$\begin{pmatrix} -(D^-)^T & \mathbb{I}_\Gamma \\ 0 & D^+ \end{pmatrix}^T \begin{pmatrix} U^- \\ U^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.2)$$

In this system, the unknowns $U^k, P^k (k = -, +)$ are the values at the global collocation points of the velocity and the pressure, D^- and $(D^+)^T$ are the discrete divergence operators derived from $(\nabla \cdot \vec{u}_N^-, q_N^-)_{-,N}$ and $(\vec{u}_N^+, \nabla q_N^+)_{+,N}$ respectively, “ T ” denotes the transposition of matrix, B^- and B^+ are the associated mass matrices, \mathbb{I}_Γ denotes the identity operator applied in the normal \vec{n}^- on the interface Γ (under implication of multiplication by the weights corresponding on Γ), $\mathbb{L}^k (k = -, +)$ is defined by $\mathbb{L}^k = \alpha B^k + \nu (D^-)^2 \delta_{-k}$, where

$$\delta_{-k} = \begin{cases} 1, & k = -, \\ 0, & k = +. \end{cases}$$

It is assumed that the boundary conditions in the viscous part are already incorporated into the matrix operators.

We use the global Uzawa procedure to solve discrete equations (4.1)-(4.2). Formally, the system (4.1)-(4.2) can be equivalently replaced by the two separated systems:

$$\begin{aligned} & \begin{pmatrix} -(D^-)^T & \mathbb{I}_\Gamma \\ 0 & D^+ \end{pmatrix}^T \begin{pmatrix} \mathbb{L}^- & 0 \\ 0 & \mathbb{L}^+ \end{pmatrix}^{-1} \begin{pmatrix} -(D^-)^T & \mathbb{I}_\Gamma \\ 0 & D^+ \end{pmatrix} \begin{pmatrix} P^- \\ P^+ \end{pmatrix} \\ &= \begin{pmatrix} -(D^-)^T & \mathbb{I}_\Gamma \\ 0 & D^+ \end{pmatrix}^T \begin{pmatrix} \mathbb{L}^- & 0 \\ 0 & \mathbb{L}^+ \end{pmatrix}^{-1} \begin{pmatrix} B^- & F^- \\ B^+ & F^+ \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} \mathbb{L}^- & 0 \\ 0 & \mathbb{L}^+ \end{pmatrix} \begin{pmatrix} U^- \\ U^+ \end{pmatrix} = \begin{pmatrix} B^- & F^- \\ B^+ & F^+ \end{pmatrix} - \begin{pmatrix} -(D^-)^T & \mathbb{I}_\Gamma \\ 0 & D^+ \end{pmatrix} \begin{pmatrix} P^- \\ P^+ \end{pmatrix}.$$

The advantage of the Uzawa procedure is that the pressure and velocity are completely decoupled in the resolution process. The apparent disadvantage is the equations in the pressure, as the matrix \mathbf{S} ,

$$\mathbf{S} = \begin{pmatrix} -(D^-)^T & \mathbb{I}_\Gamma \\ 0 & D^+ \end{pmatrix}^T \begin{pmatrix} \mathbb{L}^- & 0 \\ 0 & \mathbb{L}^+ \end{pmatrix}^{-1} \begin{pmatrix} -(D^-)^T & \mathbb{I}_\Gamma \\ 0 & D^+ \end{pmatrix} \quad (4.3)$$

will have rank equal to the number of global pressure degrees-of-freedom, and will be full due to the presence of \mathbb{L}^{-1} , where

$$\mathbb{L} = \begin{pmatrix} \mathbb{L}^- & 0 \\ 0 & \mathbb{L}^+ \end{pmatrix}. \quad (4.4)$$

Noting that \mathbf{S} is a positive definite symmetric matrix, we can solve the pressure by an inner/outer conjugate gradient iterative procedure. But an important point to note is that the matrix \mathbb{L} in \mathbf{S} is diagonal by bloc on the interface level, which means that the inner procedure is only needed in the viscous part.

§5. Numerical Results

The numerical test is used to prove the effectiveness of the global iterative method. This is done by considering an exact analytical solution: $u_1(x, y) = \sin \pi y$, $u_2(x, y) = \cos \frac{\pi}{2} x$,

$p(x, y) = x^2 + 0.25y^2$. Fig.1 presents the velocity vectors computed. Fig.2 presents the pressure contourlines. Although the interface conditions are imposed only in a weak form in the variational problem (3.3), the continuity on the interface Γ of both velocity and pressure functions is well shown in these two figures. Fig.3 shows the jumps of the velocity and pressure solutions in two sides of the interface. Fig.4 gives the errors in the velocity and the pressure as a function of polynomial degrees N , which shows that exponential convergence is obtained.

Fig.1 Computational velocity vectors

Fig.2 Computational pressure contourlines

Fig.3 A plot of the jumps
at the Gauss-Lobatto points on Γ Fig.4 A plot of the errors in $L^2(\Omega)$
as a function of polynomial degree N

§6. Discussions

(1) Some comparisons of the computational cost between the viscous/inviscid coupled resolution and the pure viscous (i.e. global Navier-Stokes equations) resolution have been done. The partial results show that the viscous/inviscid coupled model is more economical. We expect further investigation.

(2) The simulation of complex flows will produce a large and full matrix \mathbf{S} . The “simple” nested conjugate gradient algorithm, in this case, would no longer be efficient. One way to recover rapid convergence of the Uzawa algorithm is to precondition \mathbf{S} .

(3) The coupled model could be extended to the study of coupled problem between the full Navier-Stokes equations and the Euler equations:

$$\begin{cases} \frac{\partial \vec{u}^-}{\partial t} + (\vec{u}^- \cdot \nabla) \vec{u}^- - \nu \Delta \vec{u}^- + \nabla p^- = f^-, & \nabla \cdot \vec{u}^- = 0 \quad \text{in } \Omega^- \times (0, T), \\ \frac{\partial \vec{u}^+}{\partial t} + (\vec{u}^+ \cdot \nabla) \vec{u}^+ + \nabla p^+ = f^+, & \nabla \cdot \vec{u}^+ = 0 \quad \text{in } \Omega^+ \times (0, T). \end{cases} \quad (6.1)$$

A difficult point is to treat the non-linear term (convection term) where we could utilize the method of characteristics. Indeed (6.1) can also be rewritten as

$$\begin{cases} \frac{D \vec{u}^-}{Dt} - \nu \Delta \vec{u}^- + \nabla p^- = f^-, & \nabla \cdot \vec{u}^- = 0 \quad \text{in } \Omega^- \times (0, T), \\ \frac{D \vec{u}^+}{Dt} + \nabla p^+ = f^+, & \nabla \cdot \vec{u}^+ = 0 \quad \text{in } \Omega^+ \times (0, T), \end{cases} \quad (6.2)$$

where D/Dt is the total derivative in the direction \vec{u} . We could discretize (6.2) in time by an implicit scheme:

$$\begin{cases} \alpha \vec{u}^{-n+1} - \nu \Delta \vec{u}^{-n+1} + \nabla p^{-n+1} = f^{-n+1} + \alpha \vec{u}^{-n} (\chi^n(\cdot)), & \nabla \cdot \vec{u}^{-n+1} = 0 \quad \text{in } \Omega^-, \\ \alpha \vec{u}^{+n+1} + \nabla p^{+n+1} = f^{+n+1} + \alpha \vec{u}^{+n} (\chi^n(\cdot)), & \nabla \cdot \vec{u}^{+n+1} = 0 \quad \text{in } \Omega^+, \end{cases} \quad (6.3)$$

where $\alpha = 1/\Delta t$, and $\chi^n(x) = \chi(x, (n+1)\Delta t; n\Delta t)$ is the solution of

$$\frac{d\chi}{d\tau} = \vec{u}^n(\chi), \quad \chi(x, t; t) = x. \quad (6.4)$$

The time scheme is unconditionally stable, and each time iteration requires a viscous/inviscid coupled resolution plus a transport of the previous solution on the characteristics.

Several space discretization techniques without dissipation for (6.4) can be found in [7] or [8]. A rigorous error estimation is however not easy because $\nabla \cdot \vec{u}_N$ is not exactly zero and (6.4) can not be integrated exactly. We note however that, on the interface Γ , $\vec{u}^- \cdot \vec{n}^- = \vec{u}^+ \cdot \vec{n}^-$. Thus (6.4) could be solved globally in all domain Ω without any additional interface conditions on Γ .

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