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THE FOURIER SERIES EXPANSIONS OF FUNCTIONS DEFINED ON *S*-SETS

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Abstract

Let E be a compact s-sets of R^n. The authors define an orthonormal system Φ of functions on E and obtain that, for any $f(x) \in L^1(E, \mathcal{H}^s)$, the Fourier series of f, with respect to Φ, is equal to $f(x)$ at $\mathcal{H}^s$-a.e. $x \in E$. Moreover, for any $f \in L^p(E, \mathcal{H}^s)$ $(p \geq 1)$, the partial sums of the Fourier series, with respect to Φ, of f converges to f in L^p-norm.

Keywords Hausdorff measure, Fourier series, s-set, Generalized graph directed construction

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§1. Introduction

In [4, 6], we have studied the convergence of the Fourier series, with respect to an orthonormal system of functions, of each function for any $f(x) \in L^p(K, \mathcal{H}^s)$ $(p \geq 1)$, where K is a Moran fractal or a generalized Moran fractal. In [8], for any $f \in L^p(K_1, \mathcal{H}^s)$, the similar problems are discussed, where K_1 is a self-similar fractal. In this paper, for any $f \in L^p(E, \mathcal{H}^s)$, where E is an arbitrary compact s-set, we can also obtain the similar results.

In §2, we first study the Fourier series expansions of functions defined on differentiable s-sets. We define a system of functions $\Phi \subset L^\infty(E, \mathcal{H}^s)$ and Φ is orthonormal in the Hilbert space $L^2(E, \mathcal{H}^s)$. We show that for any $f(x) \in L^1(E, \mathcal{H}^s)$, the Fourier series of $f(x)$, with respect to Φ, is equal to $f(x)$ at $\mathcal{H}^s-$a.e. $x \in E$ and for any $f \in L^p(E, \mathcal{H}^s)$, the partial sums of the Fourier series of f converges to f in L^p-norm. So the results in [4, 6, 8] are completely contained in the conclusions in this paper.

In §3, as E is an arbitrary compact s-set of R^n, we give the results for the convergence of the Fourier series of functions in $L^p(E, \mathcal{H}^s)$, $1 \leq p \leq \infty$. So, on the problems of the Fourier series expansions of functions defined on s-sets, we give satisfactory solutions in some sense.

In §4, we especially discuss a class of compact s-set produced by generalized graph directed constructions.

Note. A set $E \subset R^n$ is said to be an s-set, if E is $\mathcal{H}^s$-measurable and $0 < \mathcal{H}^s(E) < \infty$, where $\mathcal{H}^s$ denotes s-dimensional Hausdorff measure. For more details about s-sets, see [1] or [2].

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§2. The Fourier Series Expansions of Functions Defined on Differentiable s-Sets

For the sake of completeness, we first give a few definitions (also see [6]).

Definition 2.1. *Let E be an $\mathcal{H}^s$-measurable set. For each $x \in E$, let $\mathcal{B}(x)$ be a collection of bounded $\mathcal{H}^s$-measurable sets with positive measure containing x such that there is at least a sequence $\{U_k\} \subset \mathcal{B}(x)$ with $|U_k| \to 0$ $(k \to \infty)$. The whole collection $\mathcal{B} = \bigcup_{x\in E} \mathcal{B}(x)$ will be called a differentiation basis for $(E, \mathcal{H}^s)$.*

Definition 2.2. *Let $\mathcal{B}$ be a differentiation basis for $(E, \mathcal{H}^s)$. For each measurable set A and for almost every $x \in E$, if $\{U_k\}$ is an arbitrary sequence of $\mathcal{B}(x)$ contracting to x, then*

$$D(E, x) = \lim_{k\to\infty} \frac{\mathcal{H}^s(A \cap U_k)}{\mathcal{H}^s(U_k)} = \mathcal{X}_A(x).$$

We call $\mathcal{B}$ a density basis, where $\mathcal{X}$ is the characteristic function.

Definition 2.3. *Let $\mathcal{B}$ be a differentiation basis for $(E, \mathcal{H}^s)$ and let $f \in L^p(E, \mathcal{H}^s)$ $(1 \le p \le \infty)$. If*

$$\lim_{k\to\infty} \left\{ \frac{1}{\mathcal{H}^s(U_k)} \int_{U_k} f d\mathcal{H}^s \ : \ \{U_k\} \subset \mathcal{B}(x),\ |U_k| \to 0 \right\} = f(x)$$

for almost every $x \in E$, then we shall say that $\mathcal{B}$ differentiates $\int f$. We write $D(\int f, x) = f(x)$.

When $\mathcal{B}$ differentiates $\int f$ for each f in a class X of functions, we shall also say that $\mathcal{B}$ differentiates X.

Definition 2.4. *Given a differentiation basis $\mathcal{B}$ for $(E, \mathcal{H}^s)$, we define the maximal operator associated to the basis $\mathcal{B}$ by*

$$Mf(x) = \sup_{U\in\mathcal{B}(x)} \frac{1}{\mathcal{H}^s(U)} \int_U |f(y)| d\mathcal{H}^s(y) \quad \text{for all} \quad x \in E$$

for every function $f \in L^1(E, \mathcal{H}^s)$.

Definition 2.5. *Let E be an s-set of R^n. We say that E is a differentiable s-set, if the following conditions are satisfied:*

(a) *There exist finite disjoint subsets A_{i_1} of $R^n, i_1 = 1, \dots, m$, such that*

$$E \subset \bigcup_{i_1=1}^{m} A_{i_1}.$$

For each A_{i_1}, there are finite disjoint subsets $A_{i_1 i_2}, 1 \le i_2 \le m_{i_1}, m_{i_1} \in N$, such that

$$A_{i_1 i_2} \subset A_{i_1} \quad \text{and} \quad E \subset \bigcup_{i_1, i_2} A_{i_1 i_2}.$$

In general, as the sets $A_{i_1\cdots i_{k-1}}$ are determined, there are finite disjoint subsets $A_{i_1\cdots i_k}$ such that $A_{i_1\cdots i_k} \subset A_{i_1\cdots i_{k-1}}$ and $E \subset \bigcup_{i_1,\dots,i_k} A_{i_1\dots i_k}$ where $i_1 = 1, \dots, m, 1 \le i_j \le m_{i_1\cdots i_{j-1}}, 1 < j \le k, m_{i_1\cdots i_{j-1}} \in N$.

(b) $|A_{i_1\cdots i_k}| \to 0$ $(k \to \infty)$, *where $|A_{i_1\cdots i_k}|$ denotes the diameter of $A_{i_1\cdots i_k}$.*

(c) $\mathcal{H}^s(E \cap A_{i_1\cdots i_k}) > 0$ $(k \ge 1)$.

$\{A_{i_1\cdots i_k} : 1 \le i_1 \le m, 1 \le i_2 \le m_{i_1}, \dots, 1 \le i_k \le m_{i_1\cdots i_{k-1}}, k \ge 1\}$ is said to be a differentiation cover of E.

Theorem 2.1. *Let E be a differentiable s-set of R^n, and assume that E is local compact. Then*

(a) *there exists a system of functions $\Phi = \{g_n(x)\}_{n\geq 1} \subset L^\infty(E,\mathcal{H}^s)$ such that Φ is orthonormal in the Hilbert space $L^2(E,\mathcal{H}^s)$;*

(b) *for any $f(x) \in L^1(E,\mathcal{H}^s)$,*

$$\sum_{m=1}^{n} \langle f, g_m\rangle g_m \to f(x) \quad at \quad \mathcal{H}^s - a.e.\ x \in E,$$

where $\langle f, g_m\rangle = \int_E f(x) g_m(x) d\mathcal{H}^s(x)$;

(c) *for any $f(x) \in L^p(E,\mathcal{H}^s), 1 \leq p \leq \infty$,*

$$\Big\| \sum_{m=1}^{n} \langle f, g_m\rangle g_m - f \Big\|_p \to 0 \quad (n\to\infty).$$

The proof of Theorem 2.1 consists of the following theorems.

Lemma 2.1. *Suppose that E is a differentiable s-set in R^n and*

$$\{A_{i_1\cdots i_k} \ : \ k\geq 1, 1\leq i_1 \leq m,\ 1\leq i_j \leq m_{i_1\cdots i_{j-1}},\ 1 < j \leq k\}$$

is a differentiation cover of E. Write

$$E_{i_1\cdots i_k} = E \cap A_{i_1\cdots i_k} \quad (k\geq 1), \quad \mathcal{A} = \bigcup_{k\geq 1} \bigcup_{i_1,\cdots,i_k} E_{i_1\cdots i_k},$$

$$\mathcal{A}(x) = \{A \ : \ A\in\mathcal{A}, x\in A\} \quad for\ all \quad x\in E.$$

Then

(a) *$\mathcal{A}$ is a differentiation cover of E;*

(b) *$\mathcal{A}$ is a density basis for $(E,\mathcal{H}^s)$.*

Proof. The proof of (a) is trivial. The proof of (b) can be finished by a method similar to that used in the proof of Theorem 3.3 in [7].

Lemma 2.2. Let E be a local compact subset of R^n and the conditions of Lemma 2.1 are satisfied. Then for any $f \in L^1(E,\mathcal{H}^s)$,

$$D\Big(\int f, x\Big) = \lim_{k\to\infty} \left\{ \frac{1}{\mathcal{H}^s(U_k)} \int_{U_k} f d\mathcal{H}^s \ : \ \{U_k\}\subset \mathcal{A}(x),\ U_k \to x \right\} = f(x) \tag{2.1}$$

at $\mathcal{H}^s$-a.e.$x \in E$.

Proof. Because E is a local compact s-set, and $\mathcal{A}$ is a density basis and Hausdorff measure is regular, the result similar to Theorem 1.4 in [3, Chp. III] is valid after the measure and Lebesgue integral are respectively replaced by the Hausdorff measure and Hausdorff integral. That is, $\mathcal{B}$ differentiates $L^\infty(E,\mathcal{H}^s)$.

For any $f \in L^1(E,\mathcal{H}^s)$ and any $x\in E$, let

$$f_k(x) = \begin{cases} f(x), & \text{if } |f(x)| < k, \\ 0, & \text{if } |f(x)| \geq k, \end{cases}$$

and $f = f_k + f^k$.

Then $D(\int f_k, x) = f_k(x)$ for $\mathcal{H}^s$-a.e. $x\in E$.

For $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathcal{H}^s\Big(\Big\{x \in E \ : \ \Big|D\Big(\int f, x\Big) - f(x)\Big| > \varepsilon\Big\}\Big) \\
&= \mathcal{H}^s\Big(\Big\{x \in E \ : \ \Big|D\Big(\int f^k, x\Big) - f^k(x)\Big| > \varepsilon\Big\}\Big) \\
&\le \mathcal{H}^s\Big(\Big\{x \in E \ : \ D\Big(\int f^k, x\Big) > \varepsilon/2\Big\}\Big) + \mathcal{H}^s(\{x \in E \ : \ f^k(x) > \varepsilon/2\}) \\
&\le \mathcal{H}^s(\{x : Mf^k(x) > \varepsilon/2\}) + \mathcal{H}^s(\{x : f^k(x) > \varepsilon/2\}).
\end{aligned}$$

The second term in the last member of this chain of inequalities tends to zero as $k \to \infty$ by the preceding hypothesis.

On the other hand, without any substantial change in the proof with respect to Theorem 3.4 in [7], we may get that for any $f \in L^1(E, \mathcal{H}^s)$ and every number $\epsilon > 0$,

$$\mathcal{H}^s(x \in E : \ Mf(x) > \epsilon) \le c\frac{\|f\|_1}{\epsilon},$$

where $c > 0$ is a constant independent of ϵ and f.

So we have that $\mathcal{H}^s(\{x \in E \ : \ Mf^k(x) > \varepsilon/2\}) \le 2c\|f^k\|_1/\varepsilon$. But $\|f^k\|_1 \to 0$ as $k \to \infty$, hence $\mathcal{H}^s(\{x \in E \ : \ |D(\int f, x) - f(x)| > \varepsilon\}) = 0$.

Noting the arbitrariness of ε, we may obtain that $D\Big(\int f, x\Big) = f(x)$ at $\mathcal{H}^s -$ a.e. $x \in E$.

The proof is finished.

Now we begin to define a collection of functions with supports on E. The meanings of the following sets $E_{i_1\cdots i_k}$ and E are the same as those in Lemma 2.1.

A function with support on E is defined by

$$g_{-1}(x) = \mathcal{H}^s(E)^{-\frac{1}{2}} \quad \text{for all} \quad x \in E. \tag{2.2}$$

$m-1$ functions $g_0^h, 1 \le h \le m-1$, with supports on the sets $\bigcup\limits_{i_1=1}^{h+1} E_{i_1} \subset E$ are defined as

$$g_0^h(x) = \begin{cases} C_h^{-\frac{1}{2}}, & \text{if} \quad x \in \bigcup\limits_{i_1=1}^{h} E_{i_1}, \\ -C_h^{-\frac{1}{2}}\mathcal{H}^s(E_{h+1})^{-1}\sum\limits_{i_1=1}^{h}\mathcal{H}^s(E_{i_1}), & \text{if} \quad x \in E_{h+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

where $C_h = \mathcal{H}^s(E_{h+1})^{-1}\sum\limits_{i_1=1}^{h}\mathcal{H}^s(E_{i_1})\sum\limits_{i_1=1}^{h+1}\mathcal{H}^s(E_{i_1})$.

Finally, for every $i_1\cdots i_k, k \ge 1$, we define $m_{i_1\cdots i_k}-1$ functions $g_{i_1\cdots i_k}^h, 1 \le h \le m_{i_1\cdots i_k}-1$, whose supports are $\bigcup\limits_{i=1}^{h+1} E_{i_1\cdots i_k i} \subset E_{i_1\cdots i_k}$. They are

$$g_{i_1\cdots i_k}^h(x) = \begin{cases} C_{i_1\cdots i_k h}^{-\frac{1}{2}}\mathcal{H}^s(E_{i_1\cdots i_k})^{-\frac{1}{2}}, & \text{if } x \in \bigcup\limits_{i=1}^{h} E_{i_1\cdots i_k i}, \\ -C_{i_1\cdots i_k h}^{-\frac{1}{2}}\mathcal{H}^s(E_{i_1\cdots i_k})^{-\frac{1}{2}}\mathcal{H}^s(E_{i_1\cdots i_k(h+1)})^{-1}\cdot & \\ \cdot\sum\limits_{i=1}^{h}\mathcal{H}^s(E_{i_1\cdots i_k i}), & \text{if } x \in E_{i_1\cdots i_k(h+1)}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

where $C_{i_1\cdots i_k h} = \mathcal{H}^s(E_{i_1\cdots i_k})^{-1}\mathcal{H}^s(E_{i_1\cdots i_k(h+1)})^{-1}\sum_{i=1}^{h+1}\mathcal{H}^s(E_{i_1\cdots i_k i})\sum_{i=1}^{h}\mathcal{H}^s(E_{i_1\cdots i_k i})$.

Let the system Φ be

$$\begin{aligned}\Phi =&\{g_{-1}\}\cup\{g_0^h : 1\le h\le m-1\}\\ &\cup\{g^h_{i_1\cdots i_k} : k\ge 1, 1\le i_1\le m, 1\le i_j\le m_{i_1\cdots i_{j-1}}, 1<j\le k, 1\le h\le m_{i_1\cdots i_k}-1\}.\end{aligned}\tag{2.5}$$

Since $\mathcal{H}^s(E)<\infty$, it is easy to show that $\Phi\subset L^\infty(E,\mathcal{H}^s)\subset L^p(E,\mathcal{H}^s)$, $p\ge 1$.

Theorem 2.2. *Let E be a differentiable s-set, then there exists a system of functions $\Phi\subset L^\infty(E,\mathcal{H}^s)$ such that Φ is orthonormal in the Hilbert space $L^2(E,\mathcal{H}^s)$.*

Proof. Let Φ be a system of functions in (2.5). Then $\Phi\subset L^\infty(E,\mathcal{H}^s)$. The proof of the orthonormality of Φ is completely similar to Theorem 2.1 in [4]. The proof is finished.

For any $f(x)\in L^1(E,\mathcal{H}^s)$, we define its Fourier series, with respect to Φ, as

$$f(x)\sim a_{-1}g_{-1}(x)+\sum_{h=1}^{m-1}a_0^h g_0^h(x)+\sum_{k=1}^{\infty}\sum_{\substack{1\le i_1\le m,\\ 1\le i_2\le m_{i_1},\\ \cdots\\ 1\le i_k\le m_{i_1\cdots i_{k-1}}}}\sum_{h=1}^{m_{i_1\cdots i_k}-1}a^h_{i_1\cdots i_k}g^h_{i_1\cdots i_k}(x),\tag{2.6}$$

where

$$a_{-1}=\langle f,g_{-1}\rangle=\int_E f(y)g_{-1}(y)\,d\mathcal{H}^s(y),\quad a_0^h=\langle f,g_0^h\rangle=\int_E f(y)g_0^h(y)\,d\mathcal{H}^s(y),$$
$$a^h_{i_1\cdots i_k}=\langle f,g^h_{i_1\cdots i_k}\rangle=\int_E f(y)g^h_{i_1\cdots i_k}(y)\,d\mathcal{H}^s(y),\quad k\ge 1,$$

$1\le h\le m_{i_1\cdots i_k}-1, 1\le i_1\le m, 1\le i_j\le m_{i_1\cdots i_{j-1}}, 1<j\le k$, are the Fourier coefficients of f with respect to Φ.

We denote the partial sums of the Fouier series (2.6) by

$$\begin{aligned}&\mathcal{S}^{j_1\cdots j_{n+1};q}_{n+1}f(x)\\ &=a_{-1}g_{-1}(x)+\sum_{h=1}^{m-1}a_0^h g_0^h(x)+\sum_{k=1}^{n}\sum_{i_1,\dots,i_k}\sum_{h=1}^{m_{i_1\cdots i_k}-1}a^h_{i_1\cdots i_k}g^h_{i_1\cdots i_k}(x)\\ &\quad+\sum_{i_1\cdots i_{n+1}\prec j_1\cdots j_{n+1}}\sum_{h=1}^{m_{i_1\cdots i_{n+1}}-1}a^h_{i_1\cdots i_{n+1}}g^h_{i_1\cdots i_{n+1}}(x)+\sum_{h=1}^{q}a^h_{j_1\cdots j_{n+1}}g^h_{j_1\cdots j_{n+1}}(x),\end{aligned}\tag{2.7}$$

where $n\ge 1, 1\le j_1\le m, 1\le j_2\le m_{j_1},\cdots,1\le j_{n+1}\le m_{j_1\cdots j_n}$, and $1\le q\le m_{j_1\cdots j_{n+1}}-1$, and $i_1\cdots i_{n+1}\prec j_1\cdots j_{n+1}$ means that if there is an $h, 1\le h\le n+1$, such that

$$i_p=j_p,\quad\text{if}\quad 1\le p<h,$$
$$i_h<j_h,$$

we always suppose $i_1\cdots i_k\prec j_1\cdots j_k j_{k+1}, k\ge 1$.

Note. In (2.7) q may be zero. If $q=0$, then the last term in the right side of (2.7) is zero.

By using the similar method used in [4], we may obtain the following lemma.

Lemma 2.3. *The meanings of $E_{i_1\cdots i_k}$ and E are the same as above. For any $n\ge 1, 1\le j_1\le m, 1\le j_k\le m_{i_1\cdots i_{k-1}}, 1<k\le n+1, 1\le q\le m_{j_1\cdots j_{n+1}}-1$, write $\alpha=i_1\cdots i_{n+1}, \beta=$*

$j_1 \cdots j_{n+1}$. *Then for any* $f \in L^1(E, \mathcal{H}^s)$, *we have*

$$\mathcal{S}_{n+1}^{\beta;q} f(x) = \begin{cases} \frac{1}{\mathcal{H}^s(E_{\alpha i})} \int_{E_{\alpha i}} f(y)\, d\mathcal{H}^s(y), & \text{if } \; x \in E_{\alpha i}, \alpha \prec \beta, 1 \le i \le m_{i_1 \cdots i_{n+1}}, \\ \frac{1}{\mathcal{H}^s(E_\beta)} \int_{E_\beta} f(y)\, d\mathcal{H}^s(y) + \frac{1}{\mathcal{H}^s(E_{\beta i})} \int_{E_{\beta i}} f(y)\, d\mathcal{H}^s(y) - & \\ - \frac{1}{\mathcal{H}^s\left(\bigcup\limits_{k=1}^{q+1} E_{\beta k}\right)} \int_{\bigcup\limits_{k=1}^{q+1} E_{\beta k}} f(y)\, d\mathcal{H}^s(y), & \text{if } \; x \in E_{\beta i}, 1 \le i \le q+1, \\ \frac{1}{\mathcal{H}^s(E_\beta)} \int_{E_\beta} f(y)\, d\mathcal{H}^s(y), & \text{if } \; x \in E_{\beta i}, q+1 < i \le m_{j_1 \cdots j_{n+1}}, \\ \frac{1}{\mathcal{H}^s(E_\alpha)} \int_{E_\alpha} f(y)\, d\mathcal{H}^s(y), & \text{if } x \in E_\alpha, \alpha \succ \beta. \end{cases}$$

Using Lemmas 2.2 and 2.3 we immediately obtain the following theorem.

Theorem 2.3. *Let E be a local compact and differentiable s-set. Then for any $f \in L^1(E, \mathcal{H}^s)$ the partial sums of its Fourier series, with respect to Φ, converge to f at $\mathcal{H}^s$-a.e. $x \in E$.*

Corollary. *The system Φ is L^2-complete, i.e. if $f \in L^2(E, \mathcal{H}^s)$ is orthogonal to every function in Φ, then $f(x) = 0$ for $\mathcal{H}^s$*-a.e. $x \in E$.

Proof. Suppose that $f \in L^2(E, \mathcal{H}^s)$ is orthogonal to every function g in Φ, i.e., $\int_E fg d\mathcal{H}^s = 0$. Then it is clear that $\mathcal{S}_{n+1}^{j_1 \cdots j_{n+1};q} f(x) = 0$ for all $x \in E$ and for every $n \ge 1$, $1 \le j_1 \le m$, $1 \le j_2 \le m_{j_1}, \cdots, 1 \le j_{n+1} \le m_{j_1 \cdots j_n}$, $1 \le q \le m_{j_1 \cdots j_{n+1}} - 1$. Then using Theorem 2.3, we have that $f(x) = 0$ at $\mathcal{H}^s - a.e. x \in E$. The proof is finished.

Since Φ is an L^2-complete system, we can obtain the same results as the classic results of the Hilbert spaces.

Theorem 2.4. *If $f(x) \in L^2(E, \mathcal{H}^s)$, and $\{a_k\}_{k \ge 1}$ are its Fourier coefficients with respect to Φ, then*

(a) $\|f\|_2^2 = \sum\limits_{k=1}^{\infty} a_k^2 < \infty$.

(b) $\left\| \mathcal{S}_{n+1}^{j_1 \cdots j_{n+1};q} f - f \right\|_2 \longrightarrow 0$.

(c) *If $F(x) \in L^2(E, \mathcal{H}^s), \{b_k\}_{k \ge 1}$ are its Fourier coefficients with respect to Φ, then*

$$(f, F) = \int\limits_E f(y) F(y)\, d\mathcal{H}^s(y) = \sum_{k=1}^{\infty} a_k b_k.$$

(d) *If $\{b_k\}_{k \ge 1}$ is a sequence of real numbers such that $\sum\limits_{k=1}^{\infty} b_k^2 < \infty$, then there exists a unique function $f \in L^2(E, \mathcal{H}^s)$, so that $\{b_k\}_{k \ge 1}$ are its Fourier coefficients with respect to Φ and f satisfies* (a) *and* (b).

Theorem 2.5. *For convenience, write the system Φ in (2.5) as $\{g_k\}_{k \ge 1}$ and let $1 \le p \le \infty$ and $\{b_k\}_{k \ge 1}$ is a sequence of real numbers which satisfies*

$$\sum_{k=1}^{\infty} |b_k| \, \|g_k\|_p < \infty. \tag{2.8}$$

Then there is a unique function $f \in L^p(E, \mathcal{H}^s)$ so that $\{b_k\}_{k \ge 1}$ are its Fourier coefficients, and

$$\left\| \mathcal{S}_{n+1}^{j_1 \cdots j_{n+1};q} f - f \right\|_p \longrightarrow 0, \tag{2.9}$$

where the meanings of $j_1 \cdots j_{n+1}, q$ and E are the same as above.

Moreover, if $f \in L^p$, its Fourier coefficients $\{a_k\}_{k\geq 1}$ satisfy (2.8)*, then the Fourier series of the function f converges to f in L^p-norm.*

The method used for the proof is similar to Theorem 3.4 in [4].

Therefore, the proof of Theorem 2.1 is finished by Theorem 2.2, Theorem 2.3 and Theorem 2.5.

§3. The Fourier Series Expansions of Functions Defined on Compact s-Sets

Theorem 3.1. *Let E be a compact s-set of R^n. Then* (a), (b) *and* (c) *of Theorem* 2.1 *are satisfied.*

Proof. Let $B_r(x)$ denote the ball of centre x and radius r so that $|B_r(x)| = 2r$. For each $x \in R^n$, write

$$\overline{D}_1^s(E,x) = \limsup_{r\to\infty} \frac{\mathcal{H}^s(E\cap B_r(x))}{(2r)^s}. \tag{3.1}$$

Then using the same steps of proving Corollary 2.5 in [1] we can obtain

$$2^{-s} \leq \overline{D}_1^s(E,x) \leq 1 \tag{3.2}$$

at almost all $x \in E$. We might as well suppose, for any $x \in E$, the inequality (3.2) is satisfied.

Fix $\varepsilon > 0$ with $2^{-s} - \varepsilon > 0$. Then for any $x \in E$, by (3.1) and (3.2), there exists $r_n \downarrow 0$ (means that r_n converges decreasingly to 0) such that

$$2^{-s} \leq \lim_{n\to\infty} \frac{\mathcal{H}^s(E\cap B_{r_n}(x))}{(2r_n)^s} \leq 1 \tag{3.3}$$

and so there exists an N_x such that as $n > N_x$,

$$\mathcal{H}^s(E\cap B_{r_n}(x)) > (2^{-s} - s)(2r_n)^s > 0. \tag{3.4}$$

Without loos of generality, we may suppose all the balls $B_{r_n}(x)$ in (3.3) satisfy the inequality (3.4).

For each $x \in E$, let $\mathcal{B} = \bigcup\limits_{n=1}^{\infty} B_{r_n}(x)$, where $B_{r_n}(x)$ satisfies (3.4) and let $\mathcal{B} = \bigcup\limits_{x\in E} \mathcal{B}(x)$. Then $\mathcal{B}$ is an open cover of E. By the finite covering theorem, there are finite balls $B_{r_1}(x_1), \cdots, B_{r_m}(x_m) \in \mathcal{B}$ such that $E \subset \bigcup\limits_{i=1}^{m} B_{r_i}(x_i)$ and we also assume that no one of them is contained in the other. If let

$$B_1' = B_{r_1}(x_1), B_2' = B_{r_2}(x_2) - B_{r_1}(x_1), \cdots, B_m' = B_{r_m}(x_m) - \Big(\bigcup_{i=1}^{m-1} B_{r_i}(x_i)\Big),$$

then B_i' $(i = 1, \cdots, m)$ are disjoint and $E \subset \bigcup\limits_{i=1}^{m} B_i'$. (Of course, the ways of dividing $\bigcup\limits_{i=1}^{m} B_{r_i}(x_i)$ into finite disjoint sets are not unique, the number of the produced sets may be not equal.)

Write $E_{i_1} = E \cap B_{i_1}'$, $i_1 = 1, \cdots, m$. Then $\{E_{i_1} : i_1 = 1, \cdots, m\}$ are disjoint and $E = \bigcup\limits_{i_1=1}^{m} E_{i_1}$.

For any $x \in \overline{E}_{i_1}$ ($\overline{E}_{i_1}$ denotes the closure of E_{i_1}), let

$$\mathcal{B}_{i_1}(x) = \left\{ B \in \mathcal{B}(x) \ : \ |B| < 2 \min_{1 \le i_1 \le m} \{r_{i_1}\}, x \in B \right\}, \quad \mathcal{B}_{i_1} = \bigcup_{x \in \overline{E}_{i_1}} \mathcal{B}_{i_1}(x).$$

Then $\mathcal{B}_{i_1}$ is an open cover of $\overline{E}_{i_1}$, and so we can choose a finite sub-cover denoted by $\{B_{r_{i_1 i_2}}(x_{i_1 i_2}) \ : \ i_1 = 1, \cdots, m, \ i_2 = 1, \cdots, m'_{i_1}\}$. We can divide $\{B_{r_{i_1 i_2}}(x_{i_1 i_2}) : \ 1 \le i_1 \le m, \ 1 \le i_2 \le m'_{i_1}\}$ into disjoint sets $\{B_{i_1 i_2} \ : \ 1 \le i_1 \le m, \ 1 \le i_2 \le m_{i_1}\}$ such that $\overline{E}_{i_1} \subset \bigcup_{i_2=1}^{m_{i_1}} B'_{i_1 i_2}$ and $B'_{i_1 i_2}$ is a subset of some $B_{r_{i_1 j}}(x_{i_1 j})(1 \le j \le m'_{i_1})$. Let

$$E_{i_1 i_2} = E_{i_1} \cap B'_{i_1 i_2}.$$

Then $E_{i_1 i_2}, \ i_1 = 1, \cdots, m, i_2 = 1, \cdots, m_{i_1}$, are disjoint and

$$E_{i_1 i_2} \subset E_{i_1}, \quad E_{i_1} = \bigcup_{i_2} E_{i_1 i_2}, \quad E = \bigcup_{i_1, i_2} E_{i_1, i_2}.$$

For any $x \in \overline{E}_{i_1 i_2}$, let

$$\mathcal{B}_{i_1 i_2}(x) = \left\{ B \in \mathcal{B}(x) \ : |B| < 2 \min_{i_1, i_2} (r_{i_1 i_2}), x \in B \right\}, \quad \mathcal{B}_{i_1 i_2} = \bigcup_{x \in \overline{E}_{i_1 i_2}} \mathcal{B}_{i_1 i_2}(x).$$

Similarly, we can obtain finite disjoint sets $B'_{i_1 i_2 i_3}$ and by letting

$$E_{i_1 i_2 i_3} = E_{i_1 i_2} \cap B'_{i_1 i_2 i_3},$$

we get a cover $\{E_{i_1 i_2 i_3} \ : \ 1 \le i_1 \le m, 1 \le i_2 \le m_{i_1}, 1 \le i_3 \le m_{i_1 i_2}\}$ of E such that $\{E_{i_1 i_2 i_3}\}$ are disjoint and

$$E_{i_1 i_2 i_3} \subset E_{i_1 i_2}, \quad E_{i_1 i_2} = \bigcup_{i_3} E_{i_1 i_2 i_3}, \quad E = \bigcup_{i_1, i_2, i_3} E_{i_1 i_2 i_3}.$$

The rest may be deduced by analogy.

In general, we obtain finite disjoint sets $E_{i_1 \cdots i_k}$ such that

$$E_{i_1 \cdots i_k} \subset E_{i_1 \cdots i_{k-1}}, \quad E_{i_1 \cdots i_{k-1}} = \bigcup_{i_k} E_{i_1 \cdots i_k}, \quad E = \bigcup_{i_1, \ldots, i_k} E_{i_1 \cdots i_k}, \tag{3.5}$$

where $k > 1, 1 \le i_1 \le m, 1 \le i_j \le m_{i_1 \cdots i_{j-1}}, 1 < j \le k$.

We might as well suppose $\mathcal{H}^s(E_{i_1 \cdots i_k}) > 0$ $(k \ge 1)$. (If not, we shall give a detailed explanation later in the remark.)

By using the definitons of (2.2), (2.3) and (2.4), we may obtain a system of functions $\Phi \subset L^\infty(E, \mathcal{H}^s)$ and Φ is orthonormal in the Hilbert space $L^2(E, \mathcal{H}^s)$. (Of course, $E_{i_1 \cdots i_k}$ in the definitions means those in (3.5).)

In addition, we can see that $|E_{i_1 \cdots i_k}| \to 0 (k \to \infty)$ from the preceding process. So when E is a compact s-set, we can also obtain the same results as Theorem 2.1 by using Theorem 2.2, Theorem 2.3 and Theorem 2.5.

The proof is finished.

Remark. If $E_{i_1 \cdots i_k}$ chosen in the proof of Theorem 3.1 satisfies $\mathcal{H}^s(E_{i_1 \cdots i_k}) = 0$, then we shall not consider this set. Finally, we obtain a subset of E denoted by E_0 and a sequence of sets $E'_{i_1 \cdots i_k}$ such that $\{E'_{i_1 \cdots i_k} \ : \ 1 \le i_1 \le n, 1 \le i_2 \le n_{i_1}, \ldots, 1 \le i_k \le n_{i_1 \cdots i_{k-1}}\}$ are disjoint and

$$\mathcal{H}^s(E'_{i_1\cdots i_k}) > 0, \quad E_0 = \bigcup_{i_1,\cdots,i_k} E'_{i_1\cdots i_k}, \quad E'_{i_1\cdots i_k} \subset E'_{i_1\cdots i_{k-1}},$$
$$E_{i_1\cdots i_{k-1}} = \bigcup_{i_k} E'_{i_1\cdots i_k}, \quad \mathcal{H}^s(E_0) = \mathcal{H}^s(E).$$

Now we define a sequence of functions on E as

$$g_{-1}(x) = \mathcal{H}^s(E)^{-\frac{1}{2}} \text{ for all } \quad x \in E,$$

$$g_0^h(x) = \begin{cases} C_h^{-\frac{1}{2}}, & \text{if } \quad x \in \bigcup_{i_1=1}^{h} E'_{i_1}, \\ -C_h^{-\frac{1}{2}} \mathcal{H}^s(E'_{h+1})^{-1} \sum_{i_1=1}^{h} \mathcal{H}^s(E'_{i_1}), & \text{if } \quad x \in E'_{h+1}, \\ 0, & \text{otherwise}, \end{cases}$$

where $C_h = \mathcal{H}^s(E'_{h+1})^{-1} \sum_{i_1=1}^{h} \mathcal{H}^s(E'_{i_1}) \sum_{i_1=1}^{h+1} \mathcal{H}^s(E'_{i_1}), 1 \le h \le n-1.$

$$g_\alpha^h(x) = \begin{cases} C_{\alpha h}^{-\frac{1}{2}} \mathcal{H}^s(E'_\alpha)^{-\frac{1}{2}}, & if \ x \in \bigcup_{i=1}^{h} E'_{\alpha i}, \\ -C_{\alpha h}^{-\frac{1}{2}} \mathcal{H}^s(E'_\alpha)^{-\frac{1}{2}} \mathcal{H}^s(E'_{\alpha(h+1)})^{-1} \sum_{i=1}^{h} \mathcal{H}^s(E'_{\alpha i}), & \text{if } x \in E'_{\alpha(h+1)}, \\ 0, & otherwise, \end{cases}$$

where $\alpha = i_1 \cdots i_k,\ 1 \le i_1 \le n, 1 \le i_j \le n_{i_1\cdots i_{j-1}}, 1 < j \le k, 1 \le h \le n_{i_1\cdots i_k} - 1$ and

$$C_{\alpha h} = \mathcal{H}^s(E'_\alpha)^{-1} \mathcal{H}^s(E'_{\alpha(h+1)})^{-1} \sum_{i=1}^{h+1} \mathcal{H}^s(E'_{\alpha i}) \sum_{i=1}^{h} \mathcal{H}^s(E'_{\alpha i}).$$

It is easy to show that

$$\{g_{-1}\} \cup \{g_0^h : 1 \le h \le n-1\}$$
$$\cup \{g^h_{i_1\cdots i_k} : k \ge 1, 1 \le i_1 \le n, 1 \le i_j \le n_{i_1\cdots i_{j-1}}, 1 < j \le k, 1 \le h \le n_{i_1\cdots i_k} - 1\}$$
$$\subset L^\infty(E, \mathcal{H}^s).$$

By using the similar to preceding steps and noting $\mathcal{H}^s(E - E_0) = 0$, we can show that Theorem 3.1 is always valid.

§4. Generalized Ratios Graph Directed Constructions

A generalized ratio graph directed construction in R^m consists of

(1) a finite sequence of nonoverlapping, compact subsets of R^m: $J_1, J_2, \cdots, J_n$ such that each J_i has a nonempty interior,

(2) a sequence of directed graph $\{G_k\}$ with vertex set consisting of the integers $1, \cdots, n$, and contract maps $T^{(k)}_{i,j}$ of $\mathbf{R}^m$, where $(i,j) \in G_k$,with contract ratios no more than $t^{(k)}_{i,j}$, such that

(a) for each k and $i, 1 \le i \le n$, there is some j such that $(i,j) \in G_k$,

(b) for each k and i, $\{T^{(k)}_{i,j}(J_j) | (i,j) \in G_k\}$ is a nonoverlapping family and

$$J_i \supset \bigcup \{T^{(k)}_{i,j}(J_j) | (i,j) \in G_k\} \tag{4.1}$$

and

(c) if the path component from G_1 to G_k rooted at the vertex i_1 is a cycle: $[i_1, \cdots, i_q, i_{q+1} = i_1]$, then

$$\prod_{k=1}^{q} t_{i_k, i_{k+1}} < 1. \tag{4.2}$$

This construction naturally determines a compact subset K of $\mathcal{R}^m$. This set, which we will term the construction object, is pieced together by the graphs G_k and applying the maps coded by the edges to the corresponding sets.

For each i, let $\mathcal{R}(J_i)$ be the space of compact subsets of J_i provided with the Hausdorff metric, ρ_H. By using the similar method of R.D.Mauldin et al.[5], we may show the following theorem.

Theorem 4.1. For each generalized graph directed construction, there exists a unique compact set K,

$$K = \bigcap_{m \geq 1} \bigcup \{T^{(1)}_{i_1, i_2} \circ \cdots \circ T^{(m)}_{i_m, i_{m+1}}(J_{i_{m+1}}) \mid (i_j, i_{j+1}) \in G_j, \quad 1 \leq j \leq m\}. \tag{4.3}$$

Let

$$G(p) = \{\sigma(1)\sigma(2)\cdots\sigma(p+1) | (\sigma(i), \sigma(i+1)) \in G_i; 1 \leq i \leq p\},$$
$$G(\infty) = \{\sigma(1)\sigma(2)\cdots | (\sigma(i), \sigma(i+1)) \in G_i; i \geq 1\}, \quad G^* = \bigcup_{p \geq 1} G(p),$$

for $\sigma \in G(\infty)$, $\sigma|p = \sigma(1)\cdots\sigma(p+1) \in G(p)$.

$$t_{\sigma|p} = \prod_{i=1}^{p} t^{(i)}_{\sigma(i), \sigma(i+1)}. \tag{4.4}$$

$$J(\sigma|p) = T^{(1)}_{\sigma(1), \sigma(2)} \circ T^{(2)}_{\sigma(2), \sigma(3)} \circ \cdots \circ T^{(p)}_{\sigma(p), \sigma(p+1)}(J_{\sigma(p+1)}). \tag{4.5}$$

Then

$$K = \bigcap_{p \geq 1} \bigcup_{\sigma \in G(p)} J(\sigma). \tag{4.6}$$

It is easy to see that the generalized graph directed construction object K includes the Moran fractals, the generalized Moran fractals, the self-affine sets and graph directed construction. By Theorem 3.1, we have

Theorem 4.2. *If the generalized graph directed construction K is an s-set, and $f \in L^1(E, \mathcal{H}^s)$, then the Fourier expansion theorem is true.*

It is difficulty to prove that the generalized graph directed construction K is an s-set in general case. Now we give a class of generalized graph directed constructions for which K is an s-set.

Example Let G be a directed graph with vertex set consisting of the integers $1, 2, \cdots, n$, and $T^{(1)}_{i,j}, T^{(2)}_{i,j}$ are similarity maps of R^m with similarity ratios $t^{(1)}_{i,j}, t^{(2)}_{i,j}$, respectively, where $(i, j) \in G$.

A sequence of similarity maps $\{\{T^{(k)}_{i,j}\}_{(i,j) \in G}\}$ is produced by $\{T^{(1)}_{i,j}\}_{(i,j) \in G}, \{T^{(2)}_{i,j}\}_{(i,j) \in G}$, in non-periodic form. Let

$$N(k) = \#\{h : \{T^{(h)}_{i,j}\}_{(i,j) \in G} = \{T^{(1)}_{i,j}\}_{(i,j) \in G};\ h \leq k\}, \tag{4.7}$$

$$a_k = \frac{N(k)}{k}. \tag{4.8}$$

The weighted incidence matrix or construction matrix $A^{(k)} = A_G^{(k)}$ associated with a graph directed construction is the $n \times n$ matrix defined by

$$A^{(k)} = [t_{i,j}^{(k)}]_{i,j \le n}, \tag{4.9}$$

where we make the convention that $t_{i,j}^{(k)} = 0$ if $(i,j) \notin G$. For each $\beta \ge 0$, let $A_\beta^{(k)} = A_{G,\beta}$ be the $n \times n$ matrix given by $(t_{i,j}^{(k)})^\beta$. Also, let $\Phi^{(k)}(\beta)$ be the spectral radius of $A_\beta^{(k)}$. Of course, according to the Frobenius-Perron theorem, $\Phi^{(k)}(\beta)$ is the largest nonnegative eigenvalue of $A_\beta^{(k)}$. Let

$$\Phi(\beta) = (\Phi^{(1)}(\beta))^a (\Phi^{(2)}(\beta))^{1-a}. \tag{4.10}$$

Theorem 4.3. *If $G_k = G$ itself is strongly connected, and satisfies:*
(1) $\sup\limits_{k \ge 1} k|a - a_k| < c < \infty$, (2) $t_{i,j}^{(1)} = r t_{i,j}^{(2)}$, *for any* $(i,j) \in G\, r < 1$,
then the Hausdorff dimension of K is α, where $\Phi(\alpha) = 1$, and K is an α-set.

Proof. It is known the $\Phi^{(k)}(\beta)$ is continuous, $\Phi(\beta)$ is continuous, too. By Theorem 2 in [5], $\Phi^{(k)}(0) > 1$, and $\lim\limits_{\beta \to \infty} \Phi^{(k)}(\beta) = 0$. So, there exists a real number α such that $\Phi(\alpha) = 1$. Since $A_\alpha^{(k)}$ is irreducible, by the Frobenius-Perron theorem, there is a unique strictly positive column vector V,

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{4.11}$$

with $\sum\limits_{i=1}^{n} v_i = 1$ and $A_\alpha^{(k)} V = \Phi^{(k)}(\alpha) V$, i.e. for each i,

$$v_i = \sum_{j=1}^{n} \frac{(t_{i,j}^{(k)})^\alpha}{\Phi^{(k)}(\alpha)} v_j = \sum_{(i,j)\in G} \frac{(t_{i,j}^{(k)})^\alpha}{\Phi^{(k)}(\alpha)} v_j. \tag{4.12}$$

Let $w_\sigma = \prod\limits_{i=1}^{|\sigma|-1} w_{\sigma(i),\sigma(i+1)}$, $w_{\sigma(k-1),\sigma(k)} = (\Phi^{(k)})^{-1}$. Then $c_1^{-1} \le w_\sigma \le c_1$, where $c_1 = \left(\frac{\Phi^{(1)}}{\Phi^{(2)}}\right)^c + \left(\frac{\Phi^{(2)}}{\Phi^{(1)}}\right)^c$.

Define a probability measure $\widehat{\mu}$ on $G(\infty)$ by setting for each $\sigma \in G^*$,

$$\widehat{\mu}(C(\sigma)) = w_\sigma t_\sigma^\alpha v_{\sigma(|\sigma|)}, \tag{4.13}$$

where

$$C(\sigma) = \{\tau \in G(\infty) : \tau||\sigma| = \sigma\}. \tag{4.14}$$

To see that Kolmogorov's consistency theorem may be applied it is sufficient to note that if $\sigma \in G^*$, then

$$\begin{aligned} \sum_{(\sigma(|\sigma|),j)\in G} \widehat{\mu}(C(\sigma * j)) &= \sum_{(\sigma(|\sigma|),j)\in G} w_{\sigma*j} t_{\sigma*j}^\alpha v_j \\ &= w_\sigma t_\sigma^\alpha \sum_{(\sigma(|\sigma|),j)\in G} w_{\sigma(|\sigma|),j} t_{|\sigma|,j}^\alpha v_j \\ &= w_\sigma t_\sigma^\alpha v_{\sigma(|\sigma|)} = \widehat{\mu}(C(\sigma)). \end{aligned}$$

First, we show that $\mathcal{H}^\alpha(K) < +\infty$. For each p, we have

$$\sum_{\sigma\in G(p)} |J_\sigma|^\alpha = \sum_{\sigma\in G(p)} t_\sigma^\alpha |J_{(|\sigma|)}|^\alpha$$

and since V is strictly positive,

$$\sum_{\sigma\in G(p)} \widehat{\mu}(C(\sigma))|J_{(|\sigma|)}|^\alpha/w_\sigma v_{\sigma(|\sigma|)} = \sup\{|J_{(|\sigma|)}|^\alpha/\{w_\sigma v_{\sigma(|\sigma|)}\}\} \sum_{\sigma\in G(p)} \widehat{\mu}(C(\sigma))$$
$$\leq c_1 \sup\{|J_i|^\alpha/v_i\} < +\infty.$$

By the similar methods in [5], we have

$$\limsup\{|J_\sigma||\sigma\in G(p)\} = 0. \tag{4.15}$$

Thus

$$\mathcal{H}^\alpha(K) \leq c_1 \sup\{|J_i|^\alpha/v_i\} < +\infty. \tag{4.16}$$

In order to show $0 < \mathcal{H}^\alpha(K)$, transfer $\widehat{\mu}$ to a probability measure on K. Let g be the map of $G(\infty)$ into R^m defined for each $\sigma \in G(\infty)$, by $\{g(\sigma)\} = \bigcap_{k=1}^{\infty} J_{\sigma|k}$. Then g is a continuous map of $G(\infty)$ onto K (see [5]). Let $\mu = \widehat{\mu}\circ g^{-1}$. We will show that there is some $c > 0$ such that if E is a Borel subset of R^d with $\mathrm{diam}E < \inf\{|J_i|\}$, then

$$\mu(E) \leq c|E|^\alpha. \tag{4.17}$$

Of course, this inequality implies $\frac{1}{c} \leq \mathcal{H}^\alpha(K)$.

Set $B = \{\sigma_i|k_i \in G^*;\ |J_{\sigma_i|k_i}| \leq |E| \leq |J_{\sigma_i|k_i-1}|$ and $E \cap J_{\sigma_i|k_i} \neq \emptyset\}$. Then

$$\mu(E) \leq \sum_{\sigma_i|k_i\in B} \widehat{\mu}(C(\sigma_i|k_i)) \leq \#B \sup_{\sigma_i|k_i\in B} w_\sigma t_{\sigma_i}^\alpha v_{\sigma_i(k_i)}$$
$$\leq \#B \sup_{\sigma_i|k_i\in B} c_1|E|^\alpha v_{\sigma_i(k_i)}/|J_{\sigma_i(k_i)}| \leq \#Bc_1|E|^\alpha \sup_{1\leq i\leq n} v_i/|J_i|.$$

By Lemma V in [5], $c_2 = \#Bc_1 \sup_{1\leq i\leq n} v_i/|J_i| < \infty$.

Therefore, (4.17) holds and Theorem 4.2 follows.

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