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# ON LARGE INCREMENTS OF $l^p$-VALUED GAUSSIAN PROCESSES**

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**Abstract**

Let $\{X_k(t), t \geq 0\}, k = 1, 2, \ldots,$ be a sequence of independent Gaussian processes with $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$. Put $\sigma(p,h) = (\sum\limits_{k=1}^{\infty} \sigma_k^p(h))^{1/p},\ p \geq 1$. The author establishes the large increment results for bounded $\sigma(p,h)$.

**Keywords** $l^p$-valued infinite dimensional Gaussian process, Large increment, a.s. limit
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## §1. Introduction

Let $\{Y(t), t \geq 0\} = \{X_k(t), t \geq 0\}_{k=1}^{\infty}$ be a sequense of independent Gaussian processes with $EX_k(t) = 0$ and stationary increments $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$, where, and throughout this paper, $\sigma_k(h)$ is assumed to be a non-decreasing continuous function for each $k \geq 1$. Then $Y(t+h) - Y(t) \in l^p$, $1 \leq p < \infty$, almost surely for fixed $t$ and $h$ if and only if $\sigma(p,h) < \infty$, where

$$\sigma(p,h) = \Big(\sum_{k=1}^{\infty} \sigma_k^p(h)\Big)^{\frac{1}{p}}, \quad p \geq 1.$$

Csörgő and Shao[2] studied almost sure path behaviour for $\{Y(t), t \geq 0\}$ based on a general result for Fernique type inequality[1] and the well-known Borell inequality. In particular, they established moduli of continuity for this kind of processes. As to large increments, the condition that $\sigma(p,h)/h^{\alpha}$ is quasi-increasing for some $\alpha > 0$ is required. But for some Gaussian processes this condition is not satisfied. For example, $\sigma(p,h)$ is bounded for the Ornstein-Uhlenbeck processes. The aim of this paper is to establish the large increment results for $\{Y(t), t \geq 0\}$ with bounded $\sigma(p,h)$.

Let $\{X(t), t \geq 0\}$ be a stationary Gaussian process with $EX(t) = 0$ and stationary increment $\sigma^2(h) = E(X(t+h) - X(t))^2$. In [3], we investigated its large increment properties. Suppose that $\sigma(h)$ is non-decreasing and

$$EX(h)X(0) \to 0, \quad \text{as} \quad h \to \infty,$$

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and

$$\int_1^\infty \sigma(e^{-x^2})\,dx < \infty.$$

Let $a_T$ be a function of $T$ with $0 < a_T \le T$ and $a_T \to \infty$ as $T \to \infty$. Suppose that $a_T = o(T^\epsilon)$ for any $\epsilon > 0$ as $T \to \infty$. Then

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|X(t+s)-X(t)|}{2\sigma_0(\log T)^{1/2}} = 1 \quad \text{a.s.},$$

$$\lim_{t\to\infty} \sup_{0\le t\le T-a_T} \frac{|X(t+a_T)-X(t)|}{2\sigma_0(\log T)^{1/2}} = 1 \quad \text{a.s.},$$

$$\limsup_{T\to\infty} \frac{|X(T+a_T)-X(T)|}{2\sigma_0(\log T)^{1/2}} = 1 \quad \text{a.s.},$$

where $\sigma_0^2 = EX^2(0)$.

We now establish the similar result for $l^p$-valued Gaussian process $\{Y(t), t \ge 0\}$. The above condition for $\{X(t), t \ge 0\}$ will be a special case of our theorem, and besides, the condition $a_T = o(T^\epsilon)$ for any $\epsilon > 0$ as $T \to \infty$ can be relaxed.

## §2. A Fernique Type Inequality

In order to prove our main result, we give a version of the Fernique type inequality of [1].

**Lemma 2.1.** *Let $\mathcal{B}$ be a seperable Banach space with norm $\|.\|$ and let $\{\Gamma(t), t \ge 1\}$ be a stochastic process with values in $\mathcal{B}$. Let $P$ be the probability measure generated by $\Gamma(\cdot)$. Assume that $\Gamma(\cdot)$ is $P$-almost surely continuous with respect to the norm and that for any $t \ge 0, h \ge 0, 0 < x^* \le x$ there exist non-negative monotone non-decreasing functions $\sigma_1(h)$ and $\sigma_2(h)$ such that*

$$P\{\|\Gamma(t+h)-\Gamma(t)\| \ge x\sigma_1(h)+\sigma_2(h)\} \le K\exp(-\gamma x^\beta) \tag{2.1}$$

*with some $K, \gamma, \beta > 0$. Then*

$$\begin{aligned} P\{&\sup_{0\le t\le T}\sup_{0\le s\le h} \|\Gamma(t+s)-\Gamma(t)\| \ge x(\sigma_1(h+d(k)^{-1}h) \\ &+\sigma_1(h,k)) + \sigma_1^*(h,k) + \sigma_2(h+d(k)^{-1}h) + \sigma_2(h,k)\} \\ &\le 4K(\frac{T}{h}+1)d(k)^2\exp(-\gamma x^\beta) \end{aligned} \tag{2.2}$$

*for any $T \ge 0$, $0 \le h \le T$, $x \ge x^*$ and $k > 0$, where*

$$d(k) = 2^{2^{2^k}},$$

$$\sigma_1(h,k) = 2^{2+\frac{1}{\beta}}\alpha^{-1}\int_{2^{k+1-\alpha}}^\infty \frac{\sigma_1(h2^{-2^y})}{y}\,dy,$$

$$\sigma_1^*(h,k) = 2(\frac{2}{\gamma})^{\frac{1}{\beta}}(1-2^{-\frac{1}{\beta}2^{k+1}(1-2^{-\alpha})})^{-1}\int_{2^{\frac{1}{\beta}2^{k+1-\alpha}}}^\infty \sigma_1(h2^{-y^\beta})dy,$$

$$\sigma_2(h,k) = 4\alpha^{-1}\int_{2^{k+1-\alpha}}^\infty \frac{\sigma_2(h2^{-2^y})}{y}\,dy$$

*for any given $0 < \alpha < 1$.*

**Proof.** Following the proof of Lemma 2.1 in [1], for any positive real number $t$ put

$$t_j = [td(j)/h]/(d(j)/h).$$

We have

$$\|\Gamma(t+s)-\Gamma(t)\| \le \|\Gamma((t+s)_k)-\Gamma(t_k)\| + \sum_{j=0}^{\infty}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\| + \sum_{j=0}^{\infty}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\| \text{ a.s.},$$

where the a.s. continuity of $\Gamma(\cdot)$ is used. Since

$$\sup_{0\le t\le T}\sup_{0\le s\le h}|(t+s)_k-t_k| \le h+d(k)^{-1}h,$$
$$\sup_{0\le t\le T}\sup_{0\le s\le h}|(t+s)_{k+j+1}-(t+s)_{k+j}| \le hd(k+j+1)^{-1},$$

we obtain from (2.1) for any $x>x^*$ and $x_j>x^*$,

$$P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le h}\|\Gamma((t+s)_k)-\Gamma(t_k)\| \ge x\sigma_1(h+d(k)^{-1}h)+\sigma_2(h+d(k)^{-1}h)\Big\} \le 2Kd(k)^2\Big(\frac{T}{h}+1\Big)\exp(-\gamma x^\beta),$$

$$P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le h}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\| \ge x_j\sigma_1(hd(k+j+1)^{-1})+\sigma_2(hd(k+j+1)^{-1})\Big\} \le 2Kd(k+j+1)\Big(\frac{T}{h}+1\Big)\exp(-\gamma x_j^\beta)$$

as well as

$$P\{\sup_{0\le t\le T}\sup_{0\le s\le h}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\| \ge x_j\sigma_1(hd(k+j+1)^{-1})+\sigma_2(hd(k+j+1)^{-1})\} \le 2Kd(k+j+1)(\frac{T}{h}+1)\exp(-\gamma x_j^\beta).$$

Now put

$$\gamma x_j^\beta = \gamma x^\beta + 2^{2^{k+j+1}}.$$

Then

$$\sum_{j=0}^{\infty} d(k+j+1)\exp(-\gamma x_j^\beta) = \sum_{j=0}^{\infty} 2^{2^{2^{k+j+1}}}\exp(-2^{2^{k+j+1}})\exp(-\gamma x^\beta) \le \exp(-\gamma x^\beta)$$

and

$$2\sum_{j=0}^{\infty} x_j\sigma_1(hd(k+j+1)^{-1}) \le 2^{1+\frac{1}{\beta}}x\sum_{j=0}^{\infty}\sigma_1(hd(k+j+1)^{-1}) + 2(\frac{2}{\gamma})^{\frac{1}{\beta}}\sum_{j=0}^{\infty}2^{\frac{1}{\beta}2^{k+j+1}}\sigma_1(hd(k+j+1)^{-1})$$

$$\le 2^{1+\frac{1}{\beta}}\alpha^{-1}x\sum_{j=0}^{\infty}\int_{2^{k+j+1-\alpha}}^{2^{k+j+1}}\frac{\sigma_1(h2^{-2^y})}{y}\,dy/ln2$$
$$+2(\frac{2}{\gamma})^{\frac{1}{\beta}}(1-2^{-\frac{1}{\beta}2^{k+1}(1-2^{-\alpha})})^{-1}\sum_{j=0}^{\infty}\int_{2^{k+j+1-\alpha}}^{2^{k+j+1}}\sigma_1(h2^{-2^y})\,d2^{\frac{1}{\beta}y}$$
$$\le 2^{2+\frac{1}{\beta}}\alpha^{-1}x\int_{2^{k+1-\alpha}}^{\infty}\frac{\sigma_1(h2^{-2^y})}{y}\,dy$$
$$+2(\frac{2}{\gamma})^{\frac{1}{\beta}}(1-2^{-\frac{1}{\beta}2^{k+1}(1-2^{-\alpha})})^{-1}\int_{2^{\frac{1}{\beta}2^{k+1-\alpha}}}^{\infty}\sigma_1(h2^{-y^\beta})\,dy$$

as well as

$$2\sum_{j=0}^{\infty}\sigma_2(hd(k+j+1)^{-1})\le 4\alpha^{-1}\int_{2^{k+j-\alpha}}^{\infty}\frac{\sigma_2(h2^{-2^y})}{y}\,dy.$$

Combining all the above inequalities yields (2.2).

**Remark 2.1.** From the proof of Lemma 2.1, it is easy to see that (2.1) implies

$$P\{\sup_{0\le s\le h}\|\Gamma(T+s)-\Gamma(T)\|\ge x(\sigma_1(h+d(k)^{-1}h)$$
$$+\sigma_1(h,k))+\sigma_1^*(h,k)+\sigma_2(h+d(k)^{-1}h)+\sigma_2(h,k)\}$$
$$\le 4Kd(k)exp(-\gamma x^\beta) \tag{2.2'}$$

for any $T\ge 0, h\ge 0, x\ge x^*$ and $k>0$.

Let

$$\{Y(t),t\ge 0\}=\{X_k(t),t\ge 0\}_{k=1}^{\infty}$$

be defined as in the beginning of Section 1. Furthermore put

$$\sigma^*(h)=\max_{k\ge 1}\sigma_k(h),$$

$$\tilde\sigma(p,h)=\begin{cases}\sigma(\frac{2p}{2-p},h), & \text{if } 1\le p<2,\\ \sigma^*(h), & \text{if } p\ge 2.\end{cases}$$

$$\delta_p^p=E|N(0,1)|^p.$$

For the $l^p$-valued process $Y(\cdot)$, (2.1) has been established by Lemma 3.2 of [2].

**Lemma 2.2.** *With* $p\ge 1$, *we have*

$$P\{\|Y(t+h)-Y(t)\|_{l^p}\ge x\tilde\sigma(p,h)+\delta_p\sigma(p,h)\}\le 2\exp(-x^2/2)$$

*for any* $t,x,h\ge 0$.

## §3. General Results for Large Increments

A function $f(x)$ on $(a,b)$ will be called quasi-increasing, if there exists $c>0$ such that $f(x)\le cf(y)$ for any $a<x<y<b$. In this paper, log means logarithm with base 2.

**Theorem 3.1.** *Let* $Y(\cdot)$ *be defined as above. let* $a_T$ *be a positive continuous quasi-increasing function of* $T$ *with* $a_T\to\infty$ *as* $T\to\infty$. *Assume that*

$$\tilde\sigma(p,T)\to\tilde\sigma_p<\infty \quad \text{and} \quad \sigma(p,T)=o\Big(\Big(\log\frac{T}{a_T}\Big)^{1/2}\Big) \quad \text{as} \quad T\to\infty, \tag{3.1}$$

*and*

$$\int_1^\infty \tilde\sigma(p, 2^{-x^2})\, dx < \infty, \qquad \int_1^\infty \sigma(p, 2^{-2^x})/x\, dx < \infty. \tag{3.2}$$

*Assume that for any* $\delta > 0$,

$$a_T \le T^{1-(\log T)^{-1/2+\delta}} \tag{3.3}$$

*and there is* $a_0 > 0$ *such that for any* $a \ge a_0$,

$$\max_{k\ge1} E(X_k(ia) - X_k((i-1)a))(X_k(ja) - X_k((j-1)a)) \le 0 \tag{3.4}$$

*for every* $j > i \ge 1$. *Then we have*

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(\frac{T}{a_T}))^{1/2}} = 1 \ a.s., \tag{3.5}$$

$$\lim_{T\to\infty} \sup_{0\le t\le T} \frac{\|Y(t+a_T) - Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(\frac{T}{a_T}))^{1/2}} = 1 \ a.s. \tag{3.6}$$

*and*

$$\limsup_{T\to\infty} \frac{\|Y(T+a_T) - Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(\frac{T}{a_T}))^{1/2}} = 1 \ a.s. \tag{3.7}$$

*If conditions* (3.3) *and* (3.4) *are replaced by*

$$a_T \le T^{1-\delta} \text{ for some } 0 < \delta < 1 \tag{3.3'}$$

*and for every* $j > i \ge 1$,

$$\limsup_{a\to\infty} \max_{k\ge1} E(X_k(ia) - X_k((i-1)a))(X_k(ja) - X_k((j-1)a)) \le 0 \tag{3.4'}$$

*respectively,* (3.5) *and* (3.6) *remain true.*

**Proof.** Note that condition (3.3) implies

$$\frac{\log(T/a_T)}{\log\log T} \to \infty \text{ as } T \to \infty. \tag{3.8}$$

At first we prove

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}} \le 1 \text{ a.s.} \tag{3.9}$$

Lemma 2.2 implies that (2.1) in Lemma 2.1 holds with $K = 2, \gamma = \frac{1}{2}, \beta = 2,\ \sigma_1(h) = \tilde\sigma(p, h)$ and $\sigma_2(h) = \delta_p\sigma(p, h)$. For given $\delta > 0$ in (3.3), let $0 < \alpha < 1$ in (2.2) and $\delta_1 > \frac{1}{2} - \delta$ such that

$$\alpha_1 := \alpha - \log(1 - \delta_1) < 1.$$

Let

$$\epsilon_T = (\log a_T)^{-\delta_1}, \quad k = \log\log\log {a_T}^{\epsilon_T}$$

in (2.2) and $k_1 = \log\log\log a_T$. Then $d(k) = a_T^{\epsilon_T}$ and

$$\begin{aligned} k + 1 - \alpha &= \log\Big[(\log\log a_T)\Big(1 + \frac{\log\epsilon_T}{\log\log a_T}\Big)\Big] + 1 - \alpha \\ &= \log\log\log a_T + \log(1 - \delta_1) + 1 - \alpha \\ &\ge k_1 + 1 - \alpha_1. \end{aligned}$$

Hence using condition (3.2) we have for large $T$

$$\begin{aligned}\sigma_1(a_T,k) &= 2^{5/2}\alpha^{-1}\int_{2^{k+1-\alpha}}^{\infty}\frac{\sigma_1(a_T2^{-2^x})}{x}dx\\ &\le 2^{5/2}\alpha^{-1}\int_1^{\infty}\frac{\sigma_1(2^{-2^{2^{k_1}(2^{1-\alpha_1}-1)x}})}{x}dx\\ &\le \frac{\epsilon}{2}\tilde{\sigma}_p\end{aligned}$$

for any given $\epsilon>0$ provided $T$ is large enough. Similarly for large $T$

$$\sigma_1^*(a_T,k) = 4(1-2^{-2^k(1-2^{-\alpha})})^{-1}\int_{2^{2^{k-\alpha}}}^{\infty}\sigma_1(a_T2^{-x^2})dx \le \epsilon,$$

$$\sigma_2(a_T,k) = 4\alpha^{-1}\int_{2^{k+1-\alpha}}^{\infty}\frac{\sigma_2(a_T2^{-2^x})}{x}dx \le \epsilon.$$

Therefore, by the fact (3.8), condition (3.1) and Lemma 2.1, it follows that for any given $c>0$ and any $T$ large enough

$$\begin{aligned}&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le ca_T}\frac{||Y(t+s)-Y(t)||_{l^p}}{\tilde{\sigma}_p(2\log(T/ca_T))^{1/2}}\ge 1+5\epsilon\Big\}\\ &\le P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le ca_T}\|Y(t+s)-Y(t)\|_{l^p}\\ &\quad\ge(1+\epsilon)\Big(2\Big[\log\frac{T}{ca_T}+\log\log T\Big]\Big)^{1/2}(\sigma_1(ca_T(1+d(k)^{-1}))+\sigma_1(ca_T,k))\\ &\quad+\sigma_1^*(ca_T,k)+\sigma_2(ca_T(1+d(k)^{-1}))+\sigma_2(ca_T,k)\Big\}\\ &\le 9\frac{T}{ca_T}{a_T}^{2\epsilon_T}\exp\Big\{-(1+\epsilon)^2\Big(\log\frac{T}{ca_T}+\log\log T\Big)\Big\}\\ &\le 9c^{2\epsilon}T^{-2\epsilon}{a_T}^{2(\epsilon+\epsilon_T)}(\log T)^{-(1+2\epsilon)}\\ &\le 9c^{2\epsilon}(\log T)^{-(1+2\epsilon)}\end{aligned}\tag{3.10}$$

since ${a_T}^{\epsilon+\epsilon_T}\le T^{\epsilon-\epsilon(\log T)^{-\frac12+\delta}+(\log T)^{-\delta_1}}\le T^{\epsilon}$ for large $T$ if (3.3) holds. Let $T_j=\theta^j$ for some $\theta>1$. Then by the Borel-Cantelli Lemma we have

$$\limsup_{j\to\infty}\sup_{0\le t\le T_j}\sup_{0\le s\le ca_{T_j}}\frac{||Y(t+s)-Y(t)||_{l^p}}{\tilde{\sigma}_p(2\log(T_j/a_{T_j}))^{1/2}}\le 1+5\epsilon\quad a.s.\tag{3.11}$$

Noting that $a_T$ is quasi-increasing we obtain (3.9) from (3.11). By recalling (3.8) the inverse inequality holds from Theorem 3.3 of [2]. So now (3.5) is proved.

In order to show (3.6), it suffices to prove

$$\liminf_{T\to\infty}\sup_{0\le t\le T}\frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}}\ge 1\quad\text{a.s.}\tag{3.12}$$

Assume that conditions (3.3) and (3.4) are satisfied. Let

$$B_{nk}=\{T: kh\le a_T<(k+1)h, n-1\le T<n\}$$

for some $h>0$,

$$a_n'=\inf\{a_T: n-1\le T<n\},\quad a_n^*=\sup\{a_T: n-1\le T<n\}.$$

Then

$$\begin{aligned}
&\liminf_{T\to\infty} \sup_{0\le t\le T} \frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}} \\
&\ge \liminf_{n\to\infty} \min_{a'_n/h-1\le k\le a^*_n/h} \inf_{T\in B_{nk}} \sup_{0\le t\le T} \frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}} \\
&\ge \liminf_{n\to\infty} \min_{a'_n/h-1\le k\le a^*_n/h} \sup_{0\le t\le n-1} \frac{\|Y(t+kh)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(n/kh))^{1/2}} \\
&\quad - \limsup_{n\to\infty} \sup_{0\le t\le n} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log((n-1)/a^*_n))^{1/2}} \\
&=: L_1 - L_2. \qquad (3.13)
\end{aligned}$$

Note that $\tilde{\sigma}(p,h)/\tilde{\sigma}_p \to 0$ as $h\to 0$ and $a^*_n \le ca_n$ since $a_T$ is quasi-increasing. Then by imitating the proof of (3.9), we have

$$L_2 \le \epsilon \quad \text{a.s.} \qquad (3.14)$$

provided $h$ is small enough.

Consider $L_1$. Assume $1\le p<2$. We have (cf. (3.2) of [2])

$$\|Y((j+1)kh)-Y(jkh)\|_{l^p} \ge \frac{\sum\limits_{v=1}^{\infty} \sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\Big(\sum\limits_{v=1}^{\infty}\sigma_v(kh)^{\frac{2p}{2-p}}\Big)^{\frac{p-1}{p}}}.$$

Let

$$\begin{aligned}
\xi(j,k) &= \frac{\sum\limits_{v=1}^{\infty} \sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\tilde{\sigma}(p,kh)\Big(\sum\limits_{v=1}^{\infty}\sigma_v(kh)^{\frac{2p}{2-p}}\Big)^{\frac{p-1}{p}}} \\
&= \frac{\sum\limits_{v=1}^{\infty} \sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\Big(\sum\limits_{v=1}^{\infty}\sigma_v(kh)^{\frac{2p}{2-p}}\Big)^{1/2}}.
\end{aligned}$$

Then, using condition (3.4), we have for $j>i\ge 1$

$$\begin{aligned}
E\xi(i,k)\xi(j,k) = {} & \Big(\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{2p}{2-p}}\Big)^{-1}\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{4(p-1)}{2-p}} \\
& \cdot E(X_v((i+1)kh)-X_v(ikh))(X_v((j+1)kh)-X_v(jkh))
\end{aligned} \qquad (3.15)$$

provided $k$ is large enough. Therefore, by the Slepian inequality, recalling the definition of $B_{nk}$ and noting condition (3.3) and $a_T$ to be quasi-increasing, we obtain that there exists $C>0$ such that for large $n$

$$\begin{aligned}
&P\Big\{\min_{a'_n/h-1\le k\le a^*_n/h} \max_{0\le j\le n/2kh} \xi(j,k) \le (1-\epsilon)(2\log\frac{n}{kh})^{1/2}\Big\} \\
&\le \sum_{k=[a'_n/h]-1}^{[a^*_n/h]} P\Big\{\max_{0\le j\le n/2kh}\xi(j,k)\le (1-\epsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}
\end{aligned}$$

$$\le \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} \Big(1-\exp\Big\{-(1-\epsilon)\log\frac{n}{kh}\Big\}\Big)^{n/2kh}$$
$$\le Ca_n \exp\Big\{-C\Big(\frac{n}{a_n}\Big)^{\epsilon}\Big\}$$
$$\le Cn^{-2}, \tag{3.16}$$

which implies

$$L_1 \ge 1-\epsilon \quad \text{a.s.} \tag{3.17}$$

Assume $p \ge 2$. Take $N_k$ such that $\sigma^2_{N_k}(kh) = \sigma^*(kh)$. Clearly

$$\frac{\|Y((j+1)kh) - Y(jkh)\|_{l^p}}{\tilde\sigma_p} \ge (1-\epsilon)\frac{X_{N_k}((j+1)kh) - X_{N_k}(jkh)}{\sigma_{N_k}(kh)}$$

for large $k$. Along the lines of the proof for the case of $1 \le p < 2$, we have (3.17) as well. Now (3.12) is proved and hence we complete the proof of (3.5) and (3.6) under conditions (3.3) and (3.4).

When conditions (3.3) and (3.4) are replaced by (3.3') and (3.4') respectively, the proof of (3.17) is similar. We consider only the case of $1 \le p < 2$. Let $\epsilon' = \frac{\delta\epsilon^2}{4(3-\delta)}$. Similarly to (3.15), we have for $k$ large enough

$$E\xi(i,k)\xi(j,k) \le \epsilon'. \tag{3.15'}$$

Let $\{\eta_i, i \ge 1\}$ and $\tau$ be independent normal random variables with means zero and $E{\eta_i}^2 = 1-\epsilon'$ and $E\tau^2 = \epsilon'$. Define $\xi_i = \eta_i + \tau$. Then $E{\xi_i}^2 = 1$ and

$$\frac{E(X_v((i+1)kh) - X_v(ikh))(X_v((j+1)kh) - X_v(jkh))}{\sigma_v^2(kh)} \le E\xi_i\xi_j, \qquad j-i \ge 1$$

for large $k$. Therefore (recalling (3.16)) we have

$$P\Big\{\min_{a_n'/h-1\le k\le a_n^*/h}\ \max_{0\le j\le n/2kh} \xi(j,k) \le (1-\epsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}$$
$$\le \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} P\Big\{\max_{0\le j\le n/2kh}\xi_j < (1-\epsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}$$
$$\le \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} \Big(P\Big\{\max_{0\le j\le n/2kh}\eta_j < (1-\frac{\epsilon}{2})\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\} + P\Big\{\tau \ge \frac{\epsilon}{2}\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}\Big)$$
$$\le \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} \Big(\Big(1-\exp\Big\{-\Big(1-\frac{\epsilon}{2}\Big)\log\frac{n}{kh}\Big\}\Big)^{\frac{n}{2kh}} + \exp\Big\{-\frac{\epsilon^2}{4\epsilon'}\log\frac{n}{kh}\Big\}\Big)$$
$$\le Ca_n\Big(e^{-C(n/a_n)^{\epsilon/2}} + \Big(\frac{n}{a_n}\Big)^{-\epsilon^2/4\epsilon'}\Big)$$
$$\le C(ne^{-Cn^{\delta\epsilon/2}} + n^{-2}),$$

which implies (3.17). The proof of (3.6) is complete.

Finally, we prove (3.7). It is enough to show

$$\limsup_{T\to\infty}\frac{||Y(T+a_T) - Y(T)||_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}} \ge 1 \quad \text{a.s.} \tag{3.18}$$

For given $\epsilon^{''} > 0$, take $a_0$ large enough such that

$$\max_{k\geq 1} \frac{E(X_k(ia_0) - X_k((i-1)a_0))(X_k(ja_0) - X_k((j-1)a_0))}{\sigma_k^2(a_0)} \leq \epsilon^{''}, \qquad j > i \geq 1. \tag{3.19}$$

Put $a'_T = a_0[a_T/a_0]$. Then

$$\begin{aligned}\limsup_{T\to\infty} \frac{\|Y(T+a_T) - Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}} &\geq \limsup_{T\to\infty} \frac{\|Y(T+a'_T) - Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a'_T))^{1/2}} \\ &\quad - \limsup_{T\to\infty} \sup_{0\leq s\leq a_0} \frac{\|Y(T+s) - Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}} \\ &=: I_1 - I_2. \end{aligned} \tag{3.20}$$

Noting Remark 2.1, along the lines of the proof of (3.9), we have

$$I_2 \leq \epsilon \quad \text{a.s.} \tag{3.21}$$

for any $\epsilon > 0$. Consider $I_1$. Let $t_0 = 1$. Define $t_k$ by $t_k = t_{k-1} + a'_{t_{k-1}}, k = 1, 2, \cdots$. Then

$$\frac{||Y(t_k + a'_{t_k}) - Y(t_k)||_{l^p}}{\tilde\sigma(p, a'_{t_k})} \geq \frac{\sum_{v=1}^{\infty} \sigma_v(a_{t_k})^{\frac{2(p-1)}{2-p}}(X_v(t_k + a'_{t_k}) - X_v(t_k))}{(\sum_{v=1}^{\infty} \sigma_v(a'_{t_k})^{\frac{2p}{2-p}})^{1/2}}.$$

(3.19) implies

$$E\eta_i\eta_j \leq \epsilon^{''}, \quad \text{for } j > i \geq 1.$$

Put $D_n = \{k : \frac{1}{2}n \leq t_k \leq n-1\}$. Obviously, by condition (3.3), for $k \in D_n, a_{t_k} = o(n)$ as $n \to \infty$. Hence

$$\sum_{k\in D_n} a_{t_k} \geq \sum_{k\in D_n} (t_k - t_{k-1}) - \max_{k\in D_n} a_{t_k} \geq \frac{1}{3}n$$

for large $n$. Let $\{\eta_i\}, \{\xi_i\}$ and $\tau$ be defined as above with $\epsilon^{''}$ instead of $\epsilon^{'}$. By the Slepian lemma we obtain that for large $n, n-1 < T \leq n$,

$$\begin{aligned} &P\Big\{\sup_{T/2\leq t\leq T} \frac{||Y(t+a'_t) - Y(t)||_{l^p}}{\tilde\sigma_p(2\log(t/a_t))^{1/2}} \leq 1-\epsilon\Big\} \\ &\leq P\Big\{\max_{k\in D_n} \xi_k/(2\log(t_k/a_{t_k}))^{1/2} \leq 1 - \frac{\epsilon}{2}\Big\} \\ &\leq \prod_{k\in D_n} P\Big\{\eta_k \leq \Big(1-\frac{\epsilon}{4}\Big)(2\log(t_k/a_{t_k}))^{1/2}\Big\} + P\{\tau \geq \frac{\epsilon}{4}(2\log(t_k/a_{t_k}))^{1/2}\} \\ &\leq \prod_{k\in D_n} \Big(1 - \exp\Big\{-\Big(1-\frac{\epsilon}{4}\Big)\log(t_k/a_{t_k})\Big\}\Big) + \exp\Big\{\frac{-\epsilon^2}{16\epsilon^{''}}\log(t_k/a_{t_k})\Big\} \\ &\leq \exp\Big\{-\sum_{k\in D_n} (a_{t_k}/t_k)^{1-\epsilon/4}\Big\} + (t_k/a_{t_k})^{-\epsilon^2/16\epsilon^{''}} \\ &\leq \exp\Big\{-\frac{1}{3}n^{(1-\frac{\epsilon}{4})(\log n)^{-1/2+\delta}}\Big\} + n^{-(\epsilon^2/16\epsilon^{''})(\log n)^{-1/2+\delta}} \to 0 \end{aligned}$$

as $n \to \infty$, which implies

$$I_1 \geq 1 - \epsilon \quad \text{a.s.} \tag{3.22}$$

Inserting (3.21) and (3.22) into (3.20) yields (3.18), and hence (3.7) is proved. This completes the proof of Theorem 3.1.

## §4. Fractional Ornstein-Uhlenbeck Processes

Let $\{Y(t), t \geq 0\} = \{X_k(t), t \geq 0\}_{k=1}^{\infty}$ be a sequence of independent fractional Ornstein-Uhlenbeck processes of order $\gamma$ with coefficients $\gamma_k$ and $\lambda_k$, where $0 < \gamma < 1, \gamma_k \geq 0, \lambda_k > 0$, i.e., the $X_k(\,\cdot\,)$ are centered stationary Gaussian processes with

$$EX_k(t)X_k(s) = \frac{\gamma_k}{2\lambda_k}(e^{-2\gamma\lambda_k(t-s)} + e^{2\gamma\lambda_k(s-t)} - |e^{\lambda_k(t-s)} - e^{\lambda_k(s-t)}|^{2\gamma}) \tag{4.1}$$

for any $t, s \geq 0$. Hence

$$\begin{aligned}\sigma_k^2(h) &= E(X_k(t+h) - X_k(t))^2 \\ &= \frac{\gamma_k}{\lambda_k}(2 + |e^{\lambda_k h} - e^{-\lambda_k h}|^{2\gamma} - e^{2\gamma\lambda_k h} - e^{-2\gamma\lambda_k h}).\end{aligned} \tag{4.2}$$

Csörgő and shao[1] studied the moduli of continuity and the laws of the iterated logarithm for $\{Y(t), t \geq 0\}$ as an $l^p$-valued process , $p \geq 1$. By the elementary calculation, it is easy to verify that condition (3.4) is satisfied from (4.1). Therefore, as a consequence of Theorem 3.1, we have the following large increment result.

**Theorem 4.1.** *Let $Y(\cdot)$ be defined as above. Assume that conditions* (3.1) *and* (3.2) *are satisfied. Let $a_T$ be defined as in Theorem* 3.1, *and assume that condition* (3.3) *is satisfied. Then* (3.5), (3.6) *and* (3.7) *hold true.*

### References

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